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GENERAL DECAY RESULT OF THE TIMOSHENKO SYSTEM IN THERMOELASTICITY OF SECOND SOUND

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Abstract. In this paper we consider a on-dimensional linear thermoelastic system of Timoshenko type, where the heat flux is given by Cattaneo's law. We consider damping terms acting on the first and the second equation and we establish a general decay estimate where the exponential and polynomial decay rates are only particular cases. We establish our result without the usual assumption of the wave speeds. Our method of proof uses the energy mrthod together with some properties of convex functions. The advantage here is that from our general estimates we can derive the exponential, polynomial or logarithmic decay rate. We also give some examples to illustrate our result.

1. Introduction

This paper aims at investigating long-term behavior of solutions to the following system:

$$\begin{cases} \rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x + \mu\varphi_t = 0 \\ \rho_2\psi_{tt} - \bar{b}\psi_{xx} + \displaystyle\int_0^t g\left(t-s\right)\left(a\left(x\right)\psi_x\left(s\right)\right)_x ds + K\left(\varphi_x + \psi\right) + b\left(x\right)h\left(\psi_t\right) + \gamma\theta_x = 0 \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0 \\ \tau_0 q_t + \delta q + \kappa\theta_x = 0. \end{cases} \tag{1}$$

where $t \in (0,\infty)$ denotes the time variable and $x \in (0,1)$ is the space variable, the function φ and ψ are the displacement of the solid elastic material, the function θ is the temperature difference, $q = q(x,t) \in \mathbb{R}$ is the heat flux, and $\rho_1, \rho_2, \rho_3, \gamma, \tau_0, \delta, \kappa, \bar{b}$ and K are positive constants and $\mu > 0$. We consider the following initial conditions:

$$\begin{aligned} \varphi(.,0) &= \varphi_0(x),\ \varphi_t(.,0) = \varphi_1(x),\ \psi(.,0) = \psi_0(x) \\ \psi_t(.,0) &= \psi_1(x),\ \theta(.,0) = \theta_0\left(x\right),\ q(.,0) = q_0(x), \end{aligned} \tag{2}$$

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and boundary conditions

$$\varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=q(0,t)=q(1,t)=0,\ \forall t\geq 0. \tag{3}$$

Before we state and prove our main result, let us first recall some results regarding the Timoshenko systems of wave equations.
In 1921, Timoshenko proposed the following system of coupled hyperbolic equations

$$\begin{cases} \rho u_{tt}=(K(u_x-\varphi))_x, & \text{in } (0,L)\times(0,+\infty) \\ I_\rho\varphi_{tt}=(EI\varphi_x)_x+K(u_x-\varphi), & \text{in } (0,L)\times(0,+\infty), \end{cases} \tag{4}$$

which describes the transverse vibration of a beam of length L in its equilibrium configuration. Here t denotes the time variable, x is the space variable along the beam, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.
System (4), together with boundary conditions of the form

$$EI\varphi_x\mid_{x=0}^{x=L}=0, \qquad K(u_x-\varphi)\mid_{x=0}^{x=L}=0$$

is conservative, and so the total energy of the beam remains constant along the time.

The subject of stability of Timoshenko-type systems has received a lot of attention in the last 10 years and several outstanding results have been proved by some of the major experts in the fields of partial deferential equations, and several results concerning uniform and asymptotic decay of energy have been established.

An important issue of research is to look for a minimum dissipation by which solutions of system (4) decay uniformly to the stable state as time goes to infinity. In this regards, several types of dissipative mechanisms have been introduced.

Kim and Renardy [10] considered (4) together with two boundary controls of the form

$$\begin{aligned} K\varphi(L,t)-K\frac{\partial u}{\partial x}(L,t) &= \alpha\frac{\partial u}{\partial t}(L,t), \quad \forall t\geq 0 \\ EI\frac{\partial\varphi}{\partial x}(L,t) &= -\beta\frac{\partial\varphi}{\partial t}(L,t), \quad \forall t\geq 0 \end{aligned}$$

and used the multiplier techniques to establish an exponential decay result for the natural energy of (4). They also provided numerical estimates to the eigenvalues of the operator associated with system (4). Raposo *et al.* [25] studied the following system

$$\begin{cases} \rho_1 u_{tt}-K(u_x-\varphi)_x+u_t=0, & \text{in } (0,L)\times(0,+\infty) \\ \rho_2\varphi_{tt}-b\varphi_{xx}+K(u_x-\varphi)+\varphi_t=0, & \text{in } (0,L)\times(0,+\infty) \end{cases} \tag{5}$$

with homogeneous Dirichlet boundary conditions, and proved that the associated energy decays exponentially. Soufyane and Wehbe [26] showed that it is possible to stabilize uniformly (4) by using a unique locally distributed feedback. They considered

$$\begin{cases} \rho u_{tt} = (K(u_x - \varphi))_x, & \text{in } (0,L)\times(0,+\infty) \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi) - b\varphi_t, & \text{in } (0,L)\times(0,+\infty) \\ u(0,t) = u(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \quad \forall t > 0, \end{cases} \tag{6}$$

where b is a positive and continuous function, which satisfies

$$b(x) \geq b_0 > 0, \quad \forall\ x\ \in\ [a_0, a_1] \subset [0, L]$$

and proved that the uniform stability of (6) holds if and only if the wave speeds are equal $\left(\dfrac{K}{\rho} = \dfrac{EI}{I_\rho}\right)$; otherwise only the asymptotic stability has been proved. Recently, Muñoz Rivera and Racke [21] obtained a similar result in a work where the damping function $b = b(x)$ is allowed to change its sign. Also, Muñoz Rivera and Racke [19] treated a nonlinear Timoshenko-type system of the form

$$\begin{cases} \rho_1 \varphi_{tt} - \sigma_1(\varphi_x, \psi)_x = 0 \\ \rho_2 \psi_{tt} - \chi(\psi_x)_x + \sigma_2(\varphi_x, \psi) + d\psi_t = 0 \end{cases}$$

in a one-dimensional bounded domain. The dissipation is produced here through a frictional damping which is only present in the equation for the rotation angle. The authors gave an alternative proof for a necessary and sufficient condition for exponential stability in the linear case and then proved a polynomial stability in general. Moreover, they investigated the global existence of small smooth solutions and exponential stability in the nonlinear case. Ammar-Khodja *et al.* [2] considered a linear Timoshenko-type system with memory of the form

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ \rho_2 \psi_{tt} - b\psi_{xx} + \displaystyle\int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) = 0 \end{cases} \tag{7}$$

in $(0,L)\times(0,+\infty)$, together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$ and g decays uniformly. Precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g decays in a polynomial rate. They also required some extra technical conditions on both g' and g'' to obtain their result. Guesmia and Messaoudi [8] proved the same result without imposing the extra technical conditions of [2] . Recently, Messaoudi and Mustafa [11] improved the results of [2] and [8] by allowing more general decaying relaxation functions and showed that the rate of decay of the solution energy is exactly the rate of decay of the relaxation function. Alabau-Boussouira [1] considered the following system

$$\begin{cases} \rho_1 u_{tt} - k(u_x + \varphi)_x = 0, & \text{in } (0,L)\times(0,+\infty) \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + k(u_x + \varphi) + \alpha(\varphi_t) = 0, & \text{in } (0,L)\times(0,+\infty) \end{cases} \tag{8}$$

associated with two different types of boundary conditions and for α a nonlinear function. Under no growth assumption on α near the origin, the author established a semi-explicit formula for the decay of the energy in the case of equal wave speeds. In the case of different wave speeds, a polynomial decay has been established for both linear and nonlinear globally Lipschitz feedbacks. System (8), with $\alpha(t)g(\psi_t)$

instead of $\alpha(\varphi_t)$, has been considered by Messaoudi and Mustafa [12]. An explicit formula for the decay rate, depending on α and g, has been given under no growth condition on g at the origin. Also, Muñoz Rivera and Fernández Sare [22], considered Timoshenko type system with past history acting only in one equation. More precisely they looked into the following problem

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \displaystyle\int_0^\infty g(t)\psi_{xx}(t-s,.)ds + K(\varphi_x + \psi) = 0 \end{cases} \tag{9}$$

together with homogenous boundary conditions, and showed that the dissipation given by the history term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal. They also proved that the solution decays polynomially for the case of different wave speeds. This work was improved by Messaoudi and Said-Houari [17], where the authors considered system (9) for g decaying polynomially, and proved polynomial stability results for the equal and nonequal wave-speed propagation under conditions on the relaxation function weaker than those in [22].

For Timoshenko systems in thermoelasticity, Rivera and Racke [18] considered

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0 & \text{in } (0, L) \times (0, +\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + k\,(\varphi_x + \psi) + \gamma\theta_x = 0 & \text{in } (0, L) \times (0, +\infty) \\ \rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0 & \text{in } (0, L) \times (0, +\infty) \end{cases} \tag{10}$$

where φ, ψ and θ are functions of (x, t) which model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively. Under appropriate conditions of $\sigma, \rho_i, b, k, \gamma$, they proved several exponential decay results for the linearized system and a non exponential stability result for the case of different wave speeds.

Modeling heat conduction with the so-called Fourier law (as in (10)), which assumes the flux q to be proportional to the gradient of the temperature θ at the same time t,

$$q + \kappa\nabla\theta, \qquad (\kappa > 0),$$

leads to the phenomenon of infinite heat propagation speed. That is, any thermal disturbance at a single point has an instantaneous effect everywhere in the medium. To overcome this physical problem, a number of modification of the basic assumption on the relation between the heat flux and the temperature have been made. The common feature of these theories is that all lead to hyperbolic differential equation and the speed of propagation becomes limited. See [4] for more details. Among them Cattaneo's law,

$$\tau q_t + q + \kappa\nabla\theta = 0, \qquad (\tau > 0, \text{ relatively small}),$$

leading to the system with *second sound*, ([27], [23], [24], [14]) and a suggestion by Green and Naghdi [7], [6], for thermoelastic systems introducing what is called thermoelasticity of type III, where the constitutive equations for the heat flux is characterized by

$$q + \kappa^* p_x + \tilde{\kappa}\nabla\theta = 0, \qquad (\tilde{\kappa} > \kappa > 0,\, p_t = \theta).$$

Messaoudi *et al.* [15] studied the following problem

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x + \mu\varphi_t = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_x = 0, \\ \rho_3\theta_t + \gamma q_x + \delta\psi_{tx} = 0, \\ \tau_0 q_t + q + \kappa\theta_x = 0, \end{cases}$$

where $(x, t) \in (0, L) \times (0, \infty)$ and $\varphi = \varphi(x, t)$ is the displacement vector, $\psi = \psi(x, t)$ is the rotation angle of the filament, $\theta = \theta(x, t)$ is the temperature difference, $q = q(x, t)$ is the heat flux vector, ρ_1, ρ_2, ρ_3, b, k, γ, δ, κ, μ, τ_0 are positive constants. The nonlinear function σ is assumed to be sufficiently smooth and satisfy

$$\sigma_{\varphi_x}(0, 0) = \sigma_\psi(0, 0) = k$$

and

$$\sigma_{\varphi_x\varphi_x}(0, 0) = \sigma_{\varphi_x\psi}(0, 0) = \sigma_{\psi\psi} = 0.$$

Several exponential decay results for both linear and nonlinear cases have been established.

In system (1)−(3) the heat conduction given by Cattaneo's law instead of the usual Fourier's one. We should note here that dissipative effects of heat conduction induced by Cattaneo's law are usually weaker than those induced by Fourier's law. This justifies the presence of the extra damping term in the second equation of (1). In fact if $\mu = a = b = 0$, Fernández Sare and Racke [5] have proved recently that (1)−(3) is no longer exponentially stable even in the case of equal propagation speed ($\rho_1/\rho_2 = K/\bar{b}$). In this paper, we show a general decay result of the total energy of system (1)−(3) (Theorem 2 below). To prove this result, we followed very carefully the method used by Guesmia and Messaoudi [8, 9] .

2. Preliminaries

In this section, we introduce some notations and some techenical lemmas to be used throughout this paper. Also, we give a local existence theorem. In order to state and prove our result, we formulate the following assumptions:

- **(H1)** a, b: $[0, 1] \to \mathbb{R}^+$ are such that

$$\begin{aligned} a &\in C^1([0, 1]), \qquad b \in L^\infty([0, 1]) \\ a &= 0 \text{ or } a(0) + a(1) > 0, \qquad \inf_{x\in[0,1]} \{a(x) + b(x)\} > 0. \end{aligned}$$

- **(H2)** $h : \mathbb{R} \to \mathbb{R}$ is a differentiable nondecreasing function such that there exist constants $\varepsilon', c_1, c_2 > 0$ and a convex and increasing function $H : \mathbb{R}^+ \to \mathbb{R}^+$ of class $C^1(\mathbb{R}^+) \cap C^2((0, \infty))$ satisfying $H(0) = 0$ and H is linear on $[0, \varepsilon']$ or $H'(0) = 0$ and $H'' > 0$ on $(0, \varepsilon']$ such that

$$\begin{cases} c_1 |s| \le h(s) \le c_2 |s| & \text{if } |s| \ge \varepsilon' \\ s^2 + h^2(s) \le H^{-1}(sh(s)) & \text{if } |s| \le \varepsilon'. \end{cases}$$

- **(H3)** $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that

$$g(0) > 0, \qquad 1 - \|a\|_\infty \int_0^\infty g(s)\, ds = l > 0.$$

- **(H4)** There exists a non-increasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying
$$g'(s) \leq -\xi(s) g(s), \qquad \forall s \geq 0.$$

Throughout this paper, we use the following notations

$$\begin{aligned}
(\phi * \psi)(t) \quad &: \quad = \int_0^t \phi(t-\tau)\psi(\tau)\,d\tau \\
(\phi \diamond \psi)(t) \quad &: \quad = \int_0^t \phi(t-\tau)\left|\psi(t)-\psi(\tau)\right| d\tau \\
(\phi \circ \psi)(t) \quad &: \quad = \int_0^t \phi(t-\tau)\int_\Omega \left|\psi(t)-\psi(\tau)\right|^2 dx d\tau.
\end{aligned}$$

The following lemma was introduced in [20].

Lemma 1. *For any function $\phi \in C^1(\mathbb{R})$ and any $\psi \in H^1(0,1)$, we have*

$$\begin{aligned}
(\phi * \psi)(t)\,\psi_t(t) \quad = \quad & -\frac{1}{2}\phi(t)\left|\psi(t)\right|^2 + \frac{1}{2}(\phi' \diamond \psi)(t) \\
& -\frac{1}{2}\frac{d}{dt}\left\{(\phi \diamond \psi)(t) - \left(\int_0^t \phi(\tau)\,d\tau\right)\left|\psi(t)\right|^2\right\}.
\end{aligned}$$

Now, we are going to prepare some materials in order to state two lemmas due to Cavalcanti and Oquendo [3]. See also [9] for the proof.

By using the fact that $a(0) > 0$ and since a is continuous, then there exists $\varepsilon > 0$ such that $\inf_{x\in[0,\varepsilon]} a(x) \geq \varepsilon$. Let us denote

$$d = \min\left\{\varepsilon, \inf_{x\in[0,1]}\{a(x)+b(x)\}\right\} > 0$$

and let $\alpha \in C^1([0,1])$ be such that $0 \leq \alpha \leq a$ and

$$\begin{cases} \alpha(x) = 0 & \text{if}\, a(x) \leq \frac{d}{4} \\ \alpha(x) = a(x) & \text{if}\, a(x) \geq \frac{d}{2} \end{cases} \tag{11}$$

To simplify the notations we introduce the following

$$g \odot v = \int_0^1 \alpha(x)\int_0^t g(s)(v(t)-v(s))\,dsdx$$

for all $v \in L^2(0,1)$. Here and in the sequel, we denote various generic positive constants by C or c.

Lemma 2. *The function α is not identically zero and satisfies*

$$\inf_{x\in[0,1]}\{\alpha(x)+b(x)\} \geq \frac{d}{2}.$$

Lemma 3. *There exists a positive constant c such that*

$$(g \odot v)^2 \leq cg \circ v_x,$$

for all $v \in H_0^1(0,1)$.

In order to make this paper self contained we state, without proof, a local existence result. The proof can be established by the classical Galerkin method.

Theorem 1. *Let* $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0,1) \times L^2(0,1)$ *and* $(\theta_0, q_0) \in L^2(0,1) \times L^2(0,1)$ *be given. Assume that* ***(H1)***–***(H4)*** *are satisfied, then problem (1)–(3) has a unique global (weak) solution satisfying*

$$\begin{array}{rcl} \varphi, \psi & \in & C\left(\mathbb{R}_+; H_0^1(0,1)\right) \cap C^1\left(\mathbb{R}_+; L^2(0,1)\right) \\ \theta, q & \in & C\left(\mathbb{R}_+; L^2(0,1)\right). \end{array}$$

3. Stability result

In this section, we show the uniform decay property of the solution of the system (1)−(3). In order to use the Poincaré inequality for θ, we introduce, as in [5],

$$\bar{\theta}(x,t) = \theta(x,t) - \int_0^1 \theta_0(x)dx.$$

Then, by the third equation in (1) we easily verify that

$$\int_0^1 \bar{\theta}(x,t)\,dx = 0,$$

for all $t \geq 0$. In this case the Poincaré inequality is applicable for $\bar{\theta}$. On the other hand, $(\varphi, \psi, \bar{\theta}, q)$ satisfies the same system (1) and the boundary conditions (3). So, in the sequel, we shall work with $\bar{\theta}$ but we write θ for simplicity.

The first-order energy, associated to (1)−(3), is then given by

$$\begin{array}{rcl} E\left(t, \varphi, \psi, \bar{\theta}, q\right) & = & \frac{1}{2}\int_0^1 \left\{\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \left(\bar{b} - a(x)\int_0^t g(s)\,ds\right)\psi_x^2\right\} dx \\ & & +\frac{1}{2}\int_0^1 \left\{K(\varphi_x + \psi)^2 + \rho_3\theta^2 + \tau_0 q^2\right\} dx + \frac{1}{2}(g \circ \psi_x). \end{array} \tag{12}$$

In what follows, we denote $E(t) = E\left(t, \varphi, \psi, \bar{\theta}, q\right)$ and $E(0) = E\left(0, \varphi_0, \psi_0, \bar{\theta}_0, q_0\right)$ for simplicity. The main result of this chapter is given by the following theorem:

Theorem 2. *Let* $(\varphi_0, \varphi_1), (\psi_0, \psi_1), \in H_0^1(0,1) \times L^2(0,1)$ *and* $(\theta_0, q_0) \in L^2(0,1) \times L^2(0,1)$ *be given. Assume that* ***(H1)***–***(H4)*** *are satisfied, then there exist positive constants* c', c'' *and* ε_0 *for which the (weak) solution of problem (1)–(3) satisfies*

$$E(t) \leq c'' H_1^{-1}\left(c' \int_0^t \xi(s)\,ds\right), \qquad \forall t \geq 0, \tag{13}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2}(s)\,ds$$

and

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } \left[0, \varepsilon'\right] \\ tH'(\varepsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on } \left(0, \varepsilon'\right] \end{cases} \tag{14}$$

and $\xi = 1$ *if* $a = 0$.

Remark 1. *The result of Theorem 2 holds true without any assumption on the wave speeds of the first two equations in (1).*

Remark 2. *The result of Theorem 2 is more general than the one obtained in (*[16] *Theorem 2). For $a = b = 0$, the result of Theorem 2 is the same as that in* [16, Theorem 2] .

To prove Theorem 2, we will use the energy method to produce a suitable Lyapunov functional. This will be established through several lemmas. A starting point is, as usual, the dissipativity inequality which states that the energy E of the entire system (1)-(3) is a non-increasing function. Of course this fact is a necessary preliminary step of stability analysis. More precisely, we have the following result:

Lemma 4. *Let $(\varphi, \psi, \theta, q)$ be the solution of (1)$-$(3), then the energy E is non-increasing function and satisfies, for all $t \geq 0$,*

$$\begin{aligned} \frac{dE(t)}{dt} &= -\delta \int_0^1 q^2 dx - \frac{1}{2} g\left(t\right) \int_0^1 a\left(x\right) \psi_x^2 dx - \int_0^1 b\left(x\right) \psi_t h\left(\psi_t\right) dx \\ &\quad + \frac{1}{2}\left(g' \circ \psi_x\right) - \mu \int_0^1 \varphi_t^2 dx, \\ &\leq -\delta \int_0^1 q^2 dx - \int_0^1 b\left(x\right) \psi_t h\left(\psi_t\right) dx + \frac{1}{2}\left(g' \circ \psi_x\right) - \mu \int_0^1 \varphi_t^2 dx \leq 0 \end{aligned} \tag{15}$$

Proof. Multiplying the first equation in (1) by φ_t, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_1 \varphi_t^2 dx + K \int_0^1 \varphi_{tx}\varphi_x dx + K \int_0^1 \varphi_{tx}\psi dx = -\mu \int_0^1 \varphi_t^2 dx. \tag{16}$$

Similarly, multiplying the second equation in (1) by ψ_t, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_2 \psi_t^2 dx + \bar{b}\int_0^1 \psi_x \psi_{tx} dx + \int_0^1 \psi_t \int_0^t g\left(t-s\right)\left(a\left(x\right)\psi_x\left(s\right)\right)_x ds dx \\ &+K \int_0^1 \psi_t \varphi_x dx + K \int_0^1 \psi_t \psi dx - \gamma \int_0^1 \psi_{tx}\theta dx \\ = &-\int_0^1 b\left(x\right)\psi_t h\left(\psi_t\right) dx. \end{aligned} \tag{17}$$

Also, multiplying the third equation in (1) by θ, we find

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_3 \theta^2 dx + \kappa \int_0^1 q_x \theta dx + \gamma \int_0^1 \psi_{tx}\theta dx = 0. \tag{18}$$

Finally, multiplying the fourth equation in (1) by q, we deduce

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \tau_0 q^2 dx - \kappa \int_0^1 \theta q_x dx = -\delta \int_0^1 q^2 dx. \tag{19}$$

Now, using Lemma 1, to handle the last term in first line of (17) and summing up (16)$-$(19), then (15) holds. □

Let us now define the functional I_1 as follows:

$$\begin{aligned} I_1\left(t\right) := &-\int_0^1 \rho_2 \alpha\left(x\right)\psi_t \int_0^t g\left(t-s\right)\left(\psi\left(t\right)-\psi\left(s\right)\right) ds dx \\ &+\frac{\gamma\tau_0}{\kappa}\int_0^1 \alpha\left(x\right) q \int_0^t g\left(t-s\right)\left(\psi\left(t\right)-\psi\left(s\right)\right) ds dx, \end{aligned}$$

for simplicity we write

$$I_1(t) := \chi_1(t) + \chi_2(t). \tag{20}$$

Then, we have the following result:

Lemma 5. *Let $(\varphi, \psi, \theta, q)$ be the solution of (1)–(3). Assume that* ***(H1)–(H4)*** *hold. Then we have, for any ε_1, $\varepsilon_1' > 0$,*

$$\begin{aligned} \frac{dI_1}{dt} \leq\ & -\left(\rho_2 \int_0^t g(s)\,ds - \varepsilon_1\left(\rho_2^2 + \int_0^t g(s)\,ds\right)\right)\int_0^1 \alpha(x)\,\psi_t^2 dx \\ & + \varepsilon_1' K^2 \int_0^1 (\varphi_x + \psi)^2 dx + \varepsilon_1 \int_0^1 b(x) h^2(\psi_t) dx \\ & + \varepsilon_1' \left(2\bar{b}^2 + 1\right)\int_0^1 \psi_x^2 dx + \left(c\varepsilon_1 + \frac{1}{\varepsilon_1}\int_0^t g(s)\,ds\right)\int_0^1 q^2 dx \\ & + c\left(\varepsilon_1' + \frac{1}{\varepsilon_1'}\right) g\circ\psi_x + c\left(\varepsilon_1 + \frac{1}{\varepsilon_1}\right) g\circ\psi_x - \frac{c}{\varepsilon_1} g'\circ\psi_x \end{aligned} \tag{21}$$

Proof. Differentiating χ_1 with respect to t to obtain

$$\begin{aligned} \chi_1'(t) =\ & -\int_0^1 \rho_2\alpha(x)\psi_{tt}\int_0^t g(t-s)\left(\psi(t) - \psi(s)\right) dsdx \\ & - \int_0^1 \rho_2 \alpha(x)\,\psi_t \int_0^t g'(t-s)\left(\psi(t) - \psi(s)\right) dsdx \\ & - \int_0^1 \rho_2\alpha(x)\,\psi_t^2 \int_0^t g(s)\,dsdx. \end{aligned} \tag{22}$$

Now, using the second equation in (1), we get

$$\begin{aligned} & -\int_0^1 \rho_2\alpha(x)\psi_{tt}\int_0^t g(t-s)\left(\psi(t) - \psi(s)\right) dsdx \\ & = \int_0^1 \bar{b}\alpha(x)\,\psi_x \int_0^t g(t-s)\left(\psi_x(t) - \psi_x(s)\right) dsdx \\ & + \int_0^t K\alpha(x)\left(\varphi_x + \psi\right)\int_0^t g(t-s)\left(\psi(t) - \psi(s)\right) dsdx \\ & - \int_0^1 \alpha(x)\,a(x)\left(\int_0^t g(t-s)\,\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)\left(\psi_x(t) - \psi_x(s)\right) ds\right) dx \\ & + \int_0^1 b(x)\,h(\psi_t)\left(\int_0^t g(t-s)\left(\psi(t) - \psi(s)\right) ds\right) dx \\ & + \int_0^1 \alpha(x)\,\gamma\theta_x\left(\int_0^t g(t-s)\left(\psi(t) - \psi(s)\right) ds\right) dx \\ & + \int_0^1 \alpha'(x)\left(\bar{b}\psi_x - a(x)\int_0^t g(s)\,\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)\left(\psi(t) - \psi(s)\right) ds\right) dx. \end{aligned} \tag{23}$$

Next, we will estimate the second term in the right-hand side of (22). So, by using Lemma 3, we have, for any $\varepsilon_1 > 0$

$$\begin{aligned} & -\int_0^1 \rho_2\alpha(x)\,\psi_t\int_0^t g'(t-s)\left(\psi(t) - \psi(s)\right) dsdx \\ & \leq \varepsilon_1 \rho_2^2 \int_0^1 \alpha(x)\,\psi_t^2 dx - \frac{c}{\varepsilon_1} g'\circ\psi_x. \end{aligned} \tag{24}$$

Also, as above we have

$$\begin{aligned}\chi_2'(t) &= \frac{\gamma\tau_0}{\kappa}\int_0^1 \alpha(x)\, q_t \int_0^t g(t-s)\,(\psi(t)-\psi(s))\,dsdx \\ &+\frac{\gamma\tau_0}{\kappa}\int_0^1 \alpha(x)\, q \int_0^t g'(t-s)\,(\psi(t)-\psi(s))\,dsdx \\ &+\frac{\gamma\tau_0}{\kappa}\int_0^1 \alpha(x)\, q\psi_t \int_0^t g(s)\,ds.\end{aligned}$$

Using the fourth equation in (1), we get

$$\begin{aligned}\chi_2'(t) &= -\frac{\gamma\delta}{\kappa}\int_0^1 \alpha(x)\, q \int_0^t g(t-s)\,(\psi(t)-\psi(s))\,dsdx \\ &-\int_0^1 \alpha(x)\,\gamma\theta_x \left(\int_0^t g(t-s)\,(\psi(t)-\psi(s))\,ds\right)dx \\ &+\frac{\gamma\tau_0}{\kappa}\int_0^1 \alpha(x)\, q \int_0^t g'(t-s)\,(\psi(t)-\psi(s))\,dsdx \\ &+\frac{\gamma\tau_0}{\kappa}\left(\int_0^t g(s)\,ds\right)\int_0^1 \alpha(x)\, q\psi_t dx.\end{aligned} \tag{25}$$

Similarly to (24), by exploiting Young's inequality, we estimate the terms in the right-hand side of (23) as follows:

$$\begin{aligned}&\int_0^1 \bar{b}\alpha(x)\,\psi_x \int_0^t g(t-s)\,(\psi_x(t)-\psi_x(s))\,dsdx \\ &\le \varepsilon_1'\bar{b}^2\int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_1'} g\circ\psi_x.\end{aligned} \tag{26}$$

Similarly,

$$\begin{aligned}&\int_0^t K\alpha(x)\,(\varphi_x+\psi)\int_0^t g(t-s)\,(\psi(t)-\psi(s))\,dsdx \\ &\le \varepsilon_1' K^2\int_0^1 (\varphi_x+\psi)^2\,dx + \frac{c}{\varepsilon_1'} g\circ\psi_x.\end{aligned} \tag{27}$$

By the same method used in [10], we have the following estimates:

$$\begin{aligned}&-\int_0^1 \alpha(x)\,a(x)\left(\int_0^t g(s)\,\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)\,(\psi_x(t)-\psi_x(s))\,ds\right)dx \\ &\le \varepsilon_1'\int_0^1 \psi_x^2 dx + c\left(\varepsilon_1' + \frac{1}{\varepsilon_1'}\right) g\circ\psi_x\end{aligned} \tag{28}$$

and

$$\begin{aligned}&\int_0^1 b(x)\,h(\psi_t)\left(\int_0^t g(t-s)\,(\psi(t)-\psi(s))\,ds\right)dx \\ &\le \varepsilon_1\int_0^1 b(x)h^2(\psi_t)dx + c\left(\varepsilon_1+\frac{1}{\varepsilon_1}\right) g\circ\psi_x.\end{aligned} \tag{29}$$

Finally,

$$\begin{aligned}&\int_0^1 \alpha'(x)\left(\bar{b}\psi_x - a(x)\int_0^t g(s)\,\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)\left(\psi(t)-\psi(s)\right)ds\right)dx\\&\leq \varepsilon_1'\bar{b}^2\int_0^1 \psi_x^2 dx + c\left(\varepsilon_1' + \frac{1}{\varepsilon_1'}\right) g\circ\psi_x.\end{aligned} \tag{30}$$

As in (24), it is easy to prove

$$\begin{aligned}&\frac{\gamma\tau_0}{\kappa}\int_0^1 \alpha(x)\, q\int_0^t g'(t-s)\left(\psi(t)-\psi(s)\right)dsdx\\&\leq \varepsilon_1\int_0^1 q^2 dx - \frac{c}{\varepsilon_1} g'\circ\psi_x.\end{aligned} \tag{31}$$

Also, we estimate the first term in the right-hand side of (25) as follows:

$$\begin{aligned}&-\frac{\gamma\delta}{\kappa}\int_0^1 \alpha(x)\, q\int_0^t g(t-s)\left(\psi(t)-\psi(s)\right)dsdx\\&\leq \left(\frac{\gamma\delta}{\kappa}\right)^2\varepsilon_1\int_0^1 q^2 dx + \frac{c}{\varepsilon_1} g\circ\psi_x\end{aligned} \tag{32}$$

and

$$\begin{aligned}&\frac{\gamma\tau_0}{\kappa}\left(\int_0^t g(s)\,ds\right)\int_0^1 \alpha(x)\, q\psi_t dx\\ \leq\ &\left(\int_0^t g(s)\,ds\right)\frac{1}{\varepsilon_1}\int_0^1 q^2 dx + \left(\int_0^t g(s)\,ds\right)c\varepsilon_1\int_0^1 \psi_t^2 dx.\end{aligned} \tag{33}$$

Consequently, by combining all the above estimates (22)−(33), the assertion of Lemma 5 is fulfilled. □

Now, as in [25], let w be the solution of

$$\begin{cases} -w_{xx} = \psi_x, \\ w(0) = w(1) = 0. \end{cases} \tag{34}$$

Then, we have the following inequalities:

Lemma 6. *The solution of (34) satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx$$

and

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

Proof. We multiply Equation (34) by w, integrate by parts and use the Cauchy-Schwarz inequality to obtain

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx.$$

Next, we differentiate (34) with respect to t and by the same procedure, we obtain

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

This completes the proof of Lemma 6. □

Let w be the solution of (34). We introduce the following functional:

$$I_2(t) := \int_0^1 \left(\rho_2 \psi_t \psi + \rho_1 \varphi_t w - \frac{\gamma \tau_0}{\kappa} \psi q\right) dx. \tag{35}$$

Then, we have the following estimate:

Lemma 7. *Let $(\varphi, \psi, \theta, q)$ be the solution of (1)–(3). Assume that **(H1)–(H4)** hold. Then we have, for any $\varepsilon_2 > 0$*

$$\begin{aligned} \frac{dI_2}{dt} \leq & -\left(\bar{b} + \frac{c\mu\varepsilon_2}{2} - 2c\varepsilon_2 - \frac{\delta\gamma\varepsilon_2}{2\kappa}\right)\int_0^1 \psi_x^2 dx + \left(\frac{\rho_1}{2\varepsilon_2} + \frac{\mu}{2\varepsilon_2}\right)\int_0^1 \varphi_t^2 dx \\ & + \left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2\kappa} + \frac{\rho_1\varepsilon_2}{2}\right)\int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon_2} g \circ \psi_x \\ & + \left(\frac{\gamma\tau_0}{2\kappa\varepsilon_2} + \frac{\delta\gamma}{2\kappa\varepsilon_2}\right)\int_0^1 q^2 dx + \frac{1}{2\varepsilon_2}\int_0^1 b(x) h^2(\psi_t) dx. \end{aligned} \tag{36}$$

Proof. By taking the derivative of I_2 with respect to t we get

$$\begin{aligned} I_2'(t) &= \int_0^1 \left(\rho_2 \psi_{tt} \psi + \rho_2 \psi_t^2\right) dx + \int_0^1 \left(\rho_1 \varphi_{tt} w + \rho_1 \varphi_t w_t\right) dx \\ & \quad - \frac{\gamma\tau_0}{\kappa}\int_0^1 (\psi_t q + \psi q_t) dx \\ &:= J_1 + J_2 + J_3. \end{aligned} \tag{37}$$

Next, using the first and the fourth equations in (1) we get

$$\begin{aligned} J_2 + J_3 &= -K\int_0^1 \varphi\psi_x dx + K\int_0^1 w_x^2 dx + \rho_1\int_0^1 \varphi_t w_t dx \\ & \quad - \frac{\gamma\tau_0}{\kappa}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa}\int_0^1 \psi q dx + \gamma\int_0^1 \psi\theta_x dx. \end{aligned} \tag{38}$$

Next, using the second equation in (1), we get

$$\begin{aligned} J_1 &= -\bar{b}\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx + \int_0^1 \psi_x \int_0^t g(t-s) a(x) \psi_x(s) ds dx \\ & \quad - K\int_0^1 \psi^2 dx - K\int_0^1 \varphi_x \psi dx - \int_0^1 b(x)\psi h(\psi_t) dx - \int_0^1 \gamma\psi\theta_x dx. \end{aligned} \tag{39}$$

From (38), (39) and by using Lemma 6, we deduce

$$\begin{aligned} I_2'(t) \leq & -\mu\int_0^1 \varphi_t w dx + \rho_1\int_0^1 \varphi_t w_t dx - \frac{\gamma\tau_0}{\kappa}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{\kappa}\int_0^1 \psi q dx \\ & - \bar{b}\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx - \int_0^1 b(x)\psi h(\psi_t) dx \\ & + \int_0^1 a(x)\psi_x \int_0^t g(t-s)\psi_x(s) ds dx. \end{aligned} \tag{40}$$

By exploiting the inequality

$$|ab| \leq \frac{\nu}{2} a^2 + \frac{1}{2\nu} b^2, \qquad a, b \in \mathbb{R},\ \nu > 0,$$

we easily find, for any $\varepsilon_2 > 0$,

$$\begin{aligned} I_2'(t) &\leq -\bar{b}\int_0^1 \psi_x^2 dx + \frac{\mu}{2}\int_0^1 \left(\frac{1}{\varepsilon_2}\varphi_t^2 + \varepsilon_2 w^2\right) + \frac{\rho_1}{2}\int_0^1 \left(\frac{1}{\varepsilon_2}\varphi_t^2 + \varepsilon_2 w_t^2\right) dx \\ &\quad + \frac{\gamma\tau_0}{2\kappa}\int_0^1 \left(\varepsilon_2\psi_t^2 + \frac{1}{\varepsilon_2}q^2\right) dx + \frac{\delta\gamma}{2\kappa}\int_0^1 \left(\varepsilon_2\psi^2 + \frac{1}{\varepsilon_2}q^2\right) dx \\ &\quad + \rho_2\int_0^1 \psi_t^2 dx - \int_0^1 b(x)\psi h(\psi_t)\, dx \qquad (41) \\ &\quad + \int_0^1 a(x)\psi_x \int_0^t g(t-s)\psi_x(s)\, ds dx. \end{aligned}$$

We now proceed to the evaluation of the last two terms in the right-hand side of (41). First, by Young's and Poicaré's inequalities we have

$$\left|\int_0^1 b(x)\psi h(\psi_t)\, dx\right| \leq \varepsilon_2 c\int_0^1 \psi_x^2 dx + \frac{1}{2\varepsilon_2}\int_0^1 b(x) h^2(\psi_t)\, dx. \qquad (42)$$

Furthermore, by Young's and Cauchy-Schwartz inequalities we have

$$\left|\int_0^1 a(x)\psi_x \int_0^t g(t-s)\psi_x(s)\, ds dx\right| \leq \varepsilon_2 c\int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_2} g\circ\psi_x. \qquad (43)$$

Then, plugging (42) and (43) into (41) and using the second inequality in Lemma 6, there fore the assertion of Lemma 7 holds. □

Now, following [16], we define the functional I_3 as follows:

$$I_3(t) := \int_0^1 \rho_1\varphi_t\varphi dx + \frac{\mu}{2}\int_0^1 \varphi^2 dx. \qquad (44)$$

Then, we have the following estimate:

Lemma 8. *Let $(\varphi, \psi, \theta, q)$ be the solution of (1)–(3). Then, for any $\varepsilon_3 > 0$, we have*

$$I_3'(t) \leq \left(\frac{K\varepsilon_3}{2} - K\right)\int_0^1 \varphi_x^2 dx + \frac{K}{2\varepsilon_3}\int_0^1 \psi_x^2 dx + \rho_1\int_0^1 \varphi_t^2 dx. \qquad (45)$$

Proof. By exploiting the first equation in (1) and using Young's inequality, we get

$$\begin{aligned} I_3'(t) &= \int_0^1 \rho_1\varphi_{tt}\varphi dx + \rho_1\int_0^1 \varphi_t^2 dx + \mu\int_0^1 \varphi_t\varphi dx \\ &= \int_0^1 K\varphi(\varphi_{xx} + \psi_x)\, dx + \rho_1\int_0^1 \varphi_t^2 dx \\ &= -K\int_0^1 \varphi_x^2 dx + K\int_0^1 \varphi\psi_x dx + \rho_1\int_0^1 \varphi_t^2 dx \\ &\leq -K\int_0^1 \varphi_x^2 dx + \frac{K}{2}\int_0^1 \left(\varepsilon_3\varphi^2 + \frac{1}{\varepsilon_3}\psi_x^2\right) dx + \rho_1\int_0^1 \varphi_t^2 dx. \end{aligned}$$

A simple use of Poincaré's inequality completes the proof of Lemma 8. □

Now, in order to obtain negative terms of $\int_0^1 \theta^2 dx$ we introduce the following functional:

$$I_4(t) := -\tau_0\rho_3\int_0^1 q(t,x)\left(\int_0^x \theta(t,y)dy\right) dx. \qquad (46)$$

Then we have the following estimate:

Lemma 9. *Let $(\varphi, \psi, \theta, q)$ be the solution of (1)–(3). Then, for any $\varepsilon_4 > 0$, we have*

$$\begin{aligned} I_4'(t) &\leq \left(-\rho_3\kappa + \frac{\varepsilon_4\rho_3\delta c}{2}\right)\int_0^1 \theta^2 dx + \frac{\varepsilon_4\tau_0\gamma}{2}\int_0^1 \psi_t^2 dx \\ &\quad + \left(\tau_0\kappa + \frac{\rho_3\delta}{2\varepsilon_4} + \frac{\tau_0\gamma}{2\varepsilon_4}\right)\int_0^1 q^2 dx. \end{aligned} \tag{47}$$

Proof. By using the fourth equation in (1), we get

$$\begin{aligned} I_4'(t) &= -\rho_3\int_0^1 \tau_0 q_t\left(\int_0^x \theta dy\right)dx - \tau_0\int_0^1 q\left(\int_0^x \rho_3\theta_t dy\right)dx \\ &= -\rho_3\int_0^1 (-\delta q - \kappa\theta_x)\left(\int_0^x \theta dy\right)dx - \tau_0\int_0^1 q\left(\int_0^x (-\kappa q_x - \gamma\psi_{tx})\,dy\right)dx \\ &= \rho_3\delta\int_0^1 q\left(\int_0^x \theta dy\right)dx + \rho_3\kappa\int_0^1 \theta_x\left(\int_0^x \theta dy\right)dx \\ &\quad +\tau_0\kappa\int_0^1 q\left(\int_0^x q_x dy\right)dx + \tau_0\gamma\int_0^1 q\left(\int_0^x \psi_{tx} dy\right)dx. \end{aligned}$$

That is

$$\begin{aligned} I_4'(t) &\leq \frac{\rho_3\delta}{2}\int_0^1\left(\varepsilon_4\left(\int_0^x \theta^2 dy\right)^2 + \frac{1}{\varepsilon_4}q^2\right)dx - \rho_3\kappa\int_0^1 \theta^2 dx \\ &\quad +\tau_0\kappa\int_0^1 q^2 dx + \frac{\tau_0\gamma}{2}\int_0^1\left(\varepsilon_4\psi_t^2 + \frac{1}{\varepsilon_4}q^2\right)dx. \end{aligned} \tag{48}$$

Consequently, the assertion of Lemma 9 immediately follows. □

Proof of Theorem 2. For $N, N_1, N_2 > 0$, we can define an auxiliary functional $\mathcal{F}$ by

$$\mathcal{F}(t) := NE(t) + N_1 I_1 + N_2 I_2 + I_3 + I_4 \tag{49}$$

and let $t_0 > 0$, and $g_0(t) = \int_0^t g(s)ds > 0$. By combining (15), (21), (36), (45) and (48), and by using the inequality

$$(\varphi_x + \psi)^2 \leq 2\varphi_x^2 + 2\psi^2$$

and Poincaré's inequality, we arrive at

$$\begin{aligned} \frac{d\mathcal{F}(t)}{dt} &\leq -N_1\left(\rho_2 g_0 - \varepsilon_1\left(\rho_2^2 + g_0\right)\right)\int_0^1 (\alpha(x) + b(x))\,\psi_t^2 dx \\ &\quad + \left(N_2\left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2\kappa} + \frac{\rho_1\varepsilon_2}{2}\right) + \frac{\tau_0\gamma\varepsilon_4}{2}\right)\int_0^1 \psi_t^2 dx - N\int_0^1 b(x)\,\psi_t h(\psi_t)\,dx \\ &\quad + \left(N_2\left(\frac{\rho_1}{2\varepsilon_2} + \frac{\mu}{2\varepsilon_2}\right) + \rho_1 - N\mu\right)\int_0^1 \varphi_t^2 dx + \left(N_1\varepsilon_1 + \frac{N_2}{2\varepsilon_2}\right)\int_0^1 b(x)h^2(\psi_t)dx \\ &\quad +N_1\left(\rho_2 g_0 - \varepsilon_1\left(\rho_2^2 + g_0\right)\right)\int_0^1 b(x)\,\psi_t^2 dx \\ &\quad + \left\{N_1\varepsilon_1'\left(2\bar{b}^2 + 1 + 2K^2\right) - N_2\left(\bar{b} - 2c\varepsilon_2 - \frac{\delta\gamma\varepsilon_2}{2\kappa}\right) + \frac{K}{2\varepsilon_3}\right\}\int_0^1 \psi_x^2 dx \\ &\quad + \left(2N_1\varepsilon_1' K^2 + \frac{K\varepsilon_3}{2} - K\right)\int_0^1 \varphi_x^2 dx + \left(-\rho_3\kappa + \frac{\varepsilon_4\rho_3\delta c}{2}\right)\int_0^1 \theta^2 dx \end{aligned}$$

$$
\begin{aligned}
&+\left\{cN_1\left(\varepsilon_1+\frac{1}{\varepsilon_1}\right)+cN_1\left(\varepsilon_1^{'}+\frac{1}{\varepsilon_1^{'}}\right)+\frac{N_2c}{\varepsilon_2}\right\}g\circ\psi_x+\left(\frac{N}{2}-\frac{cN_1}{\varepsilon_1}\right)g'\circ\psi_x\\
&+\left\{N_1\left(c\varepsilon_1+\frac{g_0}{\varepsilon_1}\right)+N_2\left(\frac{\gamma\tau_0}{2\kappa\varepsilon_2}+\frac{\delta\gamma}{2\kappa\varepsilon_2}\right)\right.\\
&\left.+\left(\tau_0\kappa+\frac{\rho_3\delta}{2\varepsilon_4}+\frac{\tau_0\gamma}{2\varepsilon_4}\right)-\delta N\right\}\int_0^1 q^2dx
\end{aligned}
$$

for all $t\geq t_0$. At this point, we have to choose our constants very carefully. First, let us take $\varepsilon_3<1$, $\varepsilon_1,\varepsilon_2$ and ε_4 small enough such that

$$
\begin{aligned}
\varepsilon_1 &\leq \min\left\{\left(\frac{\rho_2 g_0}{2}\right)/\left(\rho_2^2+g_0\right),\frac{1}{4K}\right\},\\
\varepsilon_2 &\leq \left(\frac{\bar{b}}{2}\right)/\left(2c+\frac{\delta\gamma}{2\kappa}\right)
\end{aligned}
$$

and

$$
\varepsilon_4 \leq \frac{\kappa}{\delta c}.
$$

After that, we pick N_2 large enough so that

$$
N_2\geq\frac{2K\bar{b}}{\varepsilon_3}.
$$

Now, by using Lemma 2, and choosing N_1 large enough such that

$$
\frac{N_1\rho_2 g_0}{2}>\left(N_2\left(\rho_2+\frac{\gamma\tau_0\varepsilon_2}{2k}+\frac{\rho_1\varepsilon_2}{2}\right)+\frac{\tau_0\gamma\varepsilon_4}{2}\right)\frac{2}{d}
$$

then, we can select $\varepsilon_1^{'}$ small enough such that

$$
\varepsilon_1^{'}\leq\min\left\{\frac{1}{4N_1K},\left(\frac{N_2\bar{b}}{4}\right)/N_1\left(2\bar{b}^2+1+2K^2\right)\right\}. \tag{50}
$$

Finally, we choose N large enough so that, there exist positive constants η, η_1, and η_2 such that, for $t\geq t_0$,

$$
\begin{aligned}
\frac{d\mathcal{F}(t)}{dt} \leq\ & -\eta\left\{\int_0^1\left(\alpha(x)+b(x)\right)\psi_t^2dx+\int_0^1\varphi_t^2dx\right.\\
&\left.+\int_0^1\theta^2dx+\int_0^1 q^2dx\right\}-\eta_1\int_0^1\psi_x^2dx-\eta_2\int_0^1\varphi_x^2dx\\
&+cg\circ\psi_x+c\int_0^1 b(x)\left(\psi_t^2+h^2(\psi_t)\right)dx.
\end{aligned}
$$

By the same method as in [16] (see inequality (25) in [16]), we can find $\eta_3>0$ such that, for $t\geq t_0$,

$$
\begin{aligned}
\frac{d\mathcal{F}(t)}{dt} \leq\ & -\eta_3\left\{\int_0^1\left(\alpha(x)+b(x)\right)\psi_t^2dx+\int_0^1\varphi_t^2dx+\int_0^1\psi_x^2dx\right.\\
&\left.+\int_0^1\left(\varphi_x+\psi\right)^2dx+\int_0^1\theta^2dx+\int_0^1 q^2dx\right\}\\
&+cg\circ\psi_x+c\int_0^1 b(x)\left(\psi_t^2+h^2(\psi_t)\right)dx.
\end{aligned} \tag{51}
$$

Moreover, we have the following: there exist two positive constants β_1 and β_2 depending on N, N_1, N_2, such that

$$\beta_1 E(t) \leq \mathcal{F}(t) \leq \beta_2 E(t), \qquad \forall t \geq 0. \tag{52}$$

This can be seen simply from estimate (15), (20), (35), (44), (47), (49), Young's and Poincaré's inequalities, that

$$|\mathcal{F}(t) - NE(t)| \leq CE(t), \qquad \forall t \geq 0.$$

Consequently, we can choose N large enough such that $\beta_1 = N - C > 0$ and (51) therefore (52) holds true. Our goal now is to estimate the last term in the right-hand side of (51). Following the method presented in [10], we consider the following partition of the interval $(0,1)$:

$$\Omega^+ = \left\{x \in (0,1) : |\psi_t| > \varepsilon^{'}\right\} \text{ and } \Omega^- = \left\{x \in (0,1) : |\psi_t| \leq \varepsilon^{'}\right\} \tag{53}$$

where $\varepsilon^{'}$ is defined in (H2). By using the hypothesis (H2) , we have $|\psi_t| \leq c_1^{-1}\psi_t h(\psi_t)$ on Ω^+ and therefore taking into account the estimate (15), we arrive at

$$\begin{aligned}\int_{\Omega^+} b(x)\left(\psi_t^2 + h^2(\psi_t)\right)dx &\leq c\int_{\Omega^+} b(x)\,\psi_t h(\psi_t)\,dx \\ &\leq c\int_0^1 b(x)\,\psi_t h(\psi_t)\,dx \\ &\leq -cE^{'}(t). \end{aligned}\tag{54}$$

According to (H2) , we distinguish two cases:

Case 1: H is linear on $\left[0, \varepsilon^{'}\right]$. Consequently, there exist two positive constants $c_1^{'}$ and $c_2^{'}$ such that $c_1^{'}|s| \leq |h(s)| \leq c_2^{'}|s|$, for all $s \in \mathbb{R}_+$, therefore the above inequality (54) holds on $(0,1)$. Now, from (51) and (54), we arrive at

$$\begin{aligned}\frac{d}{dt}\left(\mathcal{F}(t) + cE(t)\right) &\leq -cE(t) + cg \circ \psi_x \\ &= -cH_2(E(t)) + cg \circ \psi_x, \qquad \forall t \geq t_0, \end{aligned}\tag{55}$$

where the function H_2 is defined by (14).

Case 2: $H^{'}(0) = 0$ and $H^{''}(0) > 0$ on $\left[0, \varepsilon^{'}\right]$. Let H^* denote the dual of H in the sense of Young, then we have (see [10] for more details)

$$H^*(s) = s\left(H^{'}\right)^{-1}(s) - H\left[\left(H^{'}\right)^{-1}(s)\right], \qquad \forall s \in \mathbb{R}_+.$$

By using Jensen's inequality, we deduce

$$\begin{aligned}\int_{\Omega^-} b(x)\left(\psi_t^2 + h^2(\psi_t)\right)dx &\leq c\int_{\Omega^-} b(x)\,H^{-1}(\psi_t h(\psi_t))\,dx \\ &\leq c\int_{\Omega^-} H^{-1}(b(x)\,\psi_t h(\psi_t))\,dx \\ &\leq cH^{-1}\left(\int_{\Omega^-} b(x)\,\psi_t h(\psi_t)\,dx\right) \\ &\leq cH^{-1}\left(-cE^{'}(t)\right). \end{aligned}\tag{56}$$

Thus, it follows from (51), (54) and (56) that

$$\mathcal{F}^{'}(t) \leq -cE(t) + cH^{-1}\left(-cE^{'}(t)\right) - cE^{'}(t) + cg \circ \psi_x, \qquad \forall t \geq t_0.$$

By using Young's inequality and the fact that

$$H^*(s) \leq s(H')(s), \ E^{'}(t) \leq 0, \ H^{''} \geq 0,$$

we obtain by the same method as in [10] (we omit the details)

$$H'(\varepsilon_0 E(t))(\mathcal{F}'(t) + cE'(t) + c_0 E'(t)) \leq -cH_2(E(t)) + cg \circ \psi_x \tag{57}$$

where ε_0 is a small positive constant and c_0 is a large positive constant. Now, let us define the following functional:

$$\mathcal{L}(t) = \begin{cases} \mathcal{F}(t) + cE(t) & \text{if } H \text{ is linear on } \left[0, \varepsilon^{'}\right] \\ H^{'}(\varepsilon_0 E(t))(\mathcal{F}(t) + cE(t)) + c_0 E(t) & \text{if } H^{'}(0) = 0 \text{ and } H^{''} > 0 \text{ on } (0, \varepsilon^{'}]. \end{cases}$$

We can easily show that

$$\mathcal{L} \sim E.$$

On the other hand, by making use of (55) and (57), we easily deduce that the following inequality

$$\mathcal{L}^{'}(t) \leq -cH_2(E(t)) + cg \circ \psi_x$$

holds for all $t \geq t_0$. By using (15) and (H4), we obtain

$$\begin{aligned} (\xi(t)\mathcal{L}(t))^{'} &= \xi^{'}(t)\mathcal{L}(t) + \xi(t)\mathcal{L}^{'}(t) \\ &\leq -c\xi(t)H_2(E(t)) - cE^{'}(t). \end{aligned}$$

Next, let $\mathcal{K}(t) = \varepsilon(\xi(t)\mathcal{L}(t) + cE(t))$, where $0 < \varepsilon < \bar{\varepsilon}$ and $\bar{\varepsilon}$ is a positive constant satisfying

$$\xi(t)\mathcal{L}(t) + cE(t) \leq \frac{1}{\bar{\varepsilon}}E(t), \quad \forall t \geq 0.$$

We can also show that

$$\mathcal{K} \sim E$$

and, for $t \geq t_0$,

$$\mathcal{K}^{'}(t) \leq -c\varepsilon\xi(t)H_2(\mathcal{K}(t)).$$

A simple integration of the above inequality over (t_0, t) yields

$$\mathcal{K}(t) \leq H_1^{-1}\left(c\varepsilon\int_0^t \xi(s)\,ds + H_1(\mathcal{K}(t_0)) - c\varepsilon\int_0^{t_0}\xi(s)\,ds\right), \forall t \geq t_0,$$

where $H_1(t) = \int_t^1 \left(\frac{1}{H_2(t)}\right) ds$. Since $\lim_{t\to 0^+} H_1(t) = \infty$ and

$$0 \leq \mathcal{K}(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}}E(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}}E(0).$$

We may choose ε small enough such that

$$H_1(F(t_0)) - c\varepsilon\int_0^{t_0}\xi(s)\,ds \geq 0.$$

Therefore, $\mathcal{K}(t) \leq H_1^{-1}\left(c\varepsilon\int_0^t \xi(s)\,ds\right)$, for $t \geq t_0$. Consequently, there exist two positive constants $c^{'}$, and $c^{''}$ for which

$$\mathcal{K}(t) \leq c^{''}H_1^{-1}\left(c^{'}\int_0^t \xi(s)\,ds\right), \ \forall t \geq 0,$$

since $\mathcal{K}$ is bounded, which gives (13).
This completes the proof of the Theorem 2 $\square$

4. Examples

In this section, we give some examples to illustrate our results.

Example 1. *Let us first assume that the function h has a polynomial growth at the origin, i.e.*

$$c_1^{'} |s|^q \leq |h(s)| \leq c_2^{'} |s|^{\frac{1}{q}}, \qquad on \ \left[-\varepsilon^{'},\ \varepsilon^{'}\right]$$

where $c_1^{'}$ and $c_2^{'}$ are two positive constants and $q \geq 1$. As in [10], we obtain the following decay rate of the energy:

$$\begin{cases} E(t) \leq c^{''} e^{-c^{'} \int_0^t \xi(s)ds} & \text{if } q = 1 \\ E(t) \leq \left(c^{'} \int_0^t \xi(s)\, ds + c^{''}\right)^{-\frac{2}{q-1}} & \text{if } q > 1 \end{cases}. \tag{58}$$

In the next example, and from the general assumptions ($\boldsymbol{H}4$), we obtain several decay rates in which the exponential and polynomial rates are only particulary cases.

Example 2. *Here we consider some examples of the function g:*

• *Let $a, b, \nu > 0$,*

$$g(t) = ae^{-b(1+t)^{\nu}}.$$

then it's clear that ($\boldsymbol{H}4$) holds for $\xi(t) = b\nu(1+t)^{\min\{0,\ \nu-1\}}$. Consequently, applying the first inequality in (58), we obtain the following exponential decay:

$$E(t) \leq c^{''} e^{-c^{'} b(1+t)^{\min\{1,\ \nu\}}}.$$

•*If, for $a, b > 0$ and $\nu > 1$,*

$$g(t) = ae^{-b[\ln(1+t)]^{\nu}},$$

then, for

$$\xi(t) = \frac{b\nu\left[\ln(1+t)\right]^{\nu-1}}{1+t},$$

the first inequality in (58), gives

$$E(t) \leq c^{''} e^{-c^{'} b[\ln(1+t)]^{\nu}}.$$

• *If*

$$g(t) = \frac{a}{(2+t)^{\nu}\left[\ln(2+t)\right]^{b}},$$

where

$$a > 0 \text{ and } \begin{cases} \nu > 1 & \text{and } b \in \mathbb{R} \\ & \text{or} \\ \nu = 1 & \text{and } b > 1 \end{cases}.$$

Therefor

$$\xi(t) = \frac{\nu(\ln(2+t)) + b}{(2+t)\left[\ln(2+t)\right]^{b}},$$

and we obtain from the first inequality in (58)

$$E(t) \leq \frac{c^{''}}{\left[(2+t)^{\nu}\left[\ln(2+t)\right]^{b}\right]^{c^{'}}}.$$

Example 3. *Let us now suppose that the function h has an exponential growth at the origin, i.e.*

$$h_0\left(|s|\right) \leq |h\left(s\right)| \leq h_0^{-1}\left(|s|\right), \qquad on\ \left[-\varepsilon^{'},\ \varepsilon^{'}\right]$$

where $h_0 = (1/s)\, e^{-s^{-\gamma_1}}$ and $\gamma_1 > 0$. Then we get the same decay rate of [10], i.e.

$$E\left(t\right) \leq c^{'''}\left(\ln\left(c^{'}\int_0^t \xi\left(s\right) ds + c^{''}\right)\right)^{-2/\gamma_1}.$$

Remark 3. *We can also prove the same decay results for the following boundary conditions:*

$$\varphi_x\left(0,\ t\right) = \varphi_x\left(1,\ t\right) = \psi\left(0,t\right) = \psi\left(1,t\right) = q\left(0,t\right) = q\left(1,t\right) = 0.$$

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