



## THE RECURSIVE CONSTANT MODULUS ALGORITHM; A NEW APPROACH FOR MULTI-PATH INTERFERENCE REDUCTION

El-Barbary<sup>\*</sup> K.

### ABSTRACT

The constant modulus algorithm (CMA) has proved its ability to compensate the severe effect of the multi-path interference and co-channel signals on the constant modulus signals. The application limitation of the CMA is mainly due to its slow rate of convergence. This paper presents a new algorithm, which keeps the desired properties of the original CMA while it reduces the time required for convergence at the expense of some mathematical complexity. The proposed algorithm is derived by analogy to the recursive least squares (RLS) as a fast implementation for the least mean square (LMS) one. The algorithm shows stable operations in both cases of compensating the effect of the multi-path interference by an adaptive correction filter and separation of co-channel signals by an adaptive array followed by a signal canceller. The convergence of the proposed algorithm is an order of magnitude faster than the original CMA on the average.

**KEY WORDS:** Signal processing and communication

---

<sup>\*</sup> Egyptian Armed Forces

## I- Introduction

The ability of the CMA to correct the effect of the multi-path interference on the constant envelope signals has resulted in its consideration in many interesting applications. One of these applications is adjusting a finite impulse response (FIR) filter in such a way that the filter output is a correction version of the received distorted input [1-2]. Another application of the CMA is controlling the weights of an array combiner to produce nulls in the directions of the undesired co-channel interference signals [3]. In both cases the CMA does not rely on a reference signal or the inverse of the auto correlation matrix of the input data to perform the adaptation process. Rather the CMA relies only on a priori knowledge that the transmitted signal has a constant envelope to synthesis the presence of the multi-path interference or co-channel signals. Consequently it adaptively reduces their effects [4-5]. The slow convergence of the CMA has limited its applications in real-time systems such as cellular mobile communications and electronic reconnaissance [5]. This limitation is due to the fact that the gradient algorithm used as an implementation of the CMA, has only one degree of freedom. The previous work in development a fast CMA has concentrated on the techniques for efficient implementation of the steepest descent method. Analytic formulae have been derived to determine the optimum step size, which leads to the fastest convergence in simple environments. An alternative form of the cost function of the CMA, which reduces the mathematical complexity, is introduced in [7]. Also a block formulation for computing a block of length  $N$  (multiple of 2) of the FIR filter outputs and then adjusting the filter once per block leads to reduction of the mathematical complexity of the CMA [8]. In addition to that a modified version of the CMA, called the orthogonalized CMA (OCMA) is obtained [9] by multiplying the updating term of the original CMA by the inverse of the correlation matrix of the input data. This multiplication represents an orthogonalization process of the correlation matrix of the received data, which results in reduction of the number of iterations required for the convergence on the expense of some added mathematical complexity.

This paper presents a new rapidly converging algorithm, which keeps the basic advantages of the CMA that; no rely on reference signal nor on the correlation matrix of the input data, which may be singular. The new algorithm is based on the analogy to the recursive least squares (RLS) as a speed up version of the least mean square algorithm (LMSA) [10] so the new algorithm is called the recursive constant modulus algorithm (RCMA). Complete derivation of the RCMA is introduced in section II. The properties of the RCMA are analyzed in section III, where we compare it with the original CMA and the orthogonalized one. The performance of the CMA is evaluated through computer simulation in sections IV where we consider an interesting application of the proposed algorithm for reduction of the effect of multi-path interference. The behavior of the RCMA as an adaptive algorithm for controlling a FIR filter employed to compensate the effect of the multi-path interference on a sequence of octal phase shift keyed PSK symbols. The results show that the proposed RCMA is on the average 10 times faster than the original CMA at the expense of some mathematical complexity.

---

**II The Recursive Least Squares Equivalent to CMA.**

A finite impulse response (FIR) filter, controlled by an adaptive algorithm, is commonly used to compensate the effect of multi-path interference. This FIR has L taps as shown in Fig.(1). The objective of the adaptation algorithm is to estimate the weights vector of the FIR filter  $w$ , which optimizes a predefined performance function based on the available observation. The performance function of the recursive constant modulus algorithm (RCMA) is defined as

$$\Phi = \sum_{k=0}^N \alpha^{N-k} \varepsilon^2(k) \tag{1}$$

where  $\alpha$  is a positive weighting (forgetting) factor,  $0 < \alpha < 1$ . The error signal,  $\varepsilon(k)$  at the  $k^{\text{th}}$  iteration are defined as:

$$\varepsilon(k) = 1 - |y(k)|^2 \tag{2}$$

The desired envelope of the output of the FIR is normalized to unity, while the actual output of the FIR filter is given by:

$$y(k) = x^T(k) w(k) \tag{3}$$

Where  $x(k)$  and  $w(k)$  are the instantaneous input data vector and the weights vector of the FIR filter at the  $k^{\text{th}}$  time instant respectively. They are defined as

$$x(k) = [x_k, x_{k-1}, \dots, x_{k-L+1}]^T \tag{4}$$

$$w(k) = [w_1(k), w_2(k), \dots, w_L(k)]^T \tag{5}$$

Where T denotes the matrix transpose operator.

The performance function  $\Phi$  penalizes the weighted square difference between the desired envelope and the actual one at the output of the FIR filter. It is required to determine the weight vector  $w$ , which minimize the function,  $\Phi$  as defined in (1), thus we set the gradient of the cumulative error squares  $\Phi$  with respect to the weights vector to zero which implies to

$$\nabla_w \Phi = 2 \sum_{k=0}^N \alpha^{N-k} \varepsilon(k) \frac{\partial \varepsilon(k)}{\partial w} = 0 \tag{6}$$

The error signal could be rewritten as

$$\varepsilon(k) = 1 - |y(k)|^2 = 1 - w^H x^* x^T w \tag{7}$$

Where \* denotes the complex conjugate operation and H denotes the matrix complex

conjugate transpose operation.

The derivatives of the error signal with respect to the weights vector is given by [11-12]

$$\frac{\partial \varepsilon(k)}{\partial w} = -x^*(k)x^T(k)w(k) = y(k)x^*(k) \quad (8)$$

Substituting of equations (7), (8) into (6) we get that

$$-2 \sum_{k=0}^N \alpha^{N-k} (1 - |y(k)|^2) y(k)x^*(k) = 0 \quad (9)$$

It is clear that equation (9) is a non linear equation in the filter weights. Even if, they are linear there will be solution only for the limited case that the length of the observation equals to the number of FIR weights. Instead of solving these equations we choose the FIR weights,  $w$ , to obtain a least-squares fit between the desired output envelope and the actual one. Define the matrix  $R(N)$  and the column vector  $D(N)$  such that,

$$D(N) = R(N)w(N) \quad (10)$$

Where  $R(N)$  and  $D(N)$  are given by the forms,

$$R(N) = \sum_{k=0}^N \alpha^{N-k} S(k)S^H(k) \quad (11)$$

$$D(N) = \sum_{k=0}^N \alpha^{N-k} S(k) \quad (12)$$

where the vector  $S(k)$  is defined as:

$$S(k) = y(k)x^*(k) \quad (13)$$

A generalized solution of the system in (10) is exist if the matrix  $R(N)$  is non-singular one and it is given by

$$w(N) = R^{-1}(N)D(N) \quad (14)$$

However, the system of equations in (10) is non linear equations in the filter weights vector  $w$  since  $y(k)$  itself is a function of  $w$ . In order to solve this conflict we utilize a gradient search to find  $w$ , which minimizes the performance function  $\Phi$  in an iterative manner as follows; First suppose that we have an initial value  $w(N-1)$  for the array weights vector  $w$ , which satisfies the system in (10) at the  $(N-1)^{th}$  iteration thus

$$w(N-1) \stackrel{\Delta}{=} R^{-1}(N-1)D(N-1) \quad (15)$$

Second we note the recursive nature of both D(N) and R(N) as:

$$D(N) = \alpha D(N-1) + S^*(N) \quad (16)$$

$$R(N) = \alpha R(N-1) + S^*(N)S^T(N) \quad (17)$$

Define the matrix  $P = R^{-1}$  and take the matrix inversion of both sides of equation (17) we can verify that [10]

$$P(N) = \frac{1}{\alpha} \left[ P(N-1) - \frac{P(N-1)S^*(N)S^T(N)P(N-1)}{\alpha + S^T(N)P(N-1)S^*(N)} \right] \quad (18)$$

Before we continue we further simplified P(N) as follows; define the column vector K(N) as:

$$K(N) = \frac{1}{\alpha + \Delta(N)} P(N-1)S^*(N) \quad (19)$$

where the scalar  $\Delta(N)$  is given by

$$\Delta(N) = S^T(N)P(N-1)S^*(N) \quad (20)$$

So we can rewrite P(N) as:

$$P(N) = \frac{1}{\alpha} \left[ P(N-1) - K(N)S^T(N)P(N-1) \right] \quad (21)$$

Substituting P and D from (21),(16) and simple manipulation result that the update equation of the FIR filter weight vector as

$$w(N) = w(N-1) + K(N)(1 - |y(N)|^2) \quad (22)$$

Inspection of equation (22) we can verify that the vector of the FIR filter weights change with time by an amount equal to the error signal multiplied by the gain vector K(N). Since K(N) is L dimensional vector each element of the weights vector, in effect, is controlled by one of the elements of K(N). Consequently rapid convergence is expected for the RCMA. This is on the contrast to the steepest decent algorithm used for the original CMA where we have a fixed step size [10].

### III Properties of the recursive constant modulus algorithm (RCMA).

#### III-1 Relation to the original CMA.

The original constant modulus algorithm (CMA) looks for removing the incidental amplitude variations from the received signal, that are caused by the multi-path

interference and co-channel signals [13]. The same objective is the ultimate object of the RCMA that the optimum values of the weights vector  $\mathbf{w}$  reached by the RCMA will minimize the weighted squared difference between the desired constant envelope and the actual envelope at the output of the FIR filter. The only minor problem encountered with the correction filter is the phase roll problem which is the same for the original CMA that is; if  $\mathbf{w}_{opt}$  minimizes the performance function  $\Phi$  then  $w_{opt} e^{j\theta}$ ;  $\theta$  is group phase shift} also minimizes the performance function  $\Phi$ . This problem could be solved by utilization of a phase locked loop to compensate for this group phase shift.

### III-2 Relation to the Orthogonalized CMA.

The RCMA could be viewed as a generalized version of the orthogonalized constant modulus algorithm (OCMA) introduced by Gooch [14]. The relationship between RCMA and OCMA could be explained, by setting the weighting factor  $\alpha=1$ . By simple manipulation of the update equation, (22) of the weights vector, it will reduce for this special case to

$$w(N) = w(N-1) + P(N) \varepsilon(N) y(N) x^*(N) \quad (23)$$

This equation is similar to the update equation of the OCMA given in [14] except that the inverse of the input data auto correlation matrix ( $R_x^{-1}$ ) for the OCMA is replaced by the matrix ( $P=R^{-1}$ ) for the case of the RCMA. This difference can be explained as follows. In practical the used matrices  $R_x$  for the OCMA and  $R$  for the RCMA represent an instantaneous estimate for the correlation matrices given by,  $E\{\mathbf{x}^* \mathbf{x}^T\}$  and  $E\{\mathbf{S}^* \mathbf{S}^T\} = E\{|y(k)|^2 \mathbf{x}^* \mathbf{x}^T\}$  respectively. Thus the difference is just multiplication of the updating term by a real positive value  $|y(k)|^2$  which could be viewed as a scaling factor rather than the fixed step size  $\mu$  used for the OCMA. This scaling increases the rate of convergence in the extreme zone, where  $|y(k)|^2$  is much less than unity and it will not affect the behavior near convergence since  $|y(k)|^2 = 1$ . Moreover controlling the contribution of the arrival samples  $x(N)$  by the weighting parameter  $0 < \alpha < 1$  to the estimate of the  $P$  matrix, represents a more dependent on the recent data, which is desired weighting specially in the case of time variable channel.

The high speed convergence of the RCMA compared to the original CMA, is directly referred to the fact that multiplication by the correlation matrix releases the fundamental limitation of the gradient search used in the CMA that there is only one degree of freedom [10,15]. On the other hand the CMA has the advantage over both the OCMA and RCMA, that its simplicity of computations.

It was shown in [8] that each iteration for the CMA, requires only  $(8L+ 11)$  real mathematical operations, while for the RCMA the mathematical complexity is mainly proportional to  $8L^2$ . For this reason the RCMA is preferable in the case of FIR filters with small number of elements ( $L$ ).

#### IV. Behavior of RCMA for Reduction of Multi-path interference

The optimum correlator receiver is based on the assumption that the channel is additive white Gaussian noise (AWGN) one. In practical environments the received signal is not only corrupted by the AWGN, but also by the multi-path interference, due to the band limited channel. This added interference degrades the signal to noise ratio consequently it increases the probability of error. In this section we consider the behavior of octal PSK signal transmitted over a specular multi-path channel. The channel is represented by a three taps FIR filter, with impulse response

$$h = [0.815, 0.419, 0.419]$$

The RCMA employed to control a FIR filter with 11 taps for compensation of the degradation caused by the inter-symbol interference. The received signal-to-noise ratio (SNR) is 10 dB, thus the achievable probability of error, if no multi-path present, is (0.02). We consider the corruption by additive noise and the multi-path interference, which reduces the signal-to-interference ratio (SIR) to only 1.8 dB. Fig(2)(a,b) show the original octal PSK and the corrupted received signal constellation at the input of the FIR filter respectively. It is so clear that the probability of error is much high due to the added multi-path interference. The convergence behavior of the RCMA and the obtained probability of error are plotted, versus the number of iterations, in Fig.(3)(a,b) respectively. Finally the corrected constellation is shown on figure (4). For each obtained set of filter coefficients we compute the probability of error, as the error relative frequency, the result is shown in Fig.(3-b), where we can see that the probability of error, after 400 iterations, degraded to (0.02), which is approximately equal that one for the case of no multi-path interference. Comparison of this result with that previous one for the CMA, [9] where it takes about 5000 iterations to converge, we can safely say that the RCMA is 10 times faster than the CMA.

#### V. Conclusion.

A fast version of the constant modulus adaptive algorithm is derived. The relation of the derived RCMA is analogy to the RLS as fast version of the LMS algorithm. The RCMA is a generalization for the OCMA algorithm which keeps the desired properties of the CMA while it converges faster. The RCMA shows ability for reducing the effect of the multi-path interference on an octal phase shift keying (OPSK). The RCMA is 10 times faster than the original CMA at the price of some added mathematical complexity.

#### Reference

- [1] R. Treichier & B.G. Agee "A new approach to multi-path estimation and correction of constant modulus signals." IEEE Trans ASSP, Vol. ASS-1, No.2 April 1983.
- [2] O.W. Kwon. C.K. Un, J.C. Lee, "Performance of Constant modulus Adaptive digital filters for interference cancellation," Signal Processing, vol. 26 no.2, pp. 185-196, February 1992.
- [3] R. Gooch & B. Lundeil" An Adaptive Beam-former for Constant Modulus Signals"

- IEEE Proc. ICASSP 1986 pp 2523-2526.
- [4] J.R. Treichler and M.G. Larimore" New Processing Techniques based on constant modulus adaptive algorithm." IEEE Trans ASSP, Vol. ASSP-33, No.2, April 1985.
  - [5] R. Gusani and A. Laurenti " Evaluation of the constant modulus algorithm in blind equalization of three ray multipath fading channels" European Transactions on Telecommunications and Related Technologies, vol. 5, no.2, pp. 187-90, February-April 1995.
  - [6] K. Hilal and P. Duhamel, "Stability and Convergence Analysis of the Constant Modulus Algorithm. Comments on finite Equalization schemes and the stability of the normalized CMA," in Proc. European Signal Processing Conference (Edinburgh, UK), pp. 724-7, 13-16 Sept.1994.
  - [7] M.G. Laimore&J.R. Trichier" Convergence behavior of the constant modulus algorithm" IEEE proc. [CASSP 1983, Boston.
  - [8] B. Bensety & P. Duhamel" Fast Constant Modulus Algorithm" IEEE Proc. ICASSP 1991.
  - [9] R.H. Hughes, E.J. LawrenceandL.P. Witheres, Jr, "A robust CMA adaptive array for multiple narrow band sources" E-systems, Inc., Melpar Divislon 1992.
  - [10] J.G.Proakis 1989' Digital Communications' McGraw Hill ,NY.New york.
  - [11] R.A. Axford " Refined techniques for blind equalization of phase shift keyed (PSK) and quadrature amplitude modulated (QAM) digital communication signals" ph.D. Thesis, University of California, San Diego, CA, 1995.
  - [12] C. R. Johnson,Jr., "Admissibility in Blind Adaptive Channel Equalization, "Controlled Systems Magazine, vol. 11,pp. 3-15, January, 1991.
  - [13] R. Gusani and A. Laurenti "Convergence Analysis of the CMA Blind Equalizer,"IEEE Transaction on Communications, vol. 43, no.2-4, pp.1304-7, Feb.-April 1995.
  - [14] BJ. Sublett, R.P. Gooch & S.H. Goldberg "Co-channel Signals Separation and Bearing" IEEE Proc. ICASSP 1989.
  - [15] Z. Ding, "On Convergence Analysis of Fractionally Spaced Adaptive Blind Equalizers,"IEEE Transactions on Signal Processing, vol. 45,no.3, pp. 650-57, March 1997.
-



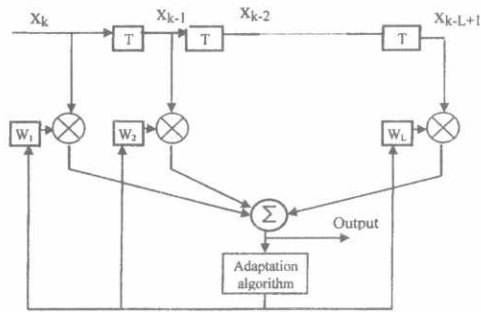


Fig.(1) FIR Adaptive Filter as interference canceller.

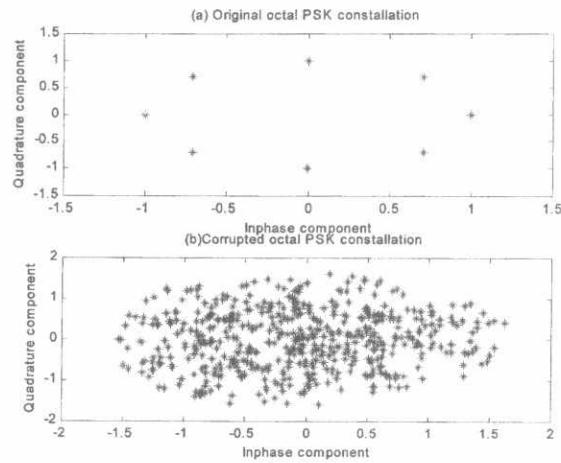


Fig.(2) Constellation of Octal PSK

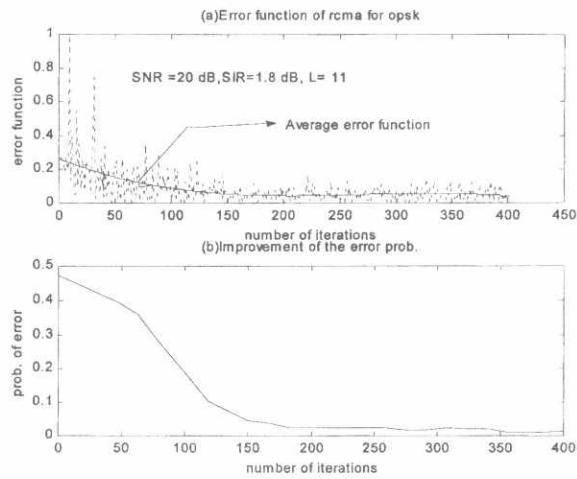


Fig. (3) (a) RCMA Performance function, (b) Error Prob.

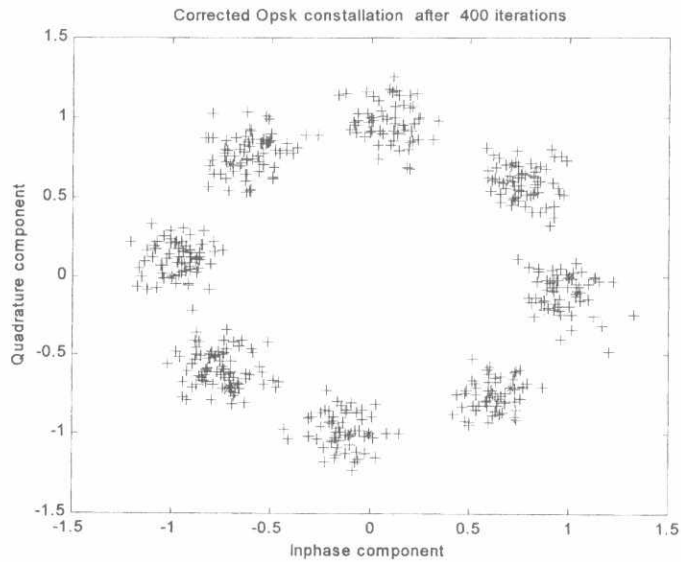


Fig.(4) Constellation of corrected signal.