Electronic Journal of Mathematical Analysis and Applications
Vol. 6(1) Jan. 2018, pp. 31-44.
ISSN: 2090-729X(online)
http://fcag-egypt.com/Journals/EJMAA/

OSCILLATION CRITERIA FOR SECOND-ORDER NONLINEAR MIXED NEUTRAL DYNAMIC EQUATIONS WITH NON POSITIVE NEUTRAL TERM ON TIME SCALES

H. A. AGWA, AHMED M. M. KHODIER AND HEBA M. ARAFA

Abstract. In this work, we establish some new oscillation results for the second-order nonlinear mixed neutral dynamic equation

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + f(t, x(\tau_1(t))) + g(t, x(\tau_2(t))) = 0,$$

where $z(t) = x(t) - p_1(t)x(\eta_1(t)) + p_2(t)x(\eta_2(t))$. Our results not only complement and generalize some existing results in [9], but also can be applied to some oscillation problems that were not covered before, we also give some examples to illustrate our main results.

1. Introduction

A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. The book by Bohner and Peterson [5] summarizes and organizes much of time scale calculus. We refer also to Bohner and Peterson [6] for advances in dynamic equations on time scales. In recent years, there has been much activities concerning oscillation and nonoscillation of the solution of various equations on time scales. We refer the reader to the papers [[2], [3],[7]-[15]] and references cited therein. In this paper, we deal with oscillation of the second order mixed nonlinear neutral dynamic equation with negative neutral term on time scales

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + f(t, x(\tau_1(t))) + g(t, x(\tau_2(t))) = 0, \tag{1}$$

where

$$z(t) = x(t) - p_1(t)x(\eta_1(t)) + p_2(t)x(\eta_2(t)) \tag{2}$$

subject to the following hypotheses:

(H_1) $\mathbb{T}$ is an unbounded above time scale and $t_0 \in \mathbb{T}$ with $t_0 > 0$. We define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \bigcap \mathbb{T}$.

2010 *Mathematics Subject Classification.* 34A21, 34C15, 34K40, 34N05.
Key words and phrases. Oscillation, mixed neutral dynamic equations, time scales, generalized Riccati technique.
Submitted Jan. 12, 2017.

(H_2) η_1, τ_1 and $\tau_2 : \mathbb{T} \to \mathbb{T}$ are rd-continuous such that $\eta_1(t) \leq t$, $\tau_1(t) \leq t$, $\tau_2(t) \geq t$, $\lim_{t\to\infty} \tau_1(t) = \infty = \lim_{t\to\infty} \eta_1(t) = \infty$ and $\eta_2 : \mathbb{T} \to \mathbb{T}$ is injective rd-continuous increasing function such that $\eta_2(t) \geq t$.

(H_3) p_1 and p_2 are non-negative rd-continuous functions on an arbitrary time scale $\mathbb{T}$ where

$$0 \leq p_1(t) \leq p_1 < 1 \text{ and } 0 \leq p_2(t) \leq p_2.$$

(H_4) r is a positive rd-continuous function such that

$$\int_{t_0}^{\infty} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} = \infty. \tag{3}$$

(H_5) $f, g \in C(\mathbb{R}\times\mathbb{T}, \mathbb{R})$ such that $uf(t,u) \geq 0$, $ug(t,u) \geq 0$, $f(t,u) \geq q_1(t)u^{\alpha}$ and $g(t,u) \geq q_2(t)u^{\beta}$ for $u \neq 0$ where q_1 and q_2 are non-negative rd-continuous functions on an arbitrary time scale $\mathbb{T}$, α and β are quotients of odd positive integers.

(H_6) γ is a quotient of odd positive integers.

Through out this paper we assume that

$$d_+(t) = \max\{0, d(t)\},\ d_-(t) = \max\{0, -d(t)\},$$

$$A(t) := \begin{cases} b_0^{\alpha-\beta} & \alpha \geq \beta \\ b_1^{\alpha-\beta}\big[\int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\big]^{\alpha-\beta} & \alpha < \beta, \end{cases} \tag{4}$$

$$C(t) := \begin{cases} b_0^{\frac{\beta}{\gamma}-1} & \frac{\beta}{\gamma} \geq 1 \\ b_1^{\frac{\beta}{\gamma}-1}\big[\int_{t_1}^{\sigma(t)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\big]^{\frac{\beta}{\gamma}-1} & \frac{\beta}{\gamma} < 1, \end{cases} \tag{5}$$

where b_0 and b_1 are positive constants, $\sigma(t)$ is the forward jump operator which is defined by $\sigma(t) = inf\{s \in \mathbb{T}, s > t\}$.

By a solution of (1), we mean a nontrivial real valued function $x(t)$ satisfies (1) for $t \in \mathbb{T}$. A solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called non-oscillatory. Eq. (1) is said to be oscillatory if all of its solutions are oscillatory. A nontrivial solution $x(t)$ is said to be almost oscillatory if either $x(t)$ is oscillatory or $x^{\Delta}(t)$ is oscillatory.

In what follows, we provide some background details which motivated our study.

L. Erbe et al. [9] considered the second-order nonlinear functional dynamic equation

$$(r(t)[(x(t) - p(t)x(\eta(t)))^{\Delta}]^{\gamma})^{\Delta} + f(t, x(g(t))) = 0, \tag{6}$$

where $\eta(t) \leq t$ and either $g(t) \geq t$ or $g(t) \leq t$ and proved that if

$$\limsup_{t\to\infty} \int_{T}^{t} \big[M(s,T_*)\delta(s)q(s) - \frac{r(s)(\delta^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\big]\Delta s = \infty, \tag{7}$$

then every solution of (6) is either oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or tends to zero where

$$M(t, T_*) := \begin{cases} 1 & g(t) \geq t \\ \theta^{\gamma}(t, T_*) & g(t) \leq t. \end{cases}$$

Qi Li et al. [10] obtained oscillation criteria for the delay differential equation

$$(r(t)((y(t)-p(t)y(\tau(t)))')^{\gamma})'+q(t)f(y(\delta(t)))=0.$$

R. Arul and V. S. Shobha [4] improved the obtained results in [10] and E. Thandapani et al. [14] obtained some results on oscillatory behavior of the second order neutral difference equation:

$$\Delta(a_n(\Delta(x_n-p_nx_{n-\tau}))^{\alpha})+q_nf(x_{n-\sigma})=0.$$

This paper is organized as follows: In Section 2, we give some lemmas that we need through our work. In Section 3, we establish some new sufficient conditions for oscillation of (1). Finally, in Section 4, we present some examples to illustrate our results.

2. Basic Lemmas

In this section, we give some lemmas that play important roles in the proof of our results.

Lemma 1 Let conditions H_1-H_6 be satisfied and $x(t)$ is a positive solution of (1). Then $z(t)$ satisfies one of the following two cases:
(C_1) $z(t)>0, z^{\Delta}(t)>0$ and $(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}\leq 0$
(C_2) $z(t)<0, z^{\Delta}(t)>0$ and $(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}\leq 0$,
for $t\geq t_1$ where $t_1\geq t_0$ is sufficiently large.

Proof. Suppose that there exists $t_1\geq t_0$ such that $x(t)>0, x(\tau_i(t))>0$ and $x(\eta_i(t))>0, i=1,2$ on $[t_1,\infty)_{\mathbb{T}}$. (when $x(t)$ is negative the proof is similar, because the transformation $x(t)=-y(t)$ transforms (1) into the same form). From (1) and H_5, it follows that

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}\leq -q_1(t)\,x^{\alpha}(\tau_1(t))-q_2(t)x^{\beta}(\tau_2(t))\leq 0 \text{ for } \quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{8}$$

Then, $r(t)(z^{\Delta}(t))^{\gamma}$ is decreasing and of one sign on $[t_1,\infty)_{\mathbb{T}}$. Hence, there exists $t_2\geq t_1$ such that $z^{\Delta}(t)>0$ or $z^{\Delta}(t)<0$ for $t\geq t_2$.
If $z^{\Delta}(t)>0$ for $t\geq t_2$, then we have (C_1) or (C_2). Now we prove that $z^{\Delta}(t)<0$ cannot occur.
If $z^{\Delta}(t)<0$ for $t\geq t_2$, then $r(t)(z^{\Delta}(t))^{\gamma}\leq -c$ for $t\geq t_2$, where $c:=-r(t_2)(z^{\Delta}(t_2))^{\gamma}>0$. Thus we conclude that

$$z(t)\leq z(t_2)-c^{\frac{1}{\gamma}}\int_{t_2}^{t}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)},$$

using (3), we have $\lim_{t\to\infty}z(t)=-\infty$. Then we have the following two possiblities
Case(a)**:** If $x(t)$ is unbounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty}t_k=\infty$ and $\lim_{k\to\infty}x(t_k)=\infty$. Assume that

$$x(t_k)=\max\{x(s):t_0\leq s\leq t_k\}.$$

Since $\lim_{t\to\infty}\eta_1(t)=\infty$, $\eta_1(t_k)>t_0$ for all sufficiently large k and $\eta_1(t)\leq t$, then

$$x(\eta_1(t_k))=max\{x(s):t_0\leq s\leq \eta_1(t_k)\}\leq max\{x(s):t_0\leq s\leq t_k\}=x(t_k), \tag{9}$$

therefore from(9) into (2), we have for all large k

$$\begin{aligned} z(t_k)&=x(t_k)-p_1(t_k)x(\eta_1(t_k))+p_2(t_k)x(\eta_2(t_k))\\ &\geq x(t_k)-p_1(t_k)x(\eta_1(t_k))\\ &\geq x(t_k)-p_1x(t_k)=(1-p_1)x(t_k)>0, \end{aligned}$$

which contradicts that $\lim_{t\to\infty} z(t) = -\infty$
Case(b)**:** If $x(t)$ is bounded, then $z(t)$ is also bounded which contradicts $\lim_{t\to\infty} z(t) = -\infty$.
Hence, $z(t)$ satisfies one of the two cases (C_1) or (C_2). This completes the proof.
Lemma 2 Assume that $x(t)$ is a positive solution of (1) and $z(t)$ satisfies case (C_2). Then $\lim_{t\to\infty} x(t) = 0$.
proof. By $z(t) < 0$ and $z^{\Delta}(t) > 0$, we deduce that

$$\lim_{t\to\infty} z(t) = l \leq 0.$$

As in the proof of Case(a) of the previous lemma, $x(t)$ is bounded, then $\lim_{t\to\infty} x(t) = a \geq 0$.
Now, if $a > 0$, then there exists $t_k \subseteq [t_2,\infty)_{\mathbb{T}}$ such that $\lim_{k\to\infty} t_k = \infty$, $\lim_{k\to\infty} x(t_k) = a > 0$ and

$$x(t_k) = \max\{x(s) : t_0 \leq s \leq t_k\},$$

then $z(t_k) \geq x(t_k) - p_1(t_k)x(\eta_1(t_k)) \geq x(t_k) - p_1 x(t_k) = (1-p_1)x(t_k)$
thus, $0 > \lim_{k\to\infty} z(t_k) > (1-p_1)a > 0$, which is a contradiction. Therefore, $a = 0$ and $\lim_{t\to\infty} x(t) = 0$.
Lemma 3 If $f(u) = bu - au^{\frac{\gamma+1}{\gamma}}$, where $a > 0$ and b are constants, then f attains its maximum value on $\mathbb{R}$ at $u^* = (\frac{b\gamma}{a(\gamma+1)})^{\gamma}$, and

$$\max_{u\in\mathbb{R}} f = f(u^*) = \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{b^{\gamma+1}}{a^{\gamma}}.$$

3. Main Results

Theorem 1 Assume that H_1-H_6 hold, $\tau_2(t) \geq \eta_2(t)$ for all $t \geq t_0$, and there exists positive real-valued Δ -differentiable functions $R(t)$ and $\delta(t)$ such that for sufficiently large T and t_1, we have

$$\frac{R(t)}{r^{\frac{1}{\gamma}}(t)\int_{t_1}^{t}\frac{1}{r^{\frac{1}{\gamma}}(s)}\Delta s} - R^{\Delta}(t) \leq 0, \tag{10}$$

and

$$\limsup_{t\to\infty}\int_T^t \big[\delta(s)\xi(s)[q_1(s)L^{\alpha}(s)A(s)+q_2(s)] - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(\delta_+^{\Delta}(s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}\big]\Delta s = \infty, \tag{11}$$

where

$$L(s) = \min\{\frac{R(\tau_1(t))}{R(t)}, \frac{R(\eta_2^{-1}(\tau_1(t)))}{R(t)}\},$$

$$\xi(t) = \min\{\frac{1}{(1+p_2(\tau_1(t))^{\alpha}}, \frac{1}{(1+p_2(\tau_2(t))^{\beta}}, \frac{1}{(1+p_2(\eta_2^{-1}(\tau_1(t)))^{\alpha}}, \frac{1}{(1+p_2(\eta_2^{-1}(\tau_2(t)))^{\beta}}\}.$$

Then, every solution of (1) is almost oscillatory on $[t_0,\infty)_{\mathbb{T}}$ or converges to zero as $t\to\infty$.
Proof. Assume that $x(t)$ is not almost oscillatory solution of (1). Then without loss of generality, there exists $t_3 \geq t_0$ such that $x(t) > 0$, $x(\tau_i(t)) > 0$ and $x(\eta_i(t)) > 0, i = 1,2$ on $[t_3,\infty)_{\mathbb{T}}$.(when $x(t)$ is negative, the proof is similar). Then from lemma 1, $z(t)$ satisfies one of the cases C_1 or C_2. Also, by the definition of not almost oscillatory we have the two possibilities:

(I) $x^{\Delta}(t) < 0$ for $t \geq t_3$
(II) $x^{\Delta}(t) > 0$ for $t \geq t_3$

Case1. Suppose that C_1 holds and $x^{\Delta}(t) < 0$, then we have

$$\begin{aligned} z(t) &= z(t_1) + \int_{t_1}^{t} \frac{(r(s)(z^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s \\ &\geq r^{\frac{1}{\gamma}}(t) z^{\Delta}(t) \int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}, \end{aligned}$$

thus

$$\begin{aligned} (\frac{z(t)}{R(t)})^{\Delta} &= \frac{z^{\Delta}(t)R(t) - z(t)R^{\Delta}(t)}{R(t)R^{\sigma}(t)} \\ &\leq \frac{z(t)}{R(t)R^{\sigma}(t)} \big[\frac{R(t)}{r^{\frac{1}{\gamma}}(t) \int_{t_1}^{t} \frac{1}{r^{\frac{1}{\gamma}}(s)} \Delta s} - R^{\Delta}(t) \big] \leq 0, \end{aligned} \tag{12}$$

then z/R is a non-increasing function. From the definition of $z(t)$, we see that

$$\begin{aligned} z(t) &< x(t) + p_2(t)x(\eta_2(t)) \\ &\leq (1 + p_2(t))x(t) for\ t \geq t_3 \end{aligned}$$

choosing $t_4 > t_3$ such that $\tau_1(t) \geq t_3$ for all $t \geq t_4$, then

$$x(\tau_1(t)) \geq \frac{1}{1 + p_2(\tau_1(t))} z(\tau_1(t)) \text{ and } x(\tau_2(t)) \geq \frac{1}{1 + p_2(\tau_2(t))} z(\tau_2(t)),\ t \geq t_4 \tag{13}$$

substituting from (13) into (8), we have

$$\begin{aligned} (r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} &\leq \frac{-q_1(t)}{(1 + p_2(\tau_1(t)))^{\alpha}} z^{\alpha}(\tau_1(t)) - \frac{q_2(t)}{(1 + p_2(\tau_2(t)))^{\beta}} z^{\beta}(\tau_2(t)) \\ &\leq -N(t)[q_1(t)z^{\alpha}(\tau_1(t)) + q_2(t)z^{\beta}(\tau_2(t))] fort \geq t_4 \end{aligned} \tag{14}$$

where $N(t) = \min\{\frac{1}{(1+p_2(\tau_1(t)))^{\alpha}}, \frac{1}{(1+p_2(\tau_2(t)))^{\beta}}\}$.
Defining the function w by

$$w = \delta(t) \frac{r(t)(z^{\Delta}(t))^{\gamma}}{z^{\beta}(t)}, \tag{15}$$

then $w(t) > 0$ and

$$\begin{aligned} w^{\Delta}(t) &= (\frac{\delta(t)}{z^{\beta}(t)})(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + r(\sigma(t))(z^{\Delta}(\sigma(t))^{\gamma}(\frac{\delta(t)}{z^{\beta}(t)})^{\Delta}. \\ &= (\frac{\delta(t)}{z^{\beta}(t)})(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma} \frac{z^{\beta}(t)\delta^{\Delta}(t) - \delta(t)(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))}. \end{aligned} \tag{16}$$

Substituting from (14) and (15) into (16), we obtained

$$\begin{aligned} w^{\Delta}(t) \leq & -\delta(t)N(t)\big[q_1(t)(\frac{z(\tau_1(t))}{z(t)})^{\alpha} z^{\alpha-\beta}(t) + q_2(t)(\frac{z(\tau_2(t))}{z(t)})^{\beta}\big] + \\ & \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))} w(\sigma(t)) - \frac{\delta(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))},\ t \geq t_4. \end{aligned} \tag{17}$$

Since $t > \tau_1(t)$ for all $t \geq t_4$, then integrating using the fact that $\frac{z(t)}{R(t)}$ is a decreasing function, therefore

$$\frac{z(\tau_1(t))}{z(t)} \geq \frac{R(\tau_1(t))}{R(t)} \text{ for all } t \geq t_4. \tag{18}$$

Using $z(t) > 0, z^{\Delta}(t) > 0$ and $(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq 0$, then there exists $t_5 \in [t_4, \infty)_{\mathbb{T}}$ and positive constants b_0 and b_1 such that

$$z(t_0) := b_0 \leq z(t) \leq b_1 \int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)},\ t \geq t_5, \tag{19}$$

hence, we have

$$z^{\alpha-\beta}(t) \geq A(t) := \begin{cases} b_0^{\alpha-\beta} & \alpha \geq \beta \\ b_1^{\alpha-\beta}\Big[\int_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\Big]^{\alpha-\beta} & \alpha < \beta. \end{cases} \tag{20}$$

Using chain rule, we get

$$(z^{\beta}(t))^{\Delta} \geq \begin{cases} \beta z^{\Delta}(t) z^{\beta-1}(t), & \beta \geq 1 \\ \beta z^{\Delta}(t)(z(\sigma(t)))^{\beta-1}, & \beta < 1 \end{cases} \tag{21}$$

since, $\sigma(t) \geq t$ and $r(t)(z^{\Delta}(t))^{\gamma}$ is a decreasing function, then

$$z^{\Delta}(t) > \frac{(r(\sigma(t)))^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)} z^{\Delta}(\sigma(t)). \tag{22}$$

Using (15), (21) and (22), then we have

$$\frac{\delta(t) r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t) z^{\beta}(\sigma(t))} \geq \frac{\beta\delta(t)}{(\delta(\sigma(t)))^{\lambda} r^{\frac{1}{\gamma}}(t)} (z(\sigma(t)))^{\frac{\beta}{\gamma}-1} w^{\lambda}(\sigma(t)), \tag{23}$$

where $\lambda = \frac{\gamma+1}{\gamma}$. Then by using (19), we have

$$(z(\sigma(t)))^{\frac{\beta}{\gamma}-1} \geq C(t) := \begin{cases} b_0^{\frac{\beta}{\gamma}-1} & \frac{\beta}{\gamma} \geq 1 \\ b_1^{\frac{\beta}{\gamma}-1}\Big[\int_{t_1}^{\sigma(t)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\Big]^{\frac{\beta}{\gamma}-1} & \frac{\beta}{\gamma} < 1, \end{cases}$$

consequently (23) becomes

$$\frac{\delta(t) r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t) z^{\beta}(\sigma(t))} \geq \frac{\beta\delta(t) C(t)}{(\delta(\sigma(t)))^{\lambda} r^{\frac{1}{\gamma}}(t)} w^{\lambda}(\sigma(t)),\ t \geq t_5. \tag{24}$$

Since $\tau_2(t) \geq t$ and $z^{\Delta}(t) > 0$, then $\frac{z(\tau_2(t))}{z(t)} \geq 1$.
Substituting from the above inequality, (18), (20) and (24) into (17), we obtain

$$\begin{aligned} w^{\Delta}(t) \leq -\delta(t) N(t)[q_1(t)(\frac{R(\tau_1(t))}{R(t)})^{\alpha} A(t) + q_2(t)] + \frac{\delta_+^{\Delta}(t)}{\delta(\sigma(t))} w(\sigma(t)) \\ - \frac{\beta\delta(t) C(t)}{(\delta(\sigma(t)))^{\lambda} r^{\frac{1}{\gamma}}(t)} w^{\lambda}(\sigma(t)), \end{aligned} \tag{25}$$

using lemma 3 and taking

$$b := \frac{\delta_+^\Delta(t)}{\delta(\sigma(t))} \text{ and } a := \frac{\beta\delta(t)C(t)}{(\delta^\sigma(t))^\lambda r^{\frac{1}{\gamma}}(t)},$$

then

$$\frac{\delta_+^\Delta(t)}{\delta(\sigma(t))}w(\sigma(t)) - \frac{\beta\delta(t)C(t)}{(\delta(\sigma(t))^\lambda r^{\frac{1}{\gamma}}(t)}w^\lambda(\sigma(t)) \le \frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}}\frac{r(t)(\delta_+^\Delta(t))^{\gamma+1}}{\delta^\gamma(t)C^\gamma(t)}. \quad (26)$$

Substituting from (26) into (25), we obtain

$$w^\Delta(t) \le -\delta(t)N(t)[q_1(t)(\frac{R(\tau_1(t))}{R(t)})^\alpha A(t)+q_2(t)]+\frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}}\frac{r(t)(\delta_+^\Delta(t))^{\gamma+1}}{\delta^\gamma(t)C^\gamma(t)}, \ t \ge t_5.$$

Integrating the above inequality from t_5 to t, we get

$$\int_{t_5}^{t} \big[\delta(s)N(s)[q_1(s)(\frac{R(\tau_1(s))}{R(s)})^\alpha A(s) + q_2(s)] - \frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}}\frac{r(s)(\delta_+^\Delta(s))^{\gamma+1}}{\delta^\gamma(s)C^\gamma(s)}$$
$$\le w(t_5) - w(t) < w(t_5)$$

which is a contradiction with (11).
Case2. Suppose that C_1 holds $x^\Delta(t) > 0$, then we have

$$\begin{aligned} z(t) &< x(t) + p_2(t)x(\eta_2(t)) \\ &\le (1+p_2(t))x(\eta_2(t)) forall\ t \ge t_5. \end{aligned}$$

Choosing t_6 sufficiently large such that $t_6 > t_5$ and $\eta_2^{-1}(t) > t_5$ for all $t \ge t_6$, then

$$x(t) \ge \frac{1}{1+p_2(\eta_2^{-1}(t))}z(\eta_2^{-1}(t))t \ge t_6.$$

Taking $t_7 > t_6$ such that $\tau_1(t) > t_6$ for all $t \ge t_7$, then

$$\begin{aligned} x(\tau_1(t)) &\ge \frac{1}{1+p_2(\eta_2^{-1}(\tau_1(t)))}z(\eta_2^{-1}(\tau_1(t))) \text{ and} \\ x(\tau_2(t)) &\ge \frac{1}{1+p_2(\eta_2^{-1}(\tau_2(t)))}z(\eta_2^{-1}(\tau_2(t))), t \ge t_7. \end{aligned} \quad (27)$$

substituting from (27) into (8), we have

$$(r(t)(z^\Delta(t))^\gamma)^\Delta \le \frac{-q_1(t)}{(1+p_2(\eta_2^{-1}(\tau_1(t))))^\alpha}z^\alpha(\eta_2^{-1}(\tau_1(t))) - \frac{q_2(t)}{(1+p_2(\eta_2^{-1}(\tau_2(t))))^\beta}z^\beta(\eta_2^{-1}(\tau_2(t))) \quad (28)$$

for all $t \ge t_7$, then using the same technique we used in Case 1, we obtain

$$\begin{aligned} w^\Delta(t) \le -\delta(t)M(t)\big[q_1(t)(\frac{z(\eta_2^{-1}(\tau_1(t)))}{z(t)})^\alpha z^{\alpha-\beta}(t) + q_2(t)(\frac{z(\eta_2^{-1}(\tau_2(t)))}{z(t)})^\beta\big] + \\ \frac{\delta_+^\Delta(t)}{\delta(\sigma(t))}w(\sigma(t)) - \frac{\beta\delta(t)C(t)}{(\delta(\sigma(t))^\lambda r^{\frac{1}{\gamma}}(t)}w^\lambda(\sigma(t)) \text{ for all } t \ge t_7, \end{aligned} \quad (29)$$

where $M(t)=\min\{\frac{1}{(1+p_2(\eta_2^{-1}(\tau_1(t))))^{\alpha}},\frac{1}{(1+p_2(\eta_2^{-1}(\tau_2(t))))^{\beta}}\}$.
Since $t\geq\eta_2^{-1}(\tau_1(t))$ for all $t\geq t_7$, then using the fact that $\frac{z(t)}{R(t)}$ is a decreasing function (see (12)), we get

$$\frac{z(\eta_2^{-1}(\tau_1(t)))}{z(t)}\geq\frac{R(\eta_2^{-1}(\tau_1(t)))}{R(t)}\text{for all}t\geq t_7. \tag{30}$$

Since $\tau_2(t)\geq\eta_2(t)$, then

$$\frac{z(\eta_2^{-1}(\tau_2(t)))}{z(t)}\geq 1 \text{ for all } t\geq t_7. \tag{31}$$

Substituting from (30) and (31) into (29), we obtain

$$w^{\Delta}(t)\leq-\delta(t)M(t)[q_1(t)(\frac{R(\eta_2^{-1}(\tau_1(t)))}{R(t)})^{\alpha}A(t)+q_2(t)]+\frac{\delta_+^{\Delta}(t)}{\delta(\sigma(t))}w(\sigma(t))- \\ \frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda}r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^{\lambda}. \tag{32}$$

Using Lemma 3 and integrating from t_7 to t, we get

$$\int_{t_7}^{t}[\delta(s)M(s)[q_1(s)(\frac{R(\eta_2^{-1}(\tau_1(s)))}{R(s)})^{\alpha}A(s)+q_2(s)]-\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(\delta_+^{\Delta}(s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}<w(t_7) \tag{33}$$

which is a contradiction with (11).
Finally, suppose that case (C_2) holds, then according to lemma 2, we have $\lim_{t\to\infty}x(t)=0$. This completes the proof.
Theorem 2 Assume that H_1-H_6 hold and $\eta_2(t)\geq\tau_2(t)$ for all $t\geq t_0$. Furthermore suppose that there exist positive real-valued Δ -differentiable functions $R(t)$ and $\delta(t)$ such that Eq. (10) is satisfied and for sufficiently large T, we have

$$\limsup_{t\to\infty}\int_{T}^{t}[\delta(s)\xi(s)[q_1(s)L^{\alpha}(s)A(s)+q_2(s)v^{\beta}(s)]-\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(\delta_+^{\Delta}(s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}]\Delta s=\infty, \tag{34}$$

where

$$L(s)=\min\{\frac{R(\tau_1(s))}{R(s)},\frac{R(\eta_2^{-1}(\tau_1(s)))}{R(s)})\}\text{ and }v(s)=\min\{1,\frac{R(\eta_2^{-1}(\tau_2(s)))}{R(s)}\}$$

Then, every solution of (1) is almost oscillatory on $[t_0,\infty)_{\mathbb{T}}$ or converges to zero as $t\to\infty$
Proof. Assume that $x(t)$ is not almost oscillatory solution of (1). Then without loss of generality, there exists $t_3\geq t_0$ such that $x(t)>0$, $x(\tau_i(t))>0$ and $x(\eta_i(t))>0, i=1,2$ on $[t_3,\infty)_{\mathbb{T}}$ (when $x(t)$ is negative, the proof is similar). Then from lemma 1, $z(t)$ satisfies one of the cases C_1 or C_2. Also, by the definition of not almost oscillatory we have the two possibilities:

(I) $x^{\Delta}(t)<0$ for $t\geq t_3$
(II) $x^{\Delta}(t)>0$ for $t\geq t_3$

Case1. Suppose that C_1 holds and $x^{\Delta}(t)<0$, then the proof is similar to that of Theorem 1. So, it is omitted.

Case2. Suppose that C_1 holds and $x^{\Delta}(t)>0$, then using the same technique that used in Case 2 of Theorem 1, until we reach to (29). Hence

$$w^{\Delta}(t)\le -\delta(t)M(t)\big[q_1(t)(\frac{z(\eta_2^{-1}(\tau_1(t)))}{z(t)})^{\alpha}z^{\alpha-\beta}(t)+q_2(t)(\frac{z(\eta_2^{-1}(\tau_2(t)))}{z(t)})^{\beta}\big]+ \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w(\sigma(t))-\frac{\delta(t)r^{\sigma}(t)(z^{\Delta\sigma}(t))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))}\text{ for all } t\ge t_7. \tag{35}$$

Since, $\eta_2(t)\ge\tau_2(t)>t_1$, then $t\ge\eta_2^{-1}(\tau_2(t))$ for all $t\ge t_7$. Using the fact that $\frac{z(t)}{R(t)}$ is decreasing, hence

$$\frac{z(\eta_2^{-1}(\tau_2(t)))}{z(t)}\ge\frac{R(\eta_2^{-1}(\tau_2(t)))}{R(t)}\text{ for all } t\ge t_7. \tag{36}$$

Substituting from (20), (24), (30) and (36) into (35), we obtain

$$w^{\Delta}(t)\le -\delta(t)M(t)[q_1(t)(\frac{R(\eta_2^{-1}(\tau_1(t)))}{R(t)})^{\alpha}A(t)+q_2(t)(\frac{R(\eta_2^{-1}(\tau_2(t)))}{R(t)})^{\beta}]+ \frac{\delta_+^{\Delta}(t)}{\delta(\sigma(t))}w(\sigma(t))-\frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda}r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^{\lambda}, \tag{37}$$

Using Lemma 3 and integrating from t_7 to t, we get

$$\int_{t_7}^{t}\big[\delta(s)M(s)[q_1(s)(\frac{R(\eta_2^{-1}(\tau_1(s)))}{R(s)})^{\alpha}A(s)+q_2(s)(\frac{R(\eta_2^{-1}(\tau_2(s)))}{R(s)})^{\beta}]- \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(\delta_+^{\Delta}(s))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}\big]\le w(t_7)-w(t)<w(t_7) \tag{38}$$

which is a contradiction with (34).
Finally, if case (C_2) holds, then according to lemma 2, we have $\lim_{t\to\infty}x(t)=0$. This completes the proof.
Theorem 3 Assume that H_1-H_6 and (10) hold, $\tau_2(t)\ge\eta_2(t)$ for all $t\ge t_0$ and there exist functions H,h such that for each fixed t, $H(t,s)$ and $h(t,s)$ are rd-continuous with respect to s on $\mathbb{D}\equiv\{(t,s):t\ge s\ge t_0\}$ such that

$$H(t,t)=0,\ t\ge t_0,H(t,s)>0,\ t>s\ge t_0, \tag{39}$$

and H has a non-positive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ satisfying

$$H^{\Delta_s}(t,s)+H(t,s)\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}=-\frac{h(t,s)}{\delta^{\sigma}(t)}(H(t,s))^{\frac{\gamma}{\gamma+1}}. \tag{40}$$

Assume that there exists a positive real-valued Δ -differentiable function $\delta(t)$ such that for sufficiently large $T\ge t_1>t_0$, we have

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\big[\delta(s)\xi(s)H(t,s)[q_1(s)L^{\alpha}(s)A(s)+q_2(s)] -\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(h_-(s,t))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}\big]\Delta s=\infty, \tag{41}$$

Then, every solution of (1) is almost oscillatory on $[t_0,\infty)_{\mathbb{T}}$ or converges to zero as $t\to\infty$.

Proof. Assume that $x(t)$ is not almost oscillatory solution of (1). Then without loss of generality, there exists $t_3\geq t_0$ such that $x(t)>0,\ x(\tau_i(t))>0$ and $x(\eta_i(t))>0, i=1,2$ on $[t_3,\infty)_{\mathbb{T}}$.(when $x(t)$ is negative, the proof is similar). Then from lemma 1, $z(t)$ satisfies one of the cases C_1 or C_2. Also, by the definition of not almost oscillatory we have the two possibilities:

(I) $x^{\Delta}(t)<0$ for $t\geq t_3$
(II) $x^{\Delta}(t)>0$ for $t\geq t_3$

Case1. Suppose that C_1 holds and $x^{\Delta}(t)<0$, then Proceeding as in the proof of Case 1 in Theorem 1 until we get (25), therefore

$$\delta(t)N(t)[q_1(t)(\frac{R(\tau_1(t))}{R(t)})^{\alpha}A(t)+q_2(t)]\leq -w^{\Delta}(t)+\frac{\delta_+^{\Delta}(t)}{\delta(\sigma(t))}w(\sigma(t))-\frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda}r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^{\lambda},$$

Multiplying both sides of the previous inequality by $H(t,s)$ and integrating from t_5 to t, we get

$$\begin{aligned}&\int_{t_5}^{t}[H(t,s)\delta(s)N(s)[q_1(s)(\frac{R(\tau_1(s))}{R(s)})^{\alpha}A(s)+q_2(s)]\Delta s\\
&\leq -\int_{t_5}^{t}H(t,s)w^{\Delta}(s)\Delta s+\int_{t_5}^{t}H(t,s)\frac{\delta_+^{\Delta}(s)}{\delta^{\sigma}(s)}w^{\sigma}(s)\Delta s-\int_{t_5}^{t}\frac{\beta H(t,s)\delta(s)C(s)}{r^{\frac{1}{\gamma}}(s)(\delta^{\sigma}(s))^{\lambda}}(w^{\sigma}(s))^{\lambda}\Delta s\\
&\leq H(t,t_5)w(t_5)+\int_{t_5}^{t}[\frac{-h(t,s)(H(t,s))^{\frac{1}{\lambda}}}{\delta^{\sigma}(s)}w^{\sigma}(s)\Delta s-\int_{t_5}^{t}\frac{\beta H(t,s)\delta(s)C(s)}{r^{\frac{1}{\gamma}}(s)(\delta^{\sigma}(s))^{\lambda}}(w^{\sigma}(s))^{\lambda}\Delta s\\
&\leq H(t,t_5)w(t_5)+\int_{t_5}^{t}[\frac{h_-(t,s)(H(t,s))^{\frac{1}{\lambda}}}{\delta^{\sigma}(s)}w^{\sigma}(s)\Delta s-\int_{t_5}^{t}\frac{\beta H(t,s)\delta(s)C(s)}{r^{\frac{1}{\gamma}}(s)(\delta^{\sigma}(s))^{\lambda}}(w^{\sigma}(s))^{\lambda}\Delta s.\end{aligned}\tag{42}$$

Using lemma 3, with

$$a:=\frac{\beta H(t,s)\delta(s)C(s)}{r^{\frac{1}{\gamma}}(s)(\delta^{\sigma}(s))^{\lambda}}\text{ and }b:=\frac{h_-(t,s)(H(t,s))^{\frac{1}{\lambda}}}{\delta^{\sigma}(s)},$$

we get:

$$\frac{h_-(t,s)(H(t,s))^{\frac{1}{\lambda}}}{\delta^{\sigma}(s)}w^{\sigma}(s)-\frac{\beta H(t,s)\delta(s)C(s)}{r^{\frac{1}{\gamma}}(s)(\delta^{\sigma}(s))^{\lambda}}(w^{\sigma}(s))^{\lambda}\leq\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(h_-(s,t))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}.\tag{43}$$

Substituting (43) into (42), we get

$$\int_{t_5}^{t}[H(t,s)\delta(s)N(s)[q_1(s)(\frac{R(\tau_1(s))}{R(s)})^{\alpha}A(s)+q_2(s)]]\Delta s$$

$$\leq H(t,t_5)w(t_5)+\int_{t_5}^{t}\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(h_-(s,t))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}\Delta s,$$

which implies

$$\frac{1}{H(t,t_5)}\int_{t_5}^{t}\big[\delta(s)N(s)H(t,s)[q_1(s)(\frac{R(\tau_1(s))}{R(s)})^{\alpha}(s,t_1)A(s)+q_2(s)]-$$

$$\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(h_-(s,t))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}\big]\Delta s\leq w(t_5),$$

this is a contradiction with (41).

Case2. Suppose that C_1 holds and $x^{\Delta}(t)>0$, then Proceeding as in the proof of Case 2 in Theorem 1 until (32), we get:

$$\delta(t)M(t)[q_1(t)(\frac{R(\eta_2^{-1}(\tau_1(t)))}{R(t)})^{\alpha}A(t)+q_2(t)]$$

$$\leq -w^{\Delta}(t)+\frac{\delta_+^{\Delta}(t)}{\delta(\sigma(t))}w(\sigma(t))-\frac{\beta\delta(t)C(t)}{(\delta(\sigma(t)))^{\lambda}r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^{\lambda}.$$

Multiplying both sides of the above inequality by $H(t,s)$, integrating from t_7 to t and following the same proof as in Case 1, we obtain

$$\frac{1}{H(t,t_7)}\int_{t_7}^{t}\big[\delta(s)M(s)H(t,s)[q_1(s)(\frac{R(\eta_2^{-1}(\tau_1(s)))}{R(s)})^{\alpha}A(s)+q_2(s)]-$$

$$\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(h_-(s,t))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}\big]\Delta s\leq w(t_7)$$

which is a contradiction with (41).

Finally, suppose that case (C_2) holds, then according to lemma 2, we get $\lim_{t\to\infty}x(t)=0$. This completes the proof.

Theorem 4 Assume that H_1-H_6 hold, $\eta_2(t)\geq\tau_2(t)$ for all $t\geq t_0$. Also, assume thst there exist functions H,h and δ defined as in Theorem 3 and satisfying Eqs. (39), (40) and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\big[\delta(s)\xi(s)H(t,s)[q_1(s)L^{\alpha}(s)A(s)+v^{\beta}(s)q_2(s)]$$

$$-\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(h_-(s,t))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}\big]\Delta s=\infty, \quad (44)$$

(45)

Then, every solution of (1) is almost oscillatory on $[t_0,\infty)_{\mathbb{T}}$ or converges to zero as $t\to\infty$.

Proof. Assume that $x(t)$ is not almost oscillatory solution of (1). Then without loss of generality, there exists $t_3 \geq t_0$ such that $x(t) > 0,\ x(\tau_i(t)) > 0$ and $x(\eta_i(t)) > 0, i = 1, 2$ on $[t_3, \infty)_{\mathbb{T}}$.(when $x(t)$ is negative, the proof is similar). Then from lemma 1, $z(t)$ satisfies one of the cases C_1 or C_2. Also, by the definition of not almost oscillatory we have the two possibilities:

(I) $x^{\Delta}(t) < 0$ for $t \geq t_3$
(II) $x^{\Delta}(t) > 0$ for $t \geq t_3$

Case1. Suppose that C_1 holds and $x^{\Delta}(t) < 0$, then the proof is similar to that of Case 1 Theorem 3. So it is omitted.
Case2. Suppose that C_1 holds and $x^{\Delta}(t) > 0$, then Proceeding as in the proof of Case 2 in Theorem 2 until (37), we get

$$\delta(t)M(t)[q_1(t)(\frac{R(\eta_2^{-1}(\tau_1(t)))}{R(t)})^{\alpha}A(t) + q_2(t)(\frac{R(\eta_2^{-1}(\tau_2(t)))}{R(t)})^{\beta}]$$
$$\leq -w^{\Delta}(t) + \frac{\delta_+^{\Delta}(t)}{\delta(\sigma(t))}w(\sigma(t)) - \frac{\beta\delta(t)C(t)}{(\delta(\sigma(t))^{\lambda}r^{\frac{1}{\gamma}}(t)}(w(\sigma(t))^{\lambda},$$

Multiplying both sides of the above inequality by $H(t, s)$, integrating from t_7 to t and following the same technique as in Case 1 Theorem 3, we obtain

$$\frac{1}{H(t,t_7)}\int_{t_7}^{t}[\delta(s)M(s)H(t,s)[q_1(s)(\frac{R(\eta_2^{-1}(\tau_1(s)))}{R(s)})^{\alpha}A(s) + q_2(s)(\frac{R(\eta_2^{-1}(\tau_2(s)))}{R(s)})^{\beta}(s,t_1)]$$
$$- \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(h_-(s,t))^{\gamma+1}}{\delta^{\gamma}(s)C^{\gamma}(s)}]\Delta s \leq w(t_7)$$

which is a contradiction with (44).
Finally, suppose that case (C_2) holds, then according to lemma 2, we get $\lim_{t\to\infty} x(t) = 0$. This completes the proof.

4. Examples

In this section, we give some examples to illustrate our main results.
Example 1 Take $\mathbb{T} = [t_1 + \Pi, \infty)_{\mathbb{R}}$ where $t_1 \geq 0$ and consider the equation

$$[x(t) - \frac{1}{2}x(t - \frac{\Pi}{2}) + 2x(t + \frac{\Pi}{2})]'' + 32x(t - \frac{\Pi}{2}) + 8x(t + \Pi) = 0 \ \text{ for all } t \geq t_0 + \Pi. \quad (46)$$

Here

$$\alpha = \beta = \gamma = 1,\ r(s) = 1,\ \eta_1(t) = \tau_1(t) = t - \frac{\Pi}{2},\ \eta_2(t) = t + \frac{\Pi}{2},\ \tau_2(t) = t + \Pi,\ p_1(t) = \frac{1}{2},$$
$$p_2(t) = 2, q_1(t) = 32 \ \text{ and } \ q_2(t) = 8$$

then substituting in (4) and (5), we obtain $A(s) = C(s) = 1$. Also

$$\int_{t_1+\Pi}^{\infty}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} = \int_{t_1+\Pi}^{\infty}\Delta s = \infty \text{ and } \eta_2^{-1}(\tau_1(t)) = t - \Pi.$$

hence $t_1 < \eta_2^{-1}(\tau_1(t)) < \tau_1(t)$, taking $\int\limits_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}$, we get $\frac{R(\eta_2^{-1}(\tau_1(s)))}{R(s)} < (\frac{R(\tau_1(s))}{R(s)})$. Consequently

$$L^{\alpha}(t) = \frac{R(\eta_2^{-1}(\tau_1(s)))}{R(s)} = \frac{\int\limits_{t_1}^{t-\Pi} \Delta s}{\int\limits_{t_1}^{t} \Delta s} = \frac{t-\Pi-t_1}{t-t_1}.$$

Also $\xi(s) = \frac{1}{3}$. Since $\eta_2(t) \leq \tau_2(t)$, then subistituting in (11) with $\delta(t) = 1$, we get

$$\limsup_{t\to\infty} \int_T^t \frac{1}{3}[32\frac{s-\Pi-t_1}{s-t_1} + 8]\Delta s = \infty \tag{47}$$

using Theorem 1, we obtain that every solution of (46) is almost oscillatory or converges to zero as $t \to \infty$. Note that $x(t) = \sin 4t$ is an almost oscillatory solution to Eq. (46).

Remark 1 The results of [4], [9] and [10] can not be applied to (46) as $p_2(t) \neq 0$ and $f(t, x(\tau_1(t))) \neq 0 \neq g(t, x(\tau_2(t)))$, but according to Theorem 1 we obtain that every solution of (46) is almost oscillatory or converges to zero as $t \to \infty$.

Example 2 Take $\mathbb{T} = [2t_1, \infty)_{\mathbb{T}}$ where $t_1 \geq 0$ and consider the equation

$$[[[x(t) - \frac{1}{2}x(\eta_1(t)) + \frac{1}{4}x(2t)]^{\Delta}]^4]^{\Delta} + q_1(t)x^9(t) + q_2(t)x^8(2t+1) = 0. \tag{48}$$

Here

$$\alpha = 9,\ \beta = 8,\ \gamma = 4,\ r(s) = 1,\ \eta_1(t) \leq t,\ \eta_2(t) = 2t,\ \tau_1(t) = t,\ \tau_2(t) = 2t+1,$$

$$p_1(t) = \frac{1}{2}, p_2(t) = \frac{1}{4}, q_1(t) = \frac{2^9(t-t_1)^9}{(t-2t_1)^9 b_0 t} \text{ and } q_2(t) = \frac{2}{t}$$

then substituting in (4) and (5), we obtain $A(s) = C(s) = b_0$,

$$\int_{2t_1}^{\infty} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} = \int_{2t_1}^{\infty} \Delta s = \infty \text{ and } \eta_2^{-1}(\tau_1(t)) = \frac{t}{2}$$

hence $t_1 < \eta_2^{-1}(\tau_1(t)) < \tau_1(t)$, taking $R(t) = \int\limits_{t_1}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}$, hence $\frac{R(\eta_2^{-1}(\tau_1(s)))}{R(s)} < (\frac{R(\tau_1(s))}{R(s)})$, consequently

$$L^{\alpha}(t) = (\frac{R(\eta_2^{-1}(\tau_1(t)))}{R(t)})^9 = (\frac{\int\limits_{t_1}^{\frac{t}{2}} \Delta s}{\int\limits_{t_1}^{t} \Delta s})^9 = (\frac{\frac{t}{2} - t_1}{t - t_1})^9 > 0.$$

Also $\xi(s) = (\frac{4}{5})^9$, since $\eta_2(t) \leq \tau_2(t)$, then substitute in (11) with $\delta(t) = 1$ we have

$$\limsup_{t\to\infty} \int_T^t (\frac{4}{5})^9 \frac{3}{s} \Delta s = \infty \tag{49}$$

using Theorem 1, we find that every solution of (48) is almost oscillatory or converges to zero as $t \to \infty$.

Remark 2 The results of [9] can not be applied to (48) as $p_2(t) \neq 0, \alpha \neq \beta \neq \gamma$ and both $f(t, x(\tau_1(t))) \neq 0 \neq g(t, x(\tau_2(t)))$. But according to Theorem 1, we obtain that every solution of (48) is almost oscillatory or converges to zero as $t \to \infty$.

References

[1] R. P. Agarwal, Said R. Grace and Donal O'Regan, Oscillation Theory for Second Order Dynamic Equations, TJ International Ltd, Padstow, Cornwall, 2003.

[2] R. P. Agarwal, D. O'Regan and S. H. Saker, Oscillation results for second-order nonlinear neutral delay dynamic equations on time scales, Applicable Analysis, 86, 1-17, 2007.

[3] R. P. Agarwal, D. O'Regan and S. H. Saker, Oscillation theorems for second-order nonlinear neutral delay dynamic equations on time scales, Acta Mathematica Sinica, 24, 1409-1432, 2008.

[4] R. Arul and V. S. Shobha, Improvement results for oscillatory behavior of second order neutral differential equations with nonpositive neutral term, British Journal of Mathematics Computer Science, 12, 1-7, 2016.

[5] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[6] M. Bohner and A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.

[7] D. X. Chen, Oscillation of second-order Emden-Fowler neutral delay dynamic equations on time scales, Mathematical and Computer Modelling, 51, 1221-1229, 2010.

[8] D. X. Chen and G. H. Lui, Oscillatory behavior of a class of second-order nonlinear dynamic equations on time sacles, International Journal of Engineering and Manufacturing, 1, 72-79, 2011.

[9] L. Erbe, Lynn, Taher S. Hassan, and Allan Peterson, Oscillation criteria for nonlinear functional neutral dynamic equations on time scales, Journal of Difference Equations and Applications, 15, 1097-1116, 2009.

[10] Qi Li, Rui Wang, Feng Chen, and Tongxing Li, Oscillation of second-order nonlinear delay differential equations with non positive neutral coefficients, Advances in Difference Equations, 35, 1-7, 2015.

[11] D. O'Regan and S. H. Saker, New oscillation criteria for second-order neutral dynamic equations on time scales via riccati substitution, Hiroshima Mathematical Journal, 42, 77-98, 2012.

[12] Y. Sahiner, Oscillation of second-order neutral delay and mixed type dynamic equations on time scales, Advances in Difference Equations, 2006, 1-9, 2006.

[13] S. H. Saker, Oscillation of second-order nonlinear neutral delay dynamic equations on time scales, Journal of Computational and Applied Mathematics, 187, 123-141, 2006.

[14] E.Thandapani, V. Balasubramanian and John R. Graef. Oscillation criteria for second order neutral difference equations with negative neutral term, International Journal of Pure and Applied Mathematics 87.2, 283-292, 2013.

[15] Tao Ji, Shuhong Tang and E. Thandapani, Oscillation of second-order neutral dynamic equations with mixed arguments, Appl. Math. Inf. Sci., 8, 2225-2228, 2014.

H. A. Agwa
Faculty of Education, Ain Shams University, Cairo, Egypt
E-mail address: hassanagwa@yahoo.com

Ahmed M. M. Khodier
Faculty of Education, Ain Shams University, Cairo, Egypt
E-mail address: khodier55@yahoo.com

Heba M. Arafa
Faculty of Education, Ain Shams University, Cairo, Egypt
E-mail address: hebaallahmohammed@edu.asu.edu.eg