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ON THE DYNAMICS OF A FRACTIONAL-ORDER RICCATI DIFFERENTIAL EQUATION WITH PERTURBED DELAY

A. M. A. EL-SAYED, S. M. SALMAN, A. A. F. ABDELFATTAH

Abstract. This paper studies the dynamics of a fractional-order Riccati differential equation with perturbed delay and introduces a novel concept of perturbed delay. The study focuses on understanding the behaviour of the solution through the application of analytical techniques to investigate the existence and uniqueness of the solution and its continuous dependence on initial conditions. Analyses of Hopf bifurcations and the local stability of fixed points are studied. The discrete system is generated by piecewise constant arguments in order to simulate the behaviour of the system under consideration. The local stability analysis of the fixed points of the discrete system is presented. Numerical simulations using bifurcation diagrams, Lyapunov exponents and phase diagrams are illustrated. This helps confirm our research and unearth more complex dynamics. The theoretical results of the fractional order Riccati differential equation with delay and its perturbed equation are compared. Our results show that, under specific conditions, the fractional-order Riccati differential equation with perturbed delay exhibits equivalent dynamical properties to the fractional-order Riccati differential equation with delay.

1. Introduction

Dynamical systems are mathematical models that describe the time evolution of a system [1]. They are used in a wide variety of fields, including physics, chemistry, biology, engineering, economics and finance [2-4]. Moreover, they can be used to model a wide variety of phenomena, including the motion of planets, the growth of populations, the spread of diseases and the behaviour of financial markets [5-8]. They can also be used to study the behaviour of complex systems, such as the human brain [9].

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Riccati differential equations find applications in numerous fields of classical and modern science and engineering, including stochastic realisation theory, network synthesis, diffusion problems, optimal filtering, controls, financial mathematics, robust stabilisation and random processes [10-15]. Another important model in physics, the Riccati differential equation is related to the Schrodinger equation of an one dimension [16].

Mathematical equations that incorporate the derivatives of an unknown function at a specific time, which is determined by the function's values at previous points in time, are called delay differential equations [17]. Furthermore, the delay differential equation may be used to characterise the dynamics of physiological systems as well as electrochemical intercalation [18-21]. Moreover, certain systems exhibit instability when subjected to a single delay, but by introducing a second delay, the system can retain its stability [22].

Despite a lengthy history in mathematics, fractional derivatives were not employed for a long time in physics. The nonlocal nature of fractional derivatives may be to blame for their lack of widespread acceptance since they defy obvious geometrical interpretation [23]. Another explanation is that fractional derivatives have various, non-equivalent meanings. Fractional calculus has lately gained a lot of interest from physicists and mathematicians. Many physical phenomena may be explained by using fractional calculus, including the nonlinear oscillation of an earthquake model, traffic flow, fluid dynamics, the continuum, diffusion wave equations and statistical mechanics. [24-26].

Consider the initial value problem of the fractional-order Riccati differential equation with delay

$$\begin{aligned} D^{\gamma}u(t) &= 1-\rho u(t)u(t-r), \qquad t\in(0,T],\\ u(t) &= u_o, \qquad t\le 0, \end{aligned} \tag{1.1}$$

where $\rho, r > 0$.

The problem (1.1) can be rewritten as

$$\begin{aligned} D^{\gamma}u(t) &= 1-\rho u(t)v(t), \qquad t\in(0,T],\\ v(t) &= u(t-r),\\ \\ u(t) &= u_o, \quad v(t)=v_o, \qquad t\le 0. \end{aligned} \tag{1.2}$$

Let there exists a perturbed delay as

$$v(t) = au(t-r) + \epsilon u(t-2r),$$

where $0 < a, \epsilon < 1$.

The problem (1.2) can considered as

$$\begin{aligned} D^{\gamma}u(t) &= 1 - \rho u(t)v(t), \qquad t \in (0,T], \\ v(t) &= au(t-r) + \epsilon u(t-2r), \\ \\ u(t) &= u_o, \quad v(t) = v_o, \qquad t \leq 0, \end{aligned} \tag{1.3}$$

The article is organized in the following structure. Section (2) contains basic definitions of differentiation and integration of fractional orders. The existence of the solution of the fractional-order Riccati differential equation with perturbed delay is discussed in Subsection (3.1). In Subsection (3.2) the continuous dependence of the solution on the initial conditions is studied. Local stability of problem (1.3) is studied in Subsection (3.3). The Hopf bifurcation analysis is performed in Subsection (3.4). The method of discretization of the differential equation of fractional-order Riccati with a perturbed delay is presented in Subsection (3.5). Local stability of the discrete system is performed in Subsection (3.6). In Subsection (3.7), we confirm the obtained results with numerical simulations. The work's summary and knowledge discussion are included in Section (4).

2. Basic Definitions

We will recall the basic definitions of (Caputo) fractional-order differentiation and integration [26-28].

Definition 2.1. *The function $f(t), t > 0$ has a fractional integral of order $\gamma \in \mathbb{R}^+$ described as follows*

$$I^{\gamma} f(t) = \int_0^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} f(\phi) d\phi.$$

Definition 2.2. *The function $f(t), t > 0$ has a fractional derivative of order $\gamma \in (0,1)$ as below*

$$D^{\gamma} f(t) = I^{1-\gamma} \frac{d}{dt} f(t).$$

3. Main Results

The problem (1.3) can be rewritten as

$$\begin{aligned} D^{\gamma}u(t) &= 1 - \rho u(t)\Big[au(t-r) + \epsilon u(t-2r)\Big], \qquad t \in (0,T], \\ u(t) &= u_o, \qquad t \leq 0. \end{aligned} \tag{3.1}$$

3.1. **Existence and uniqueness.**

Theorem 3.1. *If* $\rho < \dfrac{\Gamma(1+\gamma)}{(3a+4\epsilon)T^{\gamma}}$, *then there is a unique solution $u \in C[0,T]$ of the problem* (3.1).

Proof. Let $Q = \{\ u \in \mathbb{R} : 0 \leq u(t) \leq 1,\ t \in [0,T]\ \}$ and operator $F : C[0,T] \to C[0,T]$ by

$$
\begin{aligned}
Fu(t) &= u_o + \int_0^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} \Big(1 - \rho u(\phi)\Big[au(\phi-r) + \epsilon u(\phi-2r)\Big]\Big)\, d\phi,\\
&= u_o + \int_0^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} d\phi - \int_0^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} \rho u(\phi)\Big[au(\phi-r) + \epsilon u(\phi-2r)\Big] d\phi,\\
&= u_o + \int_0^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} d\phi - \int_0^r \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} \rho u(\phi)\Big[au(\phi-r) + \epsilon u(\phi-2r)\Big] d\phi\\
&- \int_r^{2r} \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} \rho u(\phi)\Big[au(\phi-r) + \epsilon u(\phi-2r)\Big] d\phi\\
&- \int_{2r}^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} \rho u(\phi)\Big[au(\phi-r) + \epsilon u(\phi-2r)\Big] d\phi,
\end{aligned}
$$

$$
\begin{aligned}
&= u_o + \int_0^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} d\phi - \int_0^r \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} \rho u(\phi)\Big[au_o + \epsilon u_o\Big] d\phi\\
&- \int_r^{2r} \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} \rho u(\phi)\Big[au(\phi-r) + \epsilon u_o\Big] d\phi\\
&- \int_{2r}^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} \rho u(\phi)\Big[au(\phi-r) + \epsilon u(\phi-2r)\Big] d\phi.
\end{aligned}
$$

Now for $x,\ y \in C[0,T]$, we can obtain

$$
\begin{aligned}
|Fu - Fv| &\leq \rho(a+\epsilon)u_o \int_0^r \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} |u(\phi) - v(\phi)| d\phi + \rho\epsilon u_o \int_r^{2r} \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} |u(\phi) - v(\phi)| d\phi\\
&+ \rho a \int_r^{2r} \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} |u(\phi)u(\phi-r) - v(\phi)v(\phi-r)| d\phi\\
&+ \rho a \int_{2r}^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} |u(\phi)u(\phi-r) - v(\phi)v(\phi-r)| d\phi\\
&+ \rho\epsilon \int_{2r}^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} |u(\phi)u(\phi-2r) - v(\phi)v(\phi-2r)| d\phi,\\
&\leq \rho(a+\epsilon)u_o \int_0^r \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} |u(\phi) - v(\phi)| d\phi + \rho\epsilon u_o \int_r^{2r} \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} |u(\phi) - v(\phi)| d\phi\\
&+ \rho a \int_r^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} \Big|\Big[u(\phi) - v(\phi)\Big]v(\phi-r) + u(\phi)\Big[u(\phi-r) - v(\phi-r)\Big]\Big| d\phi\\
&+ \rho\epsilon \int_{2r}^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} \Big|\Big[u(\phi) - v(\phi)\Big]v(\phi-2r) + u(\phi)\Big[u(\phi-2r) - v(\phi-2r)\Big]\Big| d\phi,
\end{aligned}
$$

then,

$$
\begin{aligned}
||Fu - Fv|| \leq\ & \frac{\rho(a+\epsilon)u_o\big(t^\gamma - (t-r)^\gamma\big)}{\Gamma(1+\gamma)}||u-v|| + \frac{\rho\epsilon u_o\big((t-r)^\gamma - (t-2r)^\gamma\big)}{\Gamma(1+\gamma)}||u-v|| \\
&+ \frac{\rho a(t-r)^\gamma}{\Gamma(1+\gamma)}\ ||u-v||\ ||v|| + \frac{\rho a(t-r)^\gamma}{\Gamma(1+\gamma)}\ ||u-v||\ ||u|| + \frac{\rho\epsilon(t-2r)^\gamma}{\Gamma(1+\gamma)}\ ||u-v||\ ||v|| \\
&+ \frac{\rho\epsilon(t-2r)^\gamma}{\Gamma(1+\gamma)}\ ||u-v||\ ||u||, \\
\leq\ & \frac{\rho(a+\epsilon)T^\gamma}{\Gamma(1+\gamma)}||u-v|| + \frac{\rho\epsilon T^\gamma}{\Gamma(1+\gamma)}||u-v|| + \frac{\rho a T^\gamma}{\Gamma(1+\gamma)}\ ||u-v|| + \frac{\rho a T^\gamma}{\Gamma(1+\gamma)}\ ||u-v|| \\
&+ \frac{\rho\epsilon T^\gamma}{\Gamma(1+\gamma)}\ ||u-v|| + \frac{\rho\epsilon T^\gamma}{\Gamma(1+\gamma)}\ ||u-v||, \\
\leq\ & \frac{\rho(3a+4\epsilon)T^\gamma}{\Gamma(1+\gamma)}\ ||u-v||.
\end{aligned}
$$

If $\rho < \dfrac{\Gamma(1+\gamma)}{(3a+4\epsilon)T^\gamma}$, then F is contraction and problem (3.1) has a unique solution $u \in C[0,T]$. □

3.2. **Continuous dependence.**

Definition 3.1. *The solution of the problem* (3.1) *depends continuously on the initial value* u_o *if* $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ *such that* $|u_o - u_o^*| \leq \delta$ *implies that* $||u - u^*|| \leq \epsilon$ *where* u^* *is the solution of the problem*

$$
\begin{aligned}
D^\gamma u(t) &= 1 - \rho u(t)\Big[au(t-r) + \epsilon u(t-2r)\Big], \qquad t \in (0,T], \\
u(t) &= u_o^*, \qquad t \leq 0.
\end{aligned}
\tag{3.2}
$$

Theorem 3.2. *If* $\rho T^\gamma(3a+4\epsilon) \neq \Gamma(1+\gamma)$, *then the unique solution of* (3.1) *depends continuously on the initial value* u_o.

Proof. Let u and u^* are the solutions of the problems (3.1) and (3.2) respectively, then

$$
\begin{aligned}
||u(t) - u^*(t)|| \leq |u_o - u_o^*| &+ \rho(a+\epsilon)\int_0^r \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}|u(\phi)u_o - u^*(\phi)u_o^*|d\phi \\
&+ \rho\epsilon\int_r^{2r} \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}|u(\phi)u_o - u^*(\phi)u_o^*|d\phi \\
&+ \rho a\int_r^{2r} \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}|u(\phi)u(\phi-r) - u^*(\phi)u^*(\phi-r)|d\phi \\
&+ \rho a\int_{2r}^{t} \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}|u(\phi)u(\phi-r) - u^*(\phi)u^*(\phi-r)|d\phi \\
&+ \rho\epsilon\int_{2r}^{t} \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}|u(\phi)u(\phi-2r) - u^*(\phi)u^*(\phi-2r)|d\phi,
\end{aligned}
$$

$$\leq |u_o - u_o^*| + \rho(a+\epsilon)\int_0^r \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}|[u(\phi)-u^*(\phi)]u_o + [u_o - u_o^*]u^*(\phi)|d\phi$$
$$+\rho\epsilon\int_r^{2r} \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}|[u(\phi)-u^*(\phi)]u_o + [u_o - u_o^*]u^*(\phi)|d\phi$$
$$+\rho a\int_r^{t} \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}|u(\phi)u(\phi-r) - u^*(\phi)u^*(\phi-r)|d\phi$$
$$+\rho\epsilon\int_{2r}^{t} \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}|u(\phi)u(\phi-2r) - u^*(\phi)u^*(\phi-2r)|d\phi,$$

$$\leq |u_o - u_o^*| + \rho(a+\epsilon)||u-u^*||\ |u_o|\int_0^r \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}d\phi$$
$$+\rho(a+\epsilon)|u_o-u_o^*|\ ||u^*||\int_0^r \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}d\phi + \rho\epsilon||u-u^*||\ |u_o|\int_r^{2r}\frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}d\phi$$
$$+\rho\epsilon|u_o-u_o^*|\ ||u^*||\int_r^{2r}\frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}d\phi$$
$$+\rho a\int_r^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}\Big|\Big[u(\phi)-u^*(\phi)\Big]u^*(\phi-r) + u(\phi)\Big[u(\phi-r)-u^*(\phi-r)\Big]\Big|d\phi$$
$$+\rho\epsilon\int_{2r}^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}\Big|\Big[u(\phi)-u^*(\phi)\Big]u^*(\phi-2r) + u(\phi)\Big[u(\phi-2r)-u^*(\phi-2r)\Big]\Big|d\phi,$$

then,

$$||u-u^*|| \leq |u_o-u_o^*| + \frac{\rho(a+\epsilon)\big(t^\gamma-(t-r)^\gamma\big)}{\Gamma(1+\gamma)}||u-u^*||\ |u_o| + \frac{\rho(a+\epsilon)\big(t^\gamma-(t-r)^\gamma\big)}{\Gamma(1+\gamma)}|u_o-u_o^*|\ ||u^*||$$
$$+\frac{\rho\epsilon\big((t-r)^\gamma-(t-2r)^\gamma\big)}{\Gamma(1+\gamma)}||u-u^*||\ |u_o| + \frac{\rho\epsilon\big((t-r)^\gamma-(t-2r)^\gamma\big)}{\Gamma(1+\gamma)}|u_o-u_o^*|\ ||u^*||$$
$$+\frac{\rho a(t-r)^\gamma}{\Gamma(1+\gamma)}||u-u^*||\ ||u^*|| + \frac{\rho a(t-r)^\gamma}{\Gamma(1+\gamma)}||u||\ ||u-u^*|| + \frac{\rho\epsilon(t-2r)^\gamma}{\Gamma(1+\gamma)}||u-u^*||\ ||u^*||$$
$$+\frac{\rho\epsilon(t-2r)^\gamma}{\Gamma(1+\gamma)}||u||\ ||u-u^*||,$$

which implies

$$||u-u^*|| \leq \frac{\Gamma(1+\gamma)+\rho T^\gamma(a+2\epsilon)}{\Gamma(1+\gamma)-\rho T^\gamma(3a+4\epsilon)}\ |u_o-u_o^*|,$$

which proves that

$$|u_o-u_o^*|\leq\delta \quad \Longrightarrow \quad ||u-u^*|| \leq \frac{\Gamma(1+\gamma)+\rho T^\gamma(a+2\epsilon)}{\Gamma(1+\gamma)-\rho T^\gamma(3a+4\epsilon)}\delta = \epsilon^*.$$

□

3.3. **The local stability of (3.1).** The local stability of the equilibrium points of problem (3.1) will be studied. Namely, $u^*_{1,2} = \pm\dfrac{1}{\sqrt{\rho(a+\epsilon)}}$, are the solutions of equation $1-\rho u[au+\epsilon u]=0$.

We will use the linearization and Routh-Hurwitz stability criterion [29] to study the local stability of the equilibrium points.

We get the linearized equation as

$$D^{\gamma}v(t) \approx \mp \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}v(t-r) \mp \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}v(t-2r)$$

and its characteristic equation is given by

$$s^{\gamma} \mp \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}e^{-sr} \mp \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}e^{-2sr} = 0. \tag{3.3}$$

To estimate the local stability of the equation (3.3) at the equilibrium points $u^*_{1,2}$ we used Routh-Hurwitz stability criterion. We get the following results.

Proposition 3.1.

(1) The equilibrium point $u_1^* = \dfrac{1}{\sqrt{\rho(a+\epsilon)}}$ *is stable.*
(2) The equilibrium point $u_2^* = \dfrac{-1}{\sqrt{\rho(a+\epsilon)}}$ *is unstable.*

3.4. **Hopf bifurcation.** The following is a discussion of the Hopf bifurcation that we cover in this part. Hopf bifurcation studies have been conducted on delayed differential equation of fractional order in [30].

Theorem 3.3. *If* $Re\left[\dfrac{ds}{d\epsilon}\right]\Big|_{\epsilon=\epsilon_*} = \dfrac{dk}{d\epsilon}\Big|_{k=0,\omega=\omega_o,\epsilon=\epsilon_*} \neq 0$, $\epsilon_* = \dfrac{-a\big(\sin(r\omega_o)+\cos(r\omega_o)\big)}{\sin(2r\omega_o)+\cos(2r\omega_o)}$,

$\omega_o = \left(\pm\dfrac{i\sqrt{a\rho}\ \sec\left(\frac{\pi\gamma}{2}\right)\sin(r\omega_o)}{\sqrt{\sin(3r\omega_o)-\sin(4r\omega_o)+\cos(r\omega_o)-1}}\right)^{1/\gamma}$, *then there is a Hopf bifurcation when*

$\epsilon = \epsilon_*$ *at the equilibrium* u_1^*, *where*

$$\begin{aligned}\frac{dk}{d\epsilon}\Big|_{k=0,\omega=\omega_o,\epsilon=\epsilon_*} &= \left[\sqrt{\rho}\Big(-a\cos(r\omega_o)+\epsilon_*\sin(2r\omega_o)+(\epsilon_*-2)\cos(2r\omega_o)\Big)\right] \\ &\div \Big[\rho r^2(a+\epsilon_*)^2\left(a^2+2a\epsilon_*\cos(r\omega_o)+\epsilon_*^2\right)+\gamma\omega^{\gamma}(a+\epsilon_*)^{3/2} \\ &+2\sqrt{\rho}r\epsilon_*(a+\epsilon_*)\cos(2r\omega_o)\Big].\end{aligned}$$

Proof. Suppose that equation (3.3) has a pure imaginary solution $s = i\omega_o,\ \omega_o \in R^+$ for a given value of a parameter $\epsilon = \epsilon_*$. Therefore, we get the following equation

$$(i\omega_o)^\gamma - \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}e^{-i\omega_o r} - \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}e^{-2i\omega_o r} = 0.$$

We can rephrase that by

$$i^\gamma\omega_o^\gamma - \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\left(\cos(r\omega_o) - i\sin(r\omega_o)\right) - \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\left(\cos(2r\omega_o) - i\sin(2r\omega_o)\right) = 0.$$

This complex equation is equivalent to the two real equations

$$\omega_o^\gamma \cos(\frac{\gamma\pi}{2}) - \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\cos(r\omega_o) - \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\cos(2r\omega_o) = 0, \tag{3.4}$$

$$\omega_o^\gamma \sin(\frac{\gamma\pi}{2}) + \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\sin(r\omega_o) + \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\sin(2r\omega_o) = 0. \tag{3.5}$$

By solving equation (3.4) and equation (3.5), we get

$$\epsilon_* = \frac{-a\big(\sin(r\omega_o) + \cos(r\omega_o)\big)}{\sin(2r\omega_o) + \cos(2r\omega_o)},$$

$$\omega_o = \left(\pm\frac{i\sqrt{a\rho}\ \sec\left(\frac{\pi\gamma}{2}\right)\sin(r\omega_o)}{\sqrt{\sin(3r\omega_o) - \sin(4r\omega_o) + \cos(r\omega_o) - 1}}\right)^{1/\gamma}.$$

In what follows, we show that condition $\mathrm{Re}\left[\frac{ds}{d\epsilon}\right]\Big|_{\epsilon=\epsilon_*} \neq 0$ is investigated. Put $s(\epsilon) = k(\epsilon) + i\omega(\epsilon)$ and use equation (3.3), we have

$$s^\gamma(\epsilon) - \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}e^{-rs(\epsilon)} - \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}e^{-2rs(\epsilon)} = 0. \tag{3.6}$$

By differentiating (3.6) with respects to ϵ, we obtain

$$\gamma s^{\gamma-1}(\epsilon)\frac{ds}{d\epsilon} + \frac{\sqrt{\rho}e^{-2rs(\epsilon)}\left(2r(a+\epsilon)\left(ae^{rs(\epsilon)} + 2\epsilon\right)\frac{ds}{d\epsilon} + a\left(e^{rs(\epsilon)} - 2\right) - \epsilon\right)}{2(a+\epsilon)^{\frac{3}{2}}} = 0,$$

then,

$$\frac{ds}{d\epsilon} = \frac{\sqrt{\rho}e^{-2rs}\left(\epsilon - a(e^{rs} - 2)\right)}{2(a+\epsilon)^{\frac{3}{2}}\gamma s^{\gamma-1} + 2r\sqrt{\rho}e^{-2rs}(a+\epsilon)(ae^{rs} + 2\epsilon)},$$

by substituting with $s(\epsilon) = k(\epsilon) + i\omega(\epsilon)$, we get

$$
\begin{aligned}
\frac{ds}{d\epsilon} = & \Big[\sqrt{\rho}e^{-2rk}\big(\cos(2r\omega) - i\sin(2r\omega)\big)\big(\epsilon - a(e^{rk}\big(\cos(r\omega) + i\sin(r\omega)\big) - 2)\big)\Big] \\
& \div \Big[2(a+\epsilon)^{\frac{3}{2}}\gamma(k+i\omega)^{\gamma-1} + 2r\sqrt{\rho}e^{-2rk}\big(\cos(2r\omega) - i\sin(2r\omega)\big)(a+\epsilon) \\
& \times (ae^{rk}\big(\cos(r\omega) + i\sin(r\omega)\big) + 2\epsilon)\Big],
\end{aligned}
\tag{3.7}
$$

by Separating equation (3.7) into its real and imaginary parts and using $\frac{ds}{d\epsilon} = \frac{dk}{d\epsilon} + i\frac{d\omega}{d\epsilon}$, we can obtain

$$
\begin{aligned}
\mathrm{Re}\Big[\frac{ds}{d\epsilon}\Big] = \frac{dk}{d\epsilon} = & \Big[\sqrt{\rho}e^{-2rk}(\epsilon - ae^{rk}\cos(r\omega_o) - 2)\cos(2r\omega_o) + \sqrt{\rho}e^{-2rk}(\epsilon - ae^{rk}\sin(r\omega_o))\sin(2r\omega_o)\Big] \\
& \div \Big[(a+\epsilon)^{3/2}\gamma(k^2+\omega^2)^{\gamma/2} + 2r\sqrt{\rho}e^{-2rk}(a+\epsilon)\epsilon\cos(2r\omega_o) + r^2\rho e^{-4rk}(a+\epsilon)^2 \\
& \times (a^2 + 2a\epsilon e^{rk}\cos(r\omega_o) + \epsilon^2)\Big],
\end{aligned}
$$

then,

$$
\frac{dk}{d\epsilon}\Big|_{k=0,\omega=\omega_o,\epsilon=\epsilon_*} = \frac{\sqrt{\rho}\Big(-a\cos(\mathrm{r}\omega_o) + \epsilon_*\sin(2\mathrm{r}\omega_o) + (\epsilon_* - 2)\cos(2\mathrm{r}\omega_o)\Big)}{\rho r^2(a+\epsilon_*)^2\,(a^2 + 2a\epsilon_*\cos(\mathrm{r}\omega_o) + \epsilon_*^2) + \gamma\omega^\gamma(a+\epsilon_*)^{3/2} + 2\sqrt{\rho}r\epsilon_*(a+\epsilon_*)\cos(2\mathrm{r}\omega_o)}.
$$

If $\mathrm{Re}\Big[\frac{ds}{d\epsilon}\Big]\Big|_{\epsilon=\epsilon_*} = \frac{dk}{d\epsilon}\Big|_{k=0,\omega=\omega_o,\epsilon=\epsilon_*} \neq 0$, hence when the parameter ϵ crosses a certain critical value

$$
\epsilon_* = \frac{-a\big(\sin(r\omega_o) + \cos(r\omega_o)\big)}{\sin(2r\omega_o) + \cos(2r\omega_o)},\ \omega_o = \left(\pm\frac{i\sqrt{a\rho}\ \sec\left(\frac{\pi\gamma}{2}\right)\sin(r\omega_o)}{\sqrt{\sin(3r\omega_o) - \sin(4r\omega_o) + \cos(r\omega_o) - 1}}\right)^{1/\gamma},
$$

the equilibrium point u_1^* undergoes Hopf bifurcation. □

Likewise, we can illustrate that the equilibrium point u_2^* undergoes Hopf bifurcation.

3.5. **The discrete system.** Dynamical systems that are formed by piecewise constant arguments studied in [31-33]. The discretization of fractional-order Riccati differential equation studied in [34].

Consider (1.3) with piecewise constant arguments as follows.

$$\begin{aligned} D^\gamma u(t) &= 1 - \rho u(r[\frac{t}{r}])v(r[\frac{t}{r}]), \qquad t \in (0, T], \\ v(r[\frac{t}{r}]) &= a u(r[\frac{t}{r}] - r) + \epsilon u(r[\frac{t}{r}] - 2r), \end{aligned} \tag{3.8}$$

$$u(t) = u_o, \quad v(t) = v_o, \qquad t \leq 0,$$

where [.] denotes the greatest integer function and r is a constant argument.

Let $t \in [nr,\ (n+1)r)$ and $n = 0, 1, 2, \ldots$. The procedure for discretization is as given below.

1) Let $t \in [0, r)$, then $[\frac{t}{r}] = 0$ and the solution of the problem (3.8) is given by

$$\begin{aligned} I^{1-\gamma} \frac{d}{dt} u(t) &= 1 - \rho u_o v_o \\ u(t) - u(0) &= \Big(1 - \rho u_o v_o\Big) \int_0^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} d\phi \\ u(t) &= u_o + \frac{t^\gamma}{\Gamma(1+\gamma)} \Big(1 - \rho u_o v_o\Big), \end{aligned}$$

$$v_o = a u_o + \epsilon u_o.$$

When $t \to r$ and $u(r) = u_1$ we get

$$\begin{aligned} u_1 &= u_o + \frac{r^\gamma}{\Gamma(1+\gamma)} (1 - \rho u_o v_o), \\ v_o &= a u_o + \epsilon u_o. \end{aligned}$$

2) Let $t \in [r, 2r)$, then $[\frac{t}{r}] = 1$ and the solution of the problem (3.8) is given by

$$\begin{aligned} I^{1-\gamma} \frac{d}{dt} u(t) &= 1 - \rho u_1 v(r) \\ u(t) - u(r) &= \Big(1 - \rho u_1 v(r)\Big) \int_r^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} d\phi \\ u(t) &= u_1 + \frac{(t-r)^\gamma}{\Gamma(1+\gamma)} \Big(1 - \rho u_1 v(r)\Big), \end{aligned}$$

$$v(r) = a u_o + \epsilon u_o.$$

When $t \to 2r$, $u(2r) = u_2$ and $v(r) = v_1$ we get

$$\begin{aligned} u_2 &= u_1 + \frac{r^\gamma}{\Gamma(1+\gamma)} (1 - \rho u_1 v_1), \\ v_1 &= a u_o + \epsilon u_o. \end{aligned}$$

3) Let $t \in [2r, 3r)$, then $[\frac{t}{r}] = 2$ and the solution of the problem (3.8) is given by

$$\begin{aligned}
I^{1-\gamma}\frac{d}{dt}u(t) &= 1 - \rho u_2 v(2r) \\
u(t) - u(2r) &= \Big(1 - \rho u_2 v(2r)\Big)\int_{2r}^{t}\frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)}d\phi \\
u(t) &= u_2 + \frac{(t-2r)^{\gamma}}{\Gamma(1+\gamma)}\Big(1 - \rho u_2 v(2r)\Big), \\
v(2r) &= au_1 + \epsilon u_o.
\end{aligned}$$

When $t \to 3r$, $u(3r) = u_3$ and $v(2r) = v_2$ we get

$$\begin{aligned}
u_3 &= u_2 + \frac{r^{\gamma}}{\Gamma(1+\gamma)}(1 - \rho u_2 v_2), \\
v_2 &= au_1 + \epsilon u_o.
\end{aligned}$$

By repeating the process we can deduce that the solution of problem (3.8) is given by

$$\begin{aligned}
u_{n+1} &= u_n + \frac{r^{\gamma}}{\Gamma(1+\gamma)}(1 - \rho u_n v_n), \\
v_{n+1} &= au_n + \epsilon u_{n-1}.
\end{aligned} \tag{3.9}$$

3.6. **The discrete system's local stability.** The system (3.9) can be rewritten as below

$$\begin{aligned}
u_{n+1} &= u_n + \frac{r^{\gamma}}{\Gamma(1+\gamma)}(1 - \rho u_n v_n), \\
v_{n+1} &= au_n + \epsilon w_n, \\
w_{n+1} &= u_n.
\end{aligned} \tag{3.10}$$

The system has two fixed points (u_1^*, v_1^*, w_1^*) and (u_2^*, v_2^*, w_2^*) where

$$u_1^* = \frac{1}{\sqrt{\rho(a+\epsilon)}}, \quad v_1^* = \frac{a+\epsilon}{\sqrt{\rho(a+\epsilon)}}, \quad w_1^* = \frac{1}{\sqrt{\rho(a+\epsilon)}},$$

$$u_2^* = \frac{-1}{\sqrt{\rho(a+\epsilon)}}, \quad v_2^* = \frac{-(a+\epsilon)}{\sqrt{\rho(a+\epsilon)}}, \quad w_2^* = \frac{-1}{\sqrt{\rho(a+\epsilon)}},$$

where $(a+\epsilon) \neq 0$, which are the solution of the following algebraic system

$$\begin{aligned}
u &= u + \frac{r^{\gamma}}{\Gamma(1+\gamma)}(1 - \rho uv), \\
v &= au + \epsilon w, \\
w &= u.
\end{aligned}$$

The Jacobian matrix associated to the system (3.10) reads

$$J(u,v,w)=\begin{bmatrix} 1-\dfrac{r^\gamma}{\Gamma(1+\gamma)}\rho v & \dfrac{-r^\gamma}{\Gamma(1+\gamma)}\rho u & 0 \\ a & 0 & \epsilon \\ 1 & 0 & 0 \end{bmatrix}.$$

What follows is stability analysis of fixed points (u_1^*, v_1^*, w_1^*) and (u_2^*, v_2^*, w_2^*).

3.6.1. *Stability analysis at* (u_1^*, v_1^*, w_1^*). The Jacobian matrix calculated at (u_1^*, v_1^*, w_1^*) reads

$$J(u_1^*, v_1^*, w_1^*)=\begin{bmatrix} 1-\dfrac{r^\gamma\sqrt{\rho}(a+\epsilon)}{\Gamma(1+\gamma)\sqrt{a+\epsilon}} & \dfrac{-r^\gamma\sqrt{\rho}}{\Gamma(1+\gamma)\sqrt{a+\epsilon}} & 0 \\ a & 0 & \epsilon \\ 1 & 0 & 0 \end{bmatrix}.$$

The characteristic equation associated to $J(u_1^*, v_1^*, w_1^*)$ is given by

$$P(\lambda)\equiv\lambda^3+\left(\frac{r^\gamma\sqrt{\rho}(a+\epsilon)}{\Gamma(1+\gamma)\sqrt{a+\epsilon}}-1\right)\lambda^2+\frac{r^\gamma a\sqrt{\rho}}{\Gamma(1+\gamma)\sqrt{a+\epsilon}}\lambda+\frac{r^\gamma\epsilon\sqrt{\rho}}{\Gamma(1+\gamma)\sqrt{a+\epsilon}}=0.$$

The Jury test described in [35] is used to establish whether or not system (3.10), at the fixed point (u_1^*, v_1^*, w_1^*), is locally stable. We find the following.

Proposition 3.2. *The fixed point* (u_1^*, v_1^*, w_1^*) *is stable if* $0<\rho<\dfrac{(a+\epsilon)\ \Gamma^2(1+\gamma)}{a^2\ r^{2\gamma}}$ *and unstable if* $\rho>\dfrac{(a+\epsilon)\ \Gamma^2(1+\gamma)}{a^2\ r^{2\gamma}}$.

3.6.2. *Stability analysis at* (u_2^*, v_2^*, w_2^*). The Jacobian matrix calculated at (u_2^*, v_2^*, w_2^*) reads

$$J(u_2^*, v_2^*, w_2^*)=\begin{bmatrix} 1+\dfrac{r^\gamma\sqrt{\rho}(a+\epsilon)}{\Gamma(1+\gamma)\sqrt{a+\epsilon}} & \dfrac{r^\gamma\sqrt{\rho}}{\Gamma(1+\gamma)\sqrt{a+\epsilon}} & 0 \\ a & 0 & \epsilon \\ 1 & 0 & 0 \end{bmatrix}.$$

The characteristic equation associated to $J(u_2^*, v_2^*, w_2^*)$ is given by

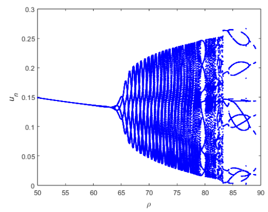
$$P(\lambda) \equiv \lambda^3 - \left(\frac{r^\gamma \sqrt{\rho}(a+\epsilon)}{\Gamma(1+\gamma)\sqrt{a+\epsilon}} + 1 \right) \lambda^2 - \frac{r^\gamma a \sqrt{\rho}}{\Gamma(1+\gamma)\sqrt{a+\epsilon}} \lambda - \frac{r^\gamma \epsilon \sqrt{\rho}}{\Gamma(1+\gamma)\sqrt{a+\epsilon}} = 0.$$

Using Jury test, the second condition not satisfied and we find the following.

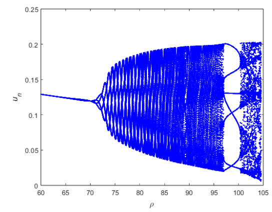
Proposition 3.3. *The fixed point* (u_2^*, v_2^*, w_2^*) *is always unstable.*

3.7. **Numerical simulations.** In this part, to validate our studies we use numerical experiments to draw out the theoretical results and show that changes in r, a, ϵ and γ affect the dynamical behaviour of the dynamical system (3.9). We have been experimenting with different values of r, a, ϵ and γ and then plotting bifurcation diagrams as a function of ρ. Moreover, for each bifurcation diagram, the maximal Lyapunov exponent is introduced below it. In Figure (1a) we start with the initial point $(0.1328, 0.1195, 0.1328)$ at $r = 0.1,\ a = 0.8,\ \epsilon = 0.1,\ \gamma = 0.85$ the system undergoes bifurcation at $\rho \simeq 63.021$. In Figure (1b) we start with the initial point $(0.1195, 0.1195, 0.1195)$ at $r = 0.1,\ a = 0.8,\ \epsilon = 0.2,\ \gamma = 0.85$ the system undergoes bifurcation at $\rho \simeq 70.024$. In Figure (1c) we start with the initial point $(0.2393, 0.2154, 0.2393)$ at $r = 0.2,\ a = 0.8,\ \epsilon = 0.1,\ \gamma = 0.85$ the system suffers a bifurcation at $\rho \simeq 19.397$. Figure (1g) illustrates that the system suffers a bifurcation at $\rho \simeq 55.327$ with initial point $(0.1344, 0.1344, 0.1344)$ and $r = 0.1,\ a = 0.9,\ \epsilon = 0.1,\ \gamma = 0.85$. Figure (1h) illustrates that the system undergoes bifurcation at $\rho \simeq 82.073$ with initial point $(0.1164, 0.1047, 0.1164)$ and $r = 0.1,\ a = 0.8,\ \epsilon = 0.1,\ \gamma = 0.90$. Figure (1i) illustrates that the system undergoes bifurcation at $\rho \simeq 76.422$ with initial point $(0.1144, 0.1144, 0.1144)$ and $r = 0.1,\ a = 0.999,\ \epsilon = 0.001,\ \gamma = 0.95$. We noticed that when $a \to 1$ and $\epsilon \to 0$ the Riccati equation with perturbed delay (1.3) will be the Riccati differential equation (1.1) as shown in Figures (1g) and (1i).

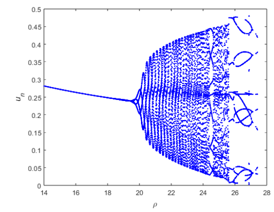
Additionally, we introduce some phase diagrams by taking $r = 0.1$, $a = 0.8$, $\epsilon = 0.2$, $\gamma = 0.85$ and initial point $= (0.1195, 0.1195, 0.1195)$ as in Figure (2). By increasing the value of ρ, the curve rotates clockwise and a period-4 orbit forms and Lyapunov exponent becomes a negative, as shown in Figures (2a)-(2d). The curve turns into a closed curve with an increase in radius and the Lyapunov exponent changes between negative and positive as in Figures (2e)-(2h). In Figure (2i) the closed curve breaks down and the Lyapunov exponent becomes positive again. The curve appears again as in Figures (2j)-(2k) and the Lyapunov exponent changes between negative and positive. In figure (2l) the curve breaks down again and Lyapunov exponent becomes positive again. Figures (2m)-(2n) show that the closed curve appears again and the Lyapunov exponent changes between negative and positive. In Figures (2o)-(2p) the closed curve breaks down and a period-7 orbit forms and Lyapunov exponent becomes negative again. In Figure (2q) show that the closed curve appears again and the Lyapunov exponent becomes positive. In Figures (2r)-(2s) the closed curve breaks down and a period-8 orbit forms and Lyapunov exponent becomes positive again. Figure (2t) shows that the closed curve appears again then disappears and Lyapunov exponent becomes positive.



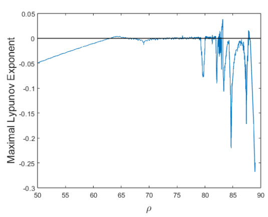
(A) $r = 0.1,\ a = 0.8,\ \epsilon = 0.1,\ \gamma = 0.85$



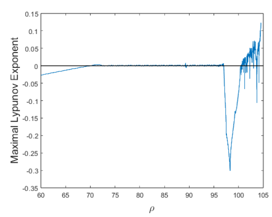
(B) $r = 0.1,\ a = 0.8,\ \epsilon = 0.2,\ \gamma = 0.85$



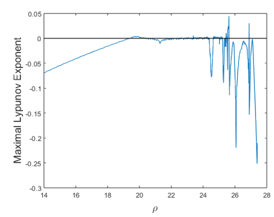
(C) $r = 0.2,\ a = 0.8,\ \epsilon = 0.1,\ \gamma = 0.85$



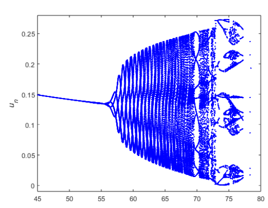
(D) $r = 0.1,\ a = 0.8,\ \epsilon = 0.1,\ \gamma = 0.85$



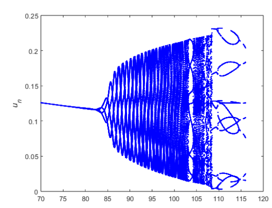
(E) $r = 0.1,\ a = 0.8,\ \epsilon = 0.2,\ \gamma = 0.85$



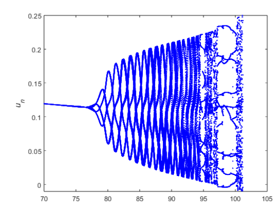
(F) $r = 0.2,\ a = 0.8,\ \epsilon = 0.1,\ \gamma = 0.85$



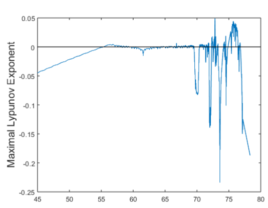
(G) $r = 0.1,\ a = 0.9,\ \epsilon = 0.1,\ \gamma = 0.85$



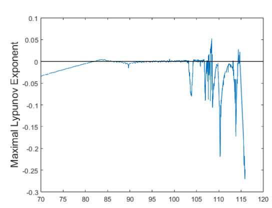
(H) $r = 0.1,\ a = 0.8,\ \epsilon = 0.1,\ \gamma = 0.90$



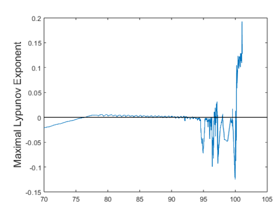
(I) $r = 0.1,\ a = 0.999,\ \epsilon = 0.001,\ \gamma = 0.95$



(J) $r = 0.1,\ a = 0.9,\ \epsilon = 0.1,\ \gamma = 0.85$

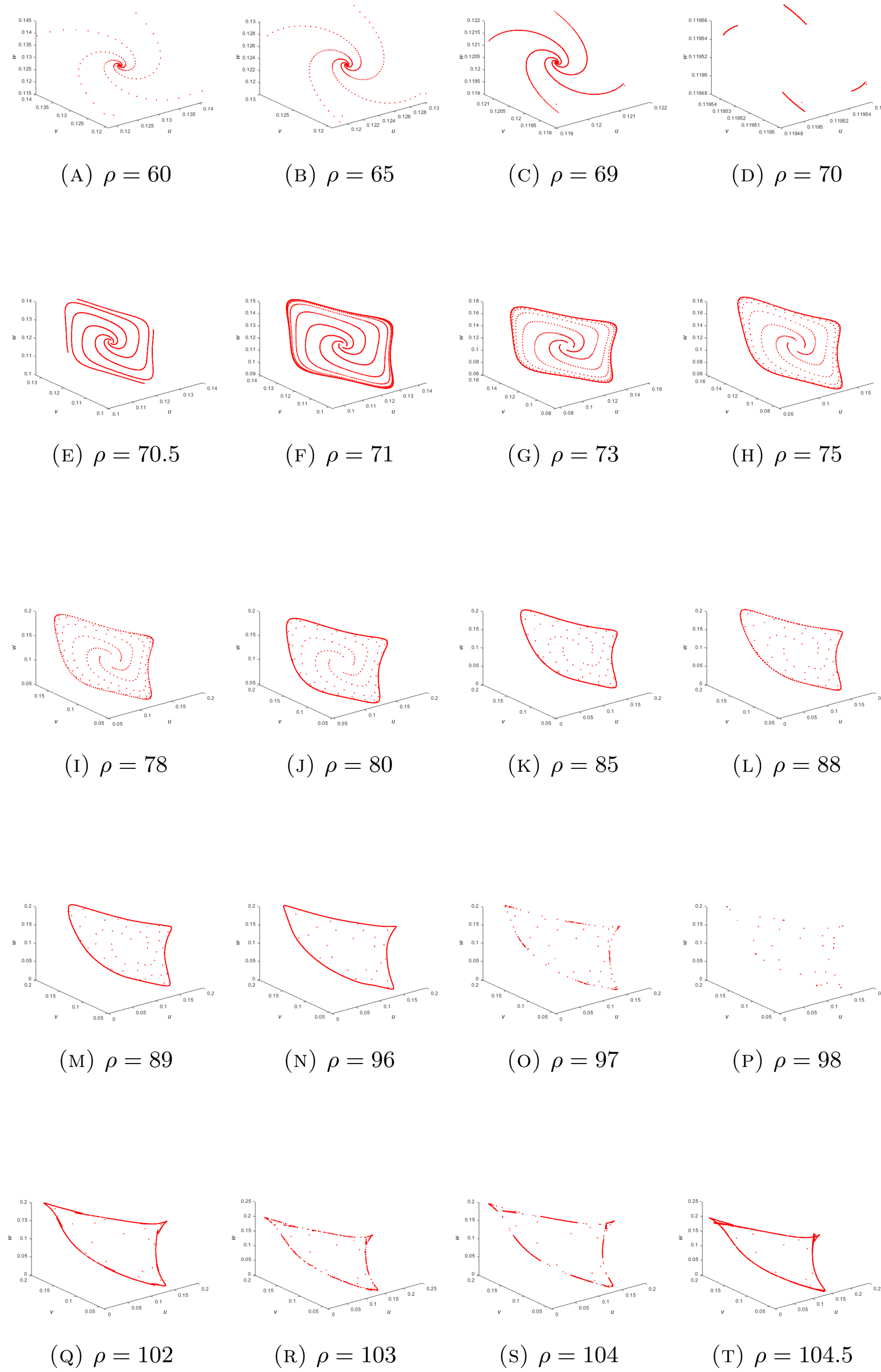


(K) $r = 0.1,\ a = 0.8,\ \epsilon = 0.1,\ \gamma = 0.90$



(L) $r = 0.1,\ a = 0.999,\ \epsilon = 0.001,\ \gamma = 0.95$

FIGURE 1. Bifurcation diagrams of system (3.9) and its corresponding maximum Lyapunov exponent



(A) $\rho = 60$ (B) $\rho = 65$ (C) $\rho = 69$ (D) $\rho = 70$

(E) $\rho = 70.5$ (F) $\rho = 71$ (G) $\rho = 73$ (H) $\rho = 75$

(I) $\rho = 78$ (J) $\rho = 80$ (K) $\rho = 85$ (L) $\rho = 88$

(M) $\rho = 89$ (N) $\rho = 96$ (O) $\rho = 97$ (P) $\rho = 98$

(Q) $\rho = 102$ (R) $\rho = 103$ (S) $\rho = 104$ (T) $\rho = 104.5$

FIGURE 2. Phase diagrams of the system (3.9) with different values of ρ

4. Conclusion

The paper discussed the dynamics of a fractional-order Riccati differential equation with perturbed delay and introduced a novel concept of perturbed delay. The study focused on understanding the behaviour of the solution through the application of analytical techniques to investigate the existence and uniqueness of the solution and its continuous dependence on initial conditions. Analyses of Hopf bifurcations and the local stability of fixed points were presented. Utilising piecewise constant arguments, the discrete system was generated in order to simulate the behaviour of the system under consideration. The local stability analysis of the fixed points of the discrete system was presented. To validate our results, numerical simulations that generated bifurcation diagrams, Lyapunov exponents and phase diagrams were used to better understand the underlying complicated dynamics. The findings of theoretical investigations of the fractional order Riccati differential equation with delay and its perturbed equation were compared. We found that the dynamical system is sensitive to shifts in r, a, γ and that even a little perturbation may cause a significant shift in the system's chaotic behaviour. Moreover, when $a \to 1$ and $\epsilon \to 0$ the problem (1.3) is equivalent to problem (1.1) with the same dynamical properties. As well as, when $\gamma \to 1$ the problem (1.1) and its perturbed equation (1.3) are equivalent to equation (1) and its perturbed equation (3) in [36] respectively, with the same dynamical properties.

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Ahmed M. A. El-Sayed
Faculty of Science, Alexandria University, Alexandria, Egypt
Email address: amasayed@alexu.edu.eg

Sanaa M. Salman
Faculty of Education, Alexandria University, Alexandria, Egypt
Email address: samastars9@alexu.edu.eg

AbdAllah A. F. AbdElfattah
Faculty of Science, Alexandria University, Alexandria, Egypt
Email address: abdallah.awad_pg@alexu.edu.eg