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MULTIPLE SOLUTIONS FOR THE NONHOMOGENOUS GJMS'S OPERATOR

KAMEL TAHRI

Abstract. Let (M,g) be a closed Riemannian maniflold, under some assumptions on f, P_g^k and h we show the existence and multiplicity of solution by The Pass Montain Theorem and Ekeland's Variational Principle of the semi-linear elliptic equation:

$$P_g^k(u) = f(x)\,|u|^{2_k^\sharp - 2}\,u + h(x) \quad \text{on } M$$

In the case of Eisteinian manifold, we obtain the existence of positive and negative solutions.

1. Introduction and Notation

Let (M,g) be a compact Riemannian manifold of dimension $n > 2k$ without boundary with $k \geq 1$. In this decade, there has been extensive analyze of the relationship between the conformally covariant operators which satisfy some invariance properties under conformal change of metric on M and their associated partial differential equations. However, in 1992 Graham, Jenne, Mason & Sparling have defined a family of conformally convariant differential operators in [6] (*GJMS-operators* in short). More precisely, *GJMS-operators* based on the ambiant metric of Graham-Fefferman [3].

Moreover, for any Riemannian metric g on M, there exists a local differential operator named: $P_g^k : C^\infty(M) \to C^\infty(M)$ such that:

$$P_g^k := \Delta_g^k + \text{lower-order terms}$$

where $\Delta_g := -div_g(\nabla_g)$ is the Laplacian Beltrami operator. One of the pertinent geometric behavior of P_g^k which is conformally convariant in the sense that: for all $\varphi \in C^\infty(M), \varphi > 0$ and $\tilde{g} := \varphi^{\frac{4}{n-2k}} g$:

$$P_g^k(\varphi u) = \varphi^{\frac{n+2k}{n-2k}} . P_{\tilde{g}}^k(u)$$

Moreover, P_g^k is self-adjoint with respect to the L^2-scalar product. A scalar invariant is associated to this operator, namely the Q-curvature, denoted as $Q_g^k \in$

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$C^{\infty}(M)$. When $k=1$, P_g^k is the conformal Laplacian and Q-curvature: Q_g^k is the scalar curvature multiplied by a constant. When $k=2$, P_g^k is the Paneitz-Branson operator defined in [11]. The Q-curvature has been introduced by Branson in [12] and generalized by Paneitz in [8]. Therefore, this geometric quantity has been obtained when we take $\varphi \equiv 1$ as

$$Q_g^k := \frac{2}{n-2k} P_g^k(1)$$

Furthermore, to figure out the problem of the prescribed Q-curvature in conformal class amounts to solve a nonlinear elliptic partial differential equations of $2k^{th}$ order. In recent years, many authors proved the prescription of the Q-curvature in $k=2$, as Djadli-Ledoux and Hebey in [14] and also Gursky and Malchiodi in [5].

However, in the specific case of *GJMS-operators* Mazumdar has proved in [7] the existence of $u \in C^{\infty}(M), u>0$ and $f \in C^{\infty}(M)$ given and he established the following results of the following equation:

$$P_g^k(u) = f(x).\left|u\right|^{2_k^\sharp - 2} u \ \text{ in } M \tag{1}$$

We define the standard norm

$$\|u\|_{H_k^2(M)} = \sum_{m=0}^{m=k} \left\|\nabla_g^m u\right\|_{L^2(M)}$$

and the space $H_k^2(M)$ as the completion of $C^{\infty}(M)$ for the norm $\|.\|_{H_k^2(M)}$.

Theorem 1.1 Let (M,g) be compact Riemannian manifold of dimension $n>2k$ without boundary with $k \geq 1$. Let $f \in C^{0,\alpha}(M)$ positive function. We assume that P_g^k is coercive on $H_k^2(M)$. Suppose that

$$\inf_{u\in S_f} \int_M P_g^k(u).u d\mu_g < \frac{1}{K(n,k).\left(\max_{x\in M} f(x)\right)^{\frac{2}{2_k^\sharp}}}$$

where

$$S_f := \left\{ u \in H_k^2(M) : \int_M f(x).\left|u\right|^{2_k^\sharp} d\mu_g = 1 \right\}$$

and

$$K(n,k) := \inf_{u\in D^{k,2}(\mathbb{R}^n)-\{0\}} \frac{\int_{\mathbb{R}^n} \left(\Delta^{\frac{k}{2}} u\right)^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{2_k^\sharp} dx\right)^{\frac{2}{2_k^\sharp}}}$$

Then there exists a solution $u \in C^{2k}(M)$ to the equation (1)

Recently, the author in has considered the Q-curvature problem with perturbation of the form:

$$P_g^k(u) = f(x)\left|u\right|^{2_k^\sharp - 2} u + h(x) \tag{2}$$

Where f is a C^{∞}-function on M with $f>0$ and h belongs to $\left(H_k^2(M)\right)^*$ such that $h \neq 0$ satisfies

$$\exists m>0, \forall u \in S_f : \int_M h(x).u d\mu_g < m.\|u\|_{P_g^k}^{1+\frac{2_k^\sharp}{2_k^\sharp - 2}} \tag{3}$$

with

$$S_f := \left\{ u \in H_k^2(M) : \int_M f(x).\left|u\right|^{2_k^\sharp} d\mu_g = 1 \right\}$$

and

$$m := \frac{2_k^\sharp - 2}{\left(2_k^\sharp - 1\right)^{1+\frac{1}{2_k^\sharp - 2}}}$$

Then, he established the following results:

Theorem 1.2 Let (M, g) be compact Riemannian manifold of dimension $n > 2k$ without boundary with $k \geq 1$. Let $f \in C^\infty(M)$ positive function and $h \in \left(H_k^2(M)\right)^*$ such that $h \neq 0$ satisfies the condition (3). We assume that P_g^k is coercive on $H_k^2(M)$. Suppose that

$$\sup_{t\geq 0} J\left(u_o + tu_\epsilon\right) < c_o + \frac{k}{n\left[K\left(n,k\right)\right]^{\frac{n}{2k}}\left[\max_{x\in M} f(x)\right]^{\frac{n-2k}{2k}}}$$

and at a point a where f atteints its maximum the following condition

$$\frac{\Delta f(a)}{f(a)} < \frac{2\left(k+4\right)n^2 - 8\left(k+1\right)n - 24k}{3\left(n-2k\right)\left(n+2\right)(n-6)} S_g(a)$$

holds. Then, the equation (2) has two non trivial solutions.

In this paper, we consider the multiplicity results of solutions of the following nonhomogenous $2k^{th}$ order elliptic equation involving GJMS's operator:

$$P_g^k(u) = f(x)\left|u\right|^{2_k^\sharp - 2} u + h(x) \tag{4}$$

Where f is a C^∞-function on M with $f > 0$ and h belongs to $L^q(M)$ such that

$$q := \frac{2_k^\sharp}{2_k^\sharp - 1} = \frac{2n}{n+2k}$$

and also, $2_k^\sharp = \frac{2n}{n-2k}$ is the critical Sobolev's exponent for the embedding $H_k^2\left(M\right) \subset L^{2_k^\sharp}\left(M\right)$.

Now we define when P_g^k is coercive, our working norm as follow: for all $u \in H_k^2\left(M\right)$:

$$\|u\|_{P_g^k}^2 := \int_M P_g^k(u).ud\mu_g$$

In recent years, there are some results of existence in certain cases concerning the GJMS-operator. The object of this paper is to establish the existence and multiplicity of solutions throughout the Ekeland's Variational Principle [4] and the Mountain Pass Theorem [1] in the critical theory, we prove the following theorem:

Theorem 1.3 Let (M, g) be a Riemannian compact smooth manifold of dimension $n > 2k$ without boundary with $k \geq 1$. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h \neq 0$ satisfying $\|h\|_q < m_o$ and supposing that the operator $u \to P_g^k(u)$ is coercive. Then, the equation (4) has at least two nontrivial solutions $u, v \in H_k^2(M)$ satisfying:

$$J(u) < 0 < J(v).$$

2. Some Preparatory Lemmas

Throughout this section, we consider the energy functional J, for each $u \in H_k^2\left(M\right)$,

$$J(u) = \frac{1}{2}\left\|u\right\|_{P_g^k}^2 - \int_M h(x).ud\mu_g - \frac{1}{2_k^\sharp}\int_M f(x).\left|u\right|^{2_k^\sharp} d\mu_g$$

Define:

$$\Phi(u) := \langle \nabla J(u),\ u \rangle$$
$$\Phi(u) = \|u\|^2_{P^k_g} - \int_M h(x).ud\mu_g - \int_M f(x).\,|u|^{2^\sharp_k}\,d\mu_g$$

and

$$\langle \nabla\Phi(u), u\rangle = 2\,\|u\|^2_{P^k_g} - \int_M h(x).ud\mu_g - 2^\sharp_k \int_M f(x).\,|u|^{2^\sharp_k}\,d\mu_g$$

Now, we use the following Sobolev inequalities proved in [7].

Theorem 2.1 Let (M,g) be compact Riemannian manifold of dimension $n > 2k$ without boundary with $k \geq 1$. Then for any $\epsilon > 0$, there exists $A_\epsilon \in \mathbb{R}$ such that for all $u \in H^2_k(M)$:

$$\left(\int_M |u|^{2^\sharp_k}\,d\mu_g\right)^{\frac{2}{2^\sharp_k}} \leq (K(n,k)+\epsilon)\int_M \left(\Delta_g^{\frac{k}{2}} u\right)^2 d\mu_g + A_\epsilon\,\|u\|^2_{H^2_{k-1}(M)}$$

Lemma 2.1 Let (M,g) be a Riemannian compact smooth manifold of dimension $n > 2k$ without boundary with $k \geq 1$. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h \neq 0$, then there exists some constants α, ρ and $m_o > 0$ such that $J(u) \geq \alpha > 0$ with $\|u\|_{P^k_g} = \rho$ for all $u \in H^2_k(M)$ and h satisfying $\|h\|_q < m_o$.

Proof. Let $u \in H^2_k(M)$:

$$J(u) = \frac{1}{2}\,\|u\|^2_{P^k_g} - \frac{1}{2^\sharp_k}\int_M f(x).\,|u|^{2^\sharp_k}\,d\mu_g - \int_M h(x).ud\mu_g$$

Using Hölder inequality, we have:

$$J(u) \geq \frac{1}{2}\,\|u\|^2_{P^k_g} - \frac{1}{2^\sharp_k}\max_{x\in M} f(x)\,\|u\|^{2^\sharp_k}_{2^\sharp_k} - \|h\|_q \cdot \|u\|_{2^\sharp_k}$$

Using Sobolev inequality, we deduce:

$$J(u) \geq \frac{1}{2}\,\|u\|^2_{P^k_g} - \frac{1}{2^\sharp_k}\max_{x\in M} f(x).\left(\max\left((K(n,k)+\epsilon)\,, A_\epsilon\right)\right)^{\frac{2^\sharp_k}{2}} . \|u\|^{2^\sharp_k}_{H^2_k(M)} -$$
$$\|h\|_q \,.\left(\max\left((K(n,k)+\epsilon)\,, A_\epsilon\right)\right)^{\frac{1}{2}} . \|u\|_{H^2_k(M)}$$

Again the coercivity of P^k_g implies that there is $\Lambda > 0$, such that:

$$J(u) \geq \frac{1}{2}\,\|u\|^2_{P^k_g} - \frac{1}{2^\sharp_k}\max_{x\in M} f(x).\left(\frac{\max\left((K(n,k)+\epsilon)\,, A_\epsilon\right)}{\Lambda}\right)^{\frac{2^\sharp_k}{2}} . \|u\|^{2^\sharp_k}_{P^k_g} -$$
$$\|h\|_q \cdot \left(\frac{\max\left((K(n,k)+\epsilon)\,, A_\epsilon\right)}{\Lambda}\right)^{\frac{1}{2}} . \|u\|_{P^k_g}$$

Thus,

$$J(u) \geq \left[\frac{1}{2}\,\|u\|_{P^k_g} - \frac{1}{2^\sharp_k}\max_{x\in M} f(x).\left(\frac{\max\left((K(n,k)+\epsilon)\,, A_\epsilon\right)}{\Lambda}\right)^{\frac{2^\sharp_k}{2}} . \|u\|^{2^\sharp_k-1}_{P^k_g} -\right.$$
$$\left.\|h\|_q \cdot \left(\frac{\max\left((K(n,k)+\epsilon)\,, A_\epsilon\right)}{\Lambda}\right)^{\frac{1}{2}}\right] . \|u\|_{P^k_g}$$

Setting for $t \geq 0$:

$$F(t) := \frac{1}{2}t - \frac{1}{2_k^\sharp} \max_{x\in M} f(x). \left(\frac{\max\left(\left(K(n,k)+\epsilon\right), A_\epsilon\right)}{\Lambda}\right)^{\frac{2_k^\sharp}{2}} .t^{2_k^\sharp - 1}$$

By continuity argument of the function $F(.)$, we see that

$$\max_{t\geq 0} F(t) = F(\rho) > 0 \quad \text{where} \quad \rho^{2_k^\sharp - 2} := \frac{1}{2.\left(2_k^\sharp - 1\right)} \left(\frac{\Lambda}{\max\left(\left(K(n,k)+\epsilon\right), A_\epsilon\right)}\right)^{\frac{2_k^\sharp}{2}} \tag{5}$$

Then, it follows from (5) that if $\|h\|_q < m_o$ such that

$$m_o := \left(\frac{\max\left(\left(K(n,k)+\epsilon\right), A_\epsilon\right)}{\Lambda}\right)^{-\frac{1}{2}} .F(\rho)$$

there exists $\alpha > 0$ such that

$$J(u)|_{\|u\|_{P_g^k} = \rho} \geq \alpha > 0.$$

$\square$

Lemma 2.2 Let (M,g) be a Riemannian compact smooth manifold of dimension $n > 2k$ without boundary with $k \geq 1$. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h \neq 0$ satisfying $\|h\|_q < m_o$. Then there exists a function $v \in H_k^2(M)$ with $\|v\|_{P_g^k} > \rho$ such that $J(v) < 0$, where ρ is given by the previous lemma.

Proof. Let $v \in H_k^2(M)$, for any $t > 0$ we have:

$$J(t.v) = \frac{t^2}{2}\|v\|_{P_g^k}^2 - \frac{t^{2_k^\sharp}}{2_k^\sharp}\int_M f(x).\,|u|^{2_k^\sharp} d\mu_g - t\int_M h(x).ud\mu_g$$

Since $2_k^\sharp > 2$, so we deduce that,

$$\lim_{t\to+\infty} J(t.v) = -\infty$$

Consequently, there exists a point $v \in H_k^2(M)$ with $\|u\|_{P_g^k} > \rho$ such that $J(v) < 0$. $\square$

Lemma 2.3 Let (M,g) be a Riemannian compact smooth manifold of dimension $n > 2k$ without boundary with $k \geq 1$. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h \neq 0$ satisfying $\|h\|_q < m_o$. Assume $(u_m)_m$ is $(PS)_c$ sequence with

$$c < \frac{k}{n.K^{\frac{n}{2k}}(n,k).\left(\max f(x)\right)^{\frac{2}{2_k^\sharp}}}$$

Then, $(u_m)_m$ is bounded in $H_k^2(M)$.

Proof. Consider a sequence $(u_m)_m$ which satisfies $J(u_m) \to c$ and $\nabla J(u_m) \to 0$. We obtain,

$$J(u_m) - \frac{1}{2_k^\sharp} \langle \nabla J(u_m), u_m \rangle = \frac{2_k^\sharp - 2}{2.2_k^\sharp} \|u_m\|_{P_g^k}^2 - \frac{2_k^\sharp - 1}{2_k^\sharp} \int_M h(x).u_m d\mu_g = c + o\,(1)$$

Using Holder and Sobolev's inequalities and by the coercivity of P_g^k implies that there is $\Lambda > 0$, such that:

$$c + o\,(1) \geq \frac{2_k^\sharp - 2}{2.2_k^\sharp} \|u_m\|_{P_g^k}^2 - \frac{2_k^\sharp - 1}{2_k^\sharp} \|h\|_q \cdot \left(\frac{\max\left(\left(K(n,k) + \epsilon \right), A_\epsilon \right)}{\Lambda} \right)^{\frac{1}{2}} \|u_m\|_{P_g^k}$$

If $\|u_m\|_{P_g^k} > 1$, then:

$$c + o\,(1) \geq \left[\frac{2_k^\sharp - 2}{2.2_k^\sharp} - \frac{2_k^\sharp - 1}{2_k^\sharp} \|h\|_q \cdot \left(\frac{\max\left(\left(K(n,k) + \epsilon \right), A_\epsilon \right)}{\Lambda} \right)^{\frac{1}{2}} \right] \cdot \|u_m\|_{P_g^k}$$

And since,

$$\|h\|_q < m_o := \frac{2_k^\sharp - 2}{2.\left(2_k^\sharp - 1 \right)} \left(\frac{\max\left(\left(K(n,k) + \epsilon \right), A_\epsilon \right)}{\Lambda} \right)^{-\frac{1}{2}}$$

Then the sequence $(u_m)_m$ is bounded in $H_k^2(M)$. □

Lemma 2.4 Let (M, g) be a Riemannian compact smooth manifold of dimension $n > 2k$ without boundary with $k \geq 1$. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h \neq 0$ satisfying $\|h\|_q < m_o$. Assume $(u_m)_m$ is a bounded Palais-Smale sequence at level c of J with

$$c < \frac{k}{n.K^{\frac{n}{2k}}(n,k).\left(\max f(x) \right)^{\frac{2}{2_k^\sharp}}}$$

Then, $(u_m)_m$ has a strongly convergent sub-sequence in $H_k^2(M)$.

Proof. Using the previous lemma, let $(u_m)_m$ be a bounded $(PS)_c$ in $H_k^2(M)$ and from the reflixivity of $H_k^2(M)$ and the compact embedding theorem, up to a subsequence noted $(u_m)_m$ there exists $u \in H_k^2(M)$ such that
(1). $u_m \to u$ weakly in $H_k^2(M)$.
(2). $u_m \to u$ strongly in $L^p(M)$ for $1 < p < 2_k^\sharp$.
(3). $u_m \to u$ a.e in M.
Then we deduce that:

$$\begin{aligned} \left| \int_M h(x) \left(u_m - u \right) d\mu_g \right| &\leq \left(\int_M |h(x)|^2 \, d\mu_g \right)^{\frac{1}{2}} \cdot \left(\int_M \left(u_m - u \right)^2 d\mu_g \right)^{\frac{1}{2}} \\ &\leq \|h\|_2 \cdot \|u_m - u\|_2 = o(1). \end{aligned}$$

After these preliminaries, we can prove that $w_m := u_m - u$ converges to 0 strongly in $H_k^2(M)$.
Using Brézis-Lieb Lemma in [2], we obtain

$$\|u_m\|_{P_g^k}^2 - \|u\|_{P_g^k}^2 = \|w_m\|_{P_g^k}^2 + o(1)$$

and

$$\int_M f(x)\left(|u_m|^{2_k^\sharp}-|u|^{2_k^\sharp}\right)d\mu_g=\int_M f(x)\,|w_m|^{2_k^\sharp}\,d\mu_g+o(1)$$

Then,

$$J(u_m)-J(u)=\frac{1}{2}\,\|w_m\|^2_{P_g^k}-\frac{1}{2_k^\sharp}\int_M f(x)\,|w_m|^{2_k^\sharp}\,d\mu_g+o(1)$$

We obtain:

$$\langle\nabla J(u_m)-\nabla J(u),(u_m-u)\rangle=\|w_m\|^2_{P_g^k}-\int_M f(x)\,|w_m|^{2_k^\sharp}\,d\mu_g=o(1)$$

That is to say

$$\|w_m\|^2_{P_g^k}=\int_M f(x)\,|w_m|^{2_k^\sharp}\,d\mu_g+o(1) \tag{6}$$

Put

$$\ell:=\limsup_m \|w_m\|_{P_g^k}$$

Using Sobolev's inequality, we have for all $w_m\in H_k^2(M)$:

$$\int_M f(x)\,|w_m|^{2_k^\sharp}\,d\mu_g\le\max_{x\in M}f(x).\int_M |w_m|^{2_k^\sharp}\,d\mu_g=\max_{x\in M}f(x).\,\|w_m\|^{2_k^\sharp}_{2_k^\sharp}$$

$$\le\max_{x\in M}f(x).\left[\max\left(\left(K(n,k)+\epsilon\right),A_\epsilon\right)\right]^{\frac{2_k^\sharp}{2}}.\,\|w_m\|^{2_k^\sharp}_{H_k^2(M)}$$

Taking account that $P_g^k:C^\infty(M)\to C^\infty(M)$ is coercive, there exists a constant $\Lambda>0$ such that:

$$\int_M f(x)\,|w_m|^{2_k^\sharp}\,d\mu_g\le\max_{x\in M}f(x).\left[\Lambda.\max\left(\left(K(n,k)+\epsilon\right),A_\epsilon\right)\right]^{\frac{2_k^\sharp}{2}}.\,\|w_m\|^{2_k^\sharp}_{P_g^k} \tag{7}$$

Consequently, we obtain from (6) and (7) that:

$$\|w_m\|^2_{P_g^k}\le\max_{x\in M}f(x).\left[\Lambda.\max\left(\left(K(n,k)+\epsilon\right),A_\epsilon\right)\right]^{\frac{2_k^\sharp}{2}}.\,\|w_m\|^{2_k^\sharp}_{P_g^k}$$

Letting $n\to+\infty$, we get:

$$\ell\le\max_{x\in M}f(x).\left[\Lambda.\max\left(\left(K(n,k)+\epsilon\right),A_\epsilon\right)\right]^{\frac{2_k^\sharp}{2}}.\ell^{2_k^\sharp}$$

Then,

$$\ell=0\ \text{ or..}\ell\ge\frac{1}{\left[\max_{x\in M}f(x)\right]^{\frac{n-2k}{n+2k}}.\left[\Lambda.\max\left(\left(K(n,k)+\epsilon\right),A_\epsilon\right)\right]^{\frac{n}{n+2k}}}$$

We deduce that: $\ell=0$ and then $w_n\to 0$ strongly in $H_k^2(M)$.
i.e. $w_n:=u_n-u\to 0$ in $H_k^2(M)$. □

3. Main Result

The following theorem is our main result.
Theorem 3.1 Let (M,g) be a Riemannian compact smooth manifold of dimension $n>2k$ without boundary with $k\geq 1$. Let f is a C^{∞}-function on M with $f>0$ and $h\in L^q(M)$ such that $h\neq 0$ satisfying $\|h\|_q<m_o$ and supposing that the operator $u\longmapsto P_g^k(u)$ is coercive. Then, the equation (4) has at least two nontrivial solutions $u,v\in H_k^2(M)$ satisfying:

$$J(u)<0<J(v).$$

The proof is based on The Mountain Pass Theorem, namely,
Theorem 3.2 Let E be a Banach space, and $J\in C^1\left(H_k^2(M);\mathbb{R}\right)$ satisfies $(P.S)_c$ condition. We suppose:
(1). There exist $\alpha>0$, $\beta>0$ such that

$$J(u)\mid_{\partial B(0;\beta)}\geq J(0)+\alpha$$

Where

$$B_\beta=\left\{u\in H_k^2(M):\|u\|_{H_k^2(M)}\leq\beta\right\}$$

(2). There is an $e\in H_2^2(M)$ and $\|\ e\ \|_{H_k^2(M)}>\beta$ such that:

$$J(e)\leq J(0)$$

Then, $J(.)$ has a critical value c which can be characterized as

$$c:=\inf_{\gamma\in\Gamma}\max_{t\in[0;1]}J(\gamma(t))$$

Where

$$\Gamma:=\left\{\gamma\in C([0;1];H_k^2(M)):\gamma(0)=0\text{ and }\gamma(1)=e\right\}$$

Then there is a sequence $(u_m)_m$ in $H_k^2(M)$ such that:

$$\begin{cases} J(u_m)\to c \ \text{ in } \ \mathbb{R} \\ \nabla J(u_m)\to 0 \ \text{ in } \ \left(H_k^2(M)\right)^* \end{cases}$$

Proof. We prove this theorem, by the following two steps:
Step 1: There exists $v\in H_k^2(M)$ satisfies:

$$J(v)>0\quad\text{and}\quad\nabla J(v)=0$$

Using Lemmas 2.1 and 2.2 and The Mountain Pass Theorem, there exists a sequence $(u_m)_m\in H_k^2(M)$ satisfying:

$$J(u_m)\to c^+\quad\text{and}\quad\nabla J(u_m)=0$$

Then, it follows from Lemmas 2.3 and 2.4 that there exists $v\in H_k^2(M)$ such that $J(v)=c>0$ and $\nabla J(v)=0$ if $\|h\|_q<m_o$.
Consequently, v is a weak solution of the equation (4).
Step 2: There exists $u\in H_k^2(M)$ such that: $J(u)<0$ and $\nabla J(u)=0$. Since $h\in L^q(M)$ such that $h\neq 0$, we can choose a function $\varphi\in H_k^2(M)$ such that:

$$\int_M h(x).\varphi(x)d\mu_g>0$$

Letting $t>0$, we have:

$$J(t.\varphi)=\frac{t^2}{2}\|\varphi\|^2_{P_g^k}-\frac{t^{2_k^\sharp}}{2_k^\sharp}\int_M f(x).|\varphi|^{2_k^\sharp}d\mu_g-t\int_M h(x).\varphi(x)d\mu_g$$

Then for $t > 0$ small enough, we get $J(t.\varphi) < 0$.
Put

$$c^- = \inf_{u \in B_\rho} J(u)$$

Where

$$B_\rho := \left\{ u \in H_k^2(M) : \|u\|_{P_g^k} \leq \rho \right\}$$

It seems that:

$$c^- = \inf_{u \in B_\rho} J(u) < 0$$

Now, applying Ekeland's Variational Principle, there exists a $(PS)_{c^-}$ sequence $(v_m)_m \in \bar{B}_\rho$ satisfying:

$$J(v_m) \to c^- \quad \text{and} \quad \nabla J(v_m) = 0$$

Using Lemmas 2.1, 2.2, 2.3 and 2.4 we obtain a sub-sequence of $(v_m)_m$ which converges strongly to $u \in H_k^2(M)$.
Consequenly, v is a weak solution of the equation (4).

□

4. Geometric Application and Multiplicity Results

When (M, g) is closed Einsteinian manifold, the GJMS operator has constant coefficients. It expresses as

$$P_g^k(u) :=_{i=1}^{i=k} \left(-\Delta_g u + \frac{(n+2i-2).(n-2i)\, S_g}{4n(n-1)} u \right)$$

where $\Delta_g := \nabla^i \nabla_i$ is the Laplace-Beltrami operator and S_g is the scalar curvature of g. As a geometric application of the study of the GJMS operator, we obtain some existence and multiplicity results of weak positive solutions on closed Einsteinian manifold under some additional assumptions as stated in the following theorem:
Theorem 4.1 Let (M, g) be a Riemannian Einsteinian compact smooth n-manifold and of positive scalar curvature with $n \geq 5$. Let f is a C^∞-function on M with $f > 0$ and $h \in L^q(M)$ such that $h > 0$ satisfying $\|h\|_q < m_o$. Then, the equation (4) has at least two non trivial solutions $u^+, u^- \in H_k^2(M)$ satisfying:

$$J(u^-) < 0 < J(u^+).$$

where

$$u^+ = \max(u, 0) \text{ and } u^- = \max(-u, 0).$$

Proof. We define the modified energies in $H_k^2(M)$ by:

$$J^+(u) = \frac{1}{2} \|u\|_{P_g^k}^2 - \int_M h(x).u^+ d\mu_g - \frac{1}{2_k^\sharp} \int_M f(x).\left(u^+\right)^{2_k^\sharp} d\mu_g$$

and

$$J^-(u) = \frac{1}{2} \|u\|_{P_g^k}^2 - \int_M h(x).u^- d\mu_g - \frac{1}{2_k^\sharp} \int_M f(x).\left(u^-\right)^{2_k^\sharp} d\mu_g$$

where

$$u^+ = \max(u, 0) \text{ and } u^- = \max(-u, 0)$$

Applying the coercitivity of P_g^k on Eisteinian Manifold (M,g) and the above similar arguments involving Montain Pass Theorem for the energies J^+ and J^- to show there exists a non trivial solution u satisfying:

$$P_g^k(u) = h(x) + f(x).\left(u^+\right)^{2_k^\sharp - 1} \text{ on } M.$$

Together with $S_g = constant > 0$, the factorization of GJMS as:

$$P_g^k(u) :=_{i=1}^{i=k} \left(-\Delta_g u + \frac{(n+2i-2).(n-2i)\,S_g}{4n(n-1)} u \right)$$

We employ strong maximum principle for elliptic equations of second order for k times to show that $u > 0$ on M and solves the equations (4) from this and Lemmas 2.1, 2.2, 2.3, 2.4 and Theorems 3.1, 3.2, we conclude that $u \in C^\infty(M)$. This completes the proof. □

5. Perspective

Problem 5.1 Let (M,g) be a Riemannian compact smooth manifold of dimension $n > 2k$ without boundary with $k \geq 1$. We set $Isom(M,g)$ the isometry group of M, and G a subgroup of $Isom(M,g)$.We assume that f and h are two smooth G-invariant functions such that $f > 0$ and $h \in L^q(M)$. We are concerned with existence of smooth G-invariant solution u to the equation:

$$P_g^k(u) = f(x)\,|u|^{2_k^\sharp - 2}\, u + h(x).$$

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Kamel Tahri, Preparatory School of Economics, Business and Management Sciences, Tlemcen, Algeria
E-mail address: tahri_kamel@yahoo.fr