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A TWO-POINT BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL EQUATION WITH SELF-REFERENCE

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Abstract. In this paper, we study the following two-point boundary value problem

$$\begin{cases} u'(t) = a(t)u(u(t)), \ t \in [-1, 1], \\ \alpha u(-1) + \beta u(1) = \gamma, \end{cases}$$

where $a(t)$ is a given continuous, non-negative function on $[-1, 1]$; α, β and γ are constants such that $\alpha + \beta \neq 0$ and other appropriate conditions. The existence of solution of this problem is proved first by the Schauder fixed-point theorem and next by a iterative procedure.

1. Introduction

The existence, uniqueness, analyticity and analytic dependence of solutions to the following equation of a one-variable unknown function $u : I \subset \mathbb{R} \to \mathbb{R}$ is well considered in [1]

$$u'(t) = u\left(u(t)\right) \tag{1}$$

This equation has attracted much attention. As a more general case than (1), Si and Cheng [3] investigated the functional-differential equation

$$u'(t) = u\left(at + bu(t)\right), \tag{2}$$

where $a \neq 1$ and $b \neq 0$ are complex numbers; the unknown $u : \mathbb{C} \to \mathbb{C}$ is a complex function. By using the power series method, analytic solutions of this equation are obtained. By generalizing (2), in [7] Cheng, Si and Wang considered the equation

$$\alpha t + \beta u'(t) = u\big(at + bu'(t)\big),$$

where a, α and b, β are complex numbers. Existence theorems are established for the analytic solutions, and systematic methods for deriving explicit solutions are also given. In [8], Staněk studied maximal solutions of the functional-differential equation

$$u(t)u'(t) = ku\left(u(t)\right) \tag{3}$$

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with $0 < |k| < 1$. Here $u : I \subset \mathbb{R} \to \mathbb{R}$ is a real unknown. This author showed that properties of maximal solutions depend on the sign of the parameter k for two separate cases $k \in (-1, 0)$ and $k \in (0, 1)$. For earlier work of Staněk than (3), see [9]–[14].

The idea of the equation (1) is developed also for partial differential equations (see [2, 4, 5, 6]).

It is emphasized that in [1, 3, 7] and [9]–[14] any boundary-value problem of (1) has not been considered.

In this paper, by associating (1) with a two-point boundary condition, we study the solution existence of the following two-point boundary-value problem:

$$\begin{cases} u'(t) = a(t)u(u(t)),\ t \in [-1, 1], \\ \alpha u(-1) + \beta u(1) = \gamma, \end{cases} \tag{4}$$

where $a(t)$, α, β and γ with $\alpha + \beta \neq 0$ are given.

2. Solution existence by Schauder fixed-point theorem

In this section we start with the definition of an operator T such that fixed point for T are solution of the problem (4).

Lemma 2.1. *Assume that $a(t)$ is a given continuous, non-negative function on $[-1, 1]$; α, β and γ are constants such that $\alpha + \beta \neq 0$. Moreover assume*

$$\int_{-1}^{1} a(s)ds \leq \frac{|\alpha + \beta| - |\gamma|}{|\alpha + \beta| + |\beta|}. \tag{5}$$

Then the problem (4) is equivalent to the following operator equation

$$u = T(u), \tag{6}$$

where the operator

$$Tu(t) := \int_{-1}^{t} a(s)u(u(s))ds - \frac{\beta}{\alpha + \beta} \int_{-1}^{1} a(s)u(u(s))ds + \frac{\gamma}{\alpha + \beta}$$

acts in the convex, closed, bounded subset $K = C([-1, 1], [-1, 1])$ of the Banach space $X = C([-1, 1]; R)$ endowed with the norm $||u|| = \max |u(t)|$.

Proof. We prove at first that if $u \in K$ then Tu is an element of K. This is easy to prove from (5). In fact if $|u(.)| \leq 1$, then

$$|Tu(t)| \leq \left(1 + \frac{|\beta|}{|\alpha + \beta|}\right) \int_{-1}^{1} a(s)ds + \frac{|\gamma|}{|\alpha + \beta|} \leq 1.$$

Now, from $(4)_1$, we deduce that

$$u(t) = u(-1) + \int_{-1}^{t} a(s)u(u(s))ds, \tag{7}$$

hence, from $(4)_2$

$$u(-1) = \frac{-\beta}{\alpha + \beta} \int_{-1}^{1} a(s)u(u(s))ds + \frac{\gamma}{\alpha + \beta}. \tag{8}$$

From (7) and (8), we obtain

$$u(t) = \int_{-1}^{t} a(s)u(u(s))ds - \frac{\beta}{\alpha + \beta} \int_{-1}^{1} a(s)u(u(s))ds + \frac{\gamma}{\alpha + \beta}. \tag{9}$$

Moreover, if (9) holds for $u \in X$ then (4) also holds. □

We have the following theorem.

Theorem 2.1. *Suppose $a(t)$ is a given continuous, non-negative function on $[-1, 1]$ satisfying (5) where α, β and γ are constants such that $\alpha+\beta \neq 0.$. Then the operator (6) has a fixed point in K.*

Proof. From the definition of K, it is clear that K is convex, closed and bounded in the Banach space X.

For $u \in K$, consider

$$T(u) := \int_{-1}^{t} a(s)u(u(s))ds - \frac{\beta}{\alpha+\beta}\int_{-1}^{1} a(s)u(u(s))ds + \frac{\gamma}{\alpha+\beta}.$$

Note that the identity $\alpha(Tu)(-1) + \beta(Tu)(1) = \gamma$ holds.

Moreover, we have that $T(K) \subseteq K$. In fact, if $u \in K$, from $|(Tu)(t)| \leq \left(1 + \frac{|\beta|}{|\alpha+\beta|}\right)\int_{-1}^{1}|a(s)|ds + \frac{|\gamma|}{|\alpha+\beta|}$, it follows

$$|(Tu)(t)| \leq \frac{[|\alpha+\beta| + |\beta|]\int_{-1}^{1}|a(s)|ds + |\gamma|}{|\alpha+\beta|} \leq 1$$

for all $t \in [-1, 1]$. Therefore the claim is proved.

Furthermore, T is continuous. Let (u_n) be a sequence in K convergent with respect to the norm $||.||_0$ to the function $u \in K$. Note that for every $n \in N$ and $t \in [-1, 1]$,

$$|u_n(u_n(t)) - u(u(t))| \leq |u_n(u_n(t)) - u(u_n(t))| + |u(u_n(t)) - u(u(t))|.$$

From the uniform convergence of (u_n) to u, for a fixed $\epsilon > 0$ there exists ν_1 such that $|u_n(\rho) - u(\rho)| \leq \frac{\epsilon}{2}$ for every ρ. And still, for the uniform continuity of u, for a fixed $\epsilon > 0$ there exists $\delta > 0$ such that $|\xi_2 - \xi_1| \leq \delta$ we have $|u(\xi_2) - u(\xi_1)| \leq \frac{\epsilon}{2}$. Hence, there exists ν_2 such that for $n > \nu_2$ we have $|u_n(\rho) - u(\rho)| \leq \delta$. So for $n > \max\{\nu_1, \nu_2\}$ we obtain that

$$|u_n(u_n(t)) - u(u(t))| \leq |u_n(u_n(t)) - u(u_n(t))| + |u(u_n(t)) - u(u(t))| < \epsilon$$

for all $t \in [-1, 1]$. This proves the continuity of T.

Since a is continuous on $[-1, 1]$, there exists $M \in \mathbb{R}$ such that $a(s) \leq M$.

We are proving that $T(K)$ is relatively compact with respect to the norm $\|.\|_0$. Let (Tu_n) be a sequence with $u_n \in K$ for all $n \in N$. It is obvious that (Tu_n) is bounded, recalled Lemma 3.2. From the continuity of u_n and a, we have that $Tu_n \in C^1$ and $(Tu_n)'(t) = a(t)u_n(u_n(t))$. Therefore, $|(Tu_n)'| \leq M$ for all t. Then (Tu_n) is an equi-bounded, equi-Lipschitz sequence. By the Ascoli-Arzelà theorem, there exists a convergent subsequence of (Tu_n).

In conclusion, we have that $T : K \to K$ is a continuous operator and $T(K)$ is relatively compact. By Schauder fixed point theorem, T has a fixed point in K. □

3. An iterative scheme for existence of solutions

The Schauder theorem applied in the previous section say that a solution of the problem (4) exists; now we consider a sequence of functions, defined by iteration, for which the uniform limit exists and is a solution of the problem (4). We need an other condition on function $a = a(t)$ and α, β, γ.

Consider the following sequence of functions $\{u_n\}_n$

$$\begin{cases} u_{n+1}(t) = \int_{-1}^{t} a(s)u_n(u_n(s))ds - \frac{\beta}{\alpha+\beta}\int_{-1}^{1} a(s)u_n(u_n(s))ds + \frac{\gamma}{\alpha+\beta}, \\ u_0(t) := \frac{\gamma}{\alpha+\beta}, \end{cases} \quad (10)$$

for all $t \in [-1, 1]$.

Assume also the following condition

$$\left|\frac{\gamma}{\alpha+\beta}\right| \leq 1, \quad (11)$$

from the definition of the operator T, as in the previous section, it is easy to prove the following lemma.

Lemma 3.2. *The sequence defined by* (10) *is equibounded and every* u_n *is a* C^1 *function provided that* $a(t)$ *is a given continuous, non-negative function on* $[-1, 1]$ *such that* (5) *and* (11) *hold.*

More precisely we have that, $\forall n \in N$,

$$|u_n(t)| \leq 1; \qquad |u_n^{'}(t))| \leq M$$

and so

$$|u_n(t_2) - u_n(t_1)| \leq M|t_2 - t_1| \qquad \forall t_2, t_1 \in [-1, 1]$$

where $M = \max_t a(t)$.

Hence we are able to prove the following theorem.

Theorem 3.2. *Suppose* $a(t)$ *is a given continuous, non-negative function on* $[-1, 1]$ *satisfying* (5) *and* (11) *where* α, β *and* γ *are constants such that* $\alpha + \beta \neq 0.$.

Assume, if $1 \leq M$,

$$\max_t \left[\frac{|\alpha|}{|\alpha+\beta|}\int_{-1}^{t} a(s)ds + \frac{|\beta|}{|\alpha+\beta|}\int_{t}^{1} a(s)ds\right] < \frac{1}{2M} \quad (12)$$

or, if α, β *are non negative,*

$$\int_{-1}^{1} a(s)ds < \frac{1}{2M} \quad (13)$$

or, if $M \leq 1$, *the same previous conditions with* M *replaced with 1. Then the sequence* (10) *is uniformly convergent to a solution of the problem* (4).

Proof. We remark that

$$|u_1(t) - u_0(t)| = \left|\frac{\gamma}{(\alpha+\beta)^2}\left[\alpha\int_{-1}^{t} a(s)ds - \beta\int_{t}^{1} a(s)ds\right]\right|,$$

hence

$$|u_1(t) - u_0(t)| \leq \frac{|\gamma|}{|\alpha+\beta|^2}\left[|\alpha|\int_{-1}^{t} a(s)ds + |\beta|\int_{t}^{1} a(s)ds\right] = \frac{|\gamma|}{|\alpha+\beta|^2}g_1(t)$$

where $g_1(t) = \left[|\alpha|\int_{-1}^{t} a(s)ds + |\beta|\int_{t}^{1} a(s)ds\right]$.

Assume that $1 \leq M$.

From

$$|u_2(t) - u_1(t)| = \left|\frac{1}{\alpha+\beta}\left[\alpha\int_{-1}^{t} a(s)[u_1(u_1(s)) - u_0(u_0(s))]ds+\right.\right.$$

$$-\beta \int_t^1 a(s)[u_1(u_1(s)) - u_0(u_0(s))]ds\Big]\Big|$$

we obtain, for the previous step, condition on M and the Lipschitz property of all u_n of the previous lemma,

$$|u_2(t) - u_1(t)| \leq \frac{M|\gamma|}{|\alpha+\beta|^3} g_2(t)$$

where

$$g_2(t) = \left[|\alpha| \int_{-1}^t a(s)g_1(s)ds + |\beta| \int_t^1 a(s)g_1(s)ds\right] +$$
$$\left[|\alpha| \int_{-1}^t a(s)g_1(u_0(s))ds + |\beta| \int_t^1 a(s)g_1(u_0(s))ds\right].$$

It is easy to prove, by induction, that for all $n \in N$ and $t \in [-1, 1]$

$$|u_{n+1}(t) - u_n(t)| \leq \frac{M^n|\gamma|}{|\alpha+\beta|^{n+2}} g_{n+1}(t)$$

where

$$g_{n+1}(t) = \left[|\alpha| \int_{-1}^t a(s)g_n(s)ds + |\beta| \int_t^1 a(s)g_n(s)ds\right] +$$
$$\left[|\alpha| \int_{-1}^t a(s)g_n(u_{n-1}(s))ds + |\beta| \int_t^1 a(s)g_n(u_{n-1}(s))ds\right].$$

Now we consider $H = \max_t g_1(t)$ and remark that

$$0 \leq g_2(t) \leq 2H^2, \qquad 0 \leq g_3(t) \leq 2^2H^3.$$

Hence, by induction, it is easy to prove that for all $n \in N$, $\quad t \in [-1, 1]$

$$0 \leq g_n(t) \leq 2^{n-1}H^n.$$

Then the following inequalities hold

$$|u_(t)n+1 - u_n(t)| \leq \frac{M^n|\gamma|}{|\alpha+\beta|^{n+2}} 2^n H^{n+1} = |\gamma| H \frac{2^2}{M^2} \left[\frac{2MH}{|\alpha+\beta|}\right]^{n+2}.$$

From condition (12) (or (13)) follows, for all $n \in N$, $\quad \left[\frac{2HM}{|\alpha+\beta|}\right] < 1$.

If we assume $M \leq 1$, the proof is analogous.

Hence the sequence $(u_n)_n$ is uniformly convergent to a function $u_\infty \in K$ and this limit is obviously a solution for the problem (4). □

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