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HARVESTING OF STAGE STRUCTURED FISHERY MODEL IN THE PRESENCE OF TOXICITY

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Abstract. In this paper, we have considered a stage structured prey-predator model in the presence of toxicity and harvesting.We have assumed that only mature predators can catch the prey and the immature predators depend on the mature predators for their food. We have discussed the existence of the equilibrium points, boundedness,local stability of the equilibrium points and also the global stability using Lyapunov function. Further, bioeconomic equilibrium has also been analysed for different cases.

1. Introduction

In recent times, due to increase in human population, demand for food has increased many fold which has led to increase in harvesting of marine life. The problem related to harvesting started with the study of authors[6, 7]. The effect of harvesting of species like fisheries were also studied by authors[5, 6, 10, 19, 20].
Another major factor which is affecting the marine life is pollution from various sources like industries etc. The great amount of toxicant enter into environment which effect the earth's ecosystem. In context of aquatic animals, contaminants like materials including plastic, paper, glass and chemicals which are released by some external sources like factories and industries directly affect the marine bio-diversity. The effect of toxicant on multi species fisheries by mathematical modelling has been studied by scholars[11, 12, 3, 21, 8, 13, 14]. [18] discussed the effect of toxicity in two multispecies fisheries by considering that the toxic part may affect the one species in the presence of other. When a toxic waste harms one organism, it can end up destroying an entire food chain of aquatic life. Further, [13] discussed the mathematical model of harvesting of two competing fish which are exploited. For example, inside Marine Reserves, the aquatic species such as the Snapper fish and the spiny lobster extinguished because of continuous increase in demands of human needs. But,in modern era, researchers are taking interest in the stage structured model of species as in real world, all population have stage structure. The dynamical phenomena of stage structured species model were studied by many authors [1, 9, 22, 16, 17, 10, 2].

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Keeping in view the above discussion, we have formulated a prey-predator fishery model with stage structure under the effect of harvesting and toxicity. For the simplicity of our model, we have only considered, stage-structure of predator species. We have assumed that the plankton denoted by X is logistically growing in the absence of consumer(fish). We have also assumed that the fish population is divided into two stage groups: juvenile and adult and are denoted by Y and Z respectively with the assumption that only adult fish are cable of preying on prey species and the juvenile predator live on their parents as they are weaker than their parents to catch the prey.
The paper is organized as follows: In section 2, we give the formulation of harvesting of stage structured predator-prey fishery model in the presence of toxicity. The existence of equilibrium points and local stability of boundary equilibrium points of the models have been discussed in section 3. In section 4, we have discussed the global stability by considering the Lyapunov function. Lastly, in section 5, the bionomic equilibrium has been analysed with the help of different cases.

2. Mathematical Model

In this section, we discuss the mathematical model describing the stage structured predator- prey model where each fishery species is effected by toxic material which is released by some external sources like factories, industries, etc. It is assumed that the prey population is growing logistically.

$$\frac{dx_1}{dt} = r_1x_1(1-\frac{x_1}{L}) - \gamma_1{x_1}^2 - \beta x_1x_3 \tag{1}$$

$$\frac{dx_2}{dt} = \alpha x_3 - \mu x_2 - r_2x_2 - \gamma_2x_2 - \delta x_2 \tag{2}$$

$$\frac{dx_3}{dt} = \theta\beta x_1x_3 - \mu x_3 - r_3x_3 - \gamma_3x_3 + \delta x_2 \tag{3}$$

where,
$x_1 = x_1(t)$ is the size of prey species at time t,
$x_2 = x_2(t)$ is the size of immature predator species at time t,
$x_3 = x_3(t)$ is the size of mature predator species at time t,
r_1 is the specific growth rate of prey species,
r_2 is the relative rate at which immature predators die out in the absence of prey,
r_3 is the relative rate at which mature predators die out in the absence of prey,
β is the rate of interaction of prey with mature predator,
α is the growth rate of immature predator because of mature predator,
μ is the death rate of predator species,
θ is the conversion rate from prey to predator,
δ is the conversion rate from immature predator to mature predator.
γ_1 is the coefficients of toxicity to the prey species.
γ_2 is the coefficients of toxicity to the immature predator.
γ_3 is the coefficients of toxicity to the mature predator.

We assume that prey species is affected by predator population if the predator population is sufficient to catch the prey species. In the Lotka-Volterra prey-predator model, the prey population is growing logistically at the rate r_1 with carrying capacity L in the absences of predator species. We consider the first term of prey species as $r_1x_1(1-\frac{x_1}{L})$ in the form of Lotka Volterra form.

The mature and immature predators have distinct growth rates r_2 and r_3 in the model. In the absence of prey population, both mature and immature predator population will decline at a rate μ. The prey species is consumed by mature predator at the rate of β. We suppose that the mature predator attack the prey at the rate of βx_1. Further, we consider different harvesting rates of each species rather than the same harvesting rate. Therefore, the system of equation can be rewritten as:

$$\frac{dx_1}{dt} = r_1x_1(1-\frac{x_1}{L}) - C_1E_1x_1 - \gamma_1{x_1}^2 - \beta x_1x_3 \quad (4)$$

$$\frac{dx_2}{dt} = \alpha x_3 - \mu x_2 - C_2E_2x_2 - r_2x_2 - \gamma_2x_2 - \delta x_2 \quad (5)$$

$$\frac{dx_3}{dt} = \theta\beta x_1x_3 - \mu x_3 - C_2E_3x_3 - r_3x_3 - \gamma_3x_3 + \delta x_2 \quad (6)$$

where C_1 and C_2 are the catching capability coefficients of the prey and predator species respectively.E_1, E_2 and E_3 are the harvesting effort rate of prey, mature predator and immature predator species.

The term $\gamma_1x_1^2$ measures the effect of toxicity on the prey species. Since, $\frac{d(\gamma_1x_1^2)}{dx_1} = 2\gamma_1x_1 > 0$ and $\frac{d^2(\gamma_1x_1^2)}{dx_1^2} = 2\gamma_1 > 0$ showing the accelerated growth in the production of toxic substance of the prey species and more of the prey species consume the toxic substance.

3. Boundedness

In this section, we have shown the boundedness of the system by the following theorem.

Theorem The system will be bounded in the following set:
$\{(x_1,x_2,x_3) : S \le x_1+x_2+x_3 \le T\}$ where $S = \frac{F}{(C+D)}$ and $T = \frac{r_1L}{A-(\alpha+\theta\beta L)}$

Proof. :Let$V = x_1 + x_2 + x_3$. Then,

$$\frac{dV}{dt} = r_1x_1(1-\frac{x_1}{L}) - C_1E_1x_1 - \gamma_1{x_1}^2 - \beta x_1x_3 + \alpha x_3 - \mu x_2 - C_2E_2x_2 - r_2x_2 - \gamma_2x_2 - \delta x_2 + \theta\beta x_1x_3 - \mu x_3 - C_2E_3x_3 - r_3x_3 - \gamma_3x_3 + \delta x_2$$

$$\frac{dV}{dt} \le r_1x_1 - C_1E_1x_1 + \alpha x_3 - C_2E_2x_2 - \delta x_2 + \theta\beta x_1x_3 - C_2E_3x_3$$

$$\frac{dV}{dt} \le r_1L - AV + (\alpha+\theta\beta L)x_3$$

Then, $V \le \frac{r_1L}{A-(\alpha+\theta\beta L)}$, where $A = min(C_1E_1, C_2E_2, C_2E_3)$

Now consider,

$$\frac{dx_1}{dt} = r_1x_1(1-\frac{x_1}{L}) - C_1E_1x_1 - \gamma_1{x_1}^2 - \beta x_1x_3$$

$$\frac{dx_1}{dt} \geq r_1x_1 - \frac{r_1F}{L}x_1 - C_1E_1F - r_1F^2 - \beta F^2$$

where $F = \frac{r_1L}{A-(\alpha+\theta\beta L)}$

$$\frac{dx_1}{dt} \geq r_1x_1 - \frac{r_1F}{L}x_1 - P$$

where $P = C_1E_1F + r_1F^2 + \beta F^2$

$$\frac{dx_1}{dt} + (\frac{r_1F}{L} - r_1)x_1 \geq P$$

which implies, $x_1 \geq \frac{P}{(\frac{r_1F}{L}-r_1)}$ as $t \to \infty$

Now let us consider, $M = x_1 + x_2 + x_3$

$$\frac{dM}{dt} = r_1x_1(1 - \frac{x_1}{L}) - C_1E_1x_1 - \gamma_1{x_1}^2 - \beta x_1x_3 + \alpha x_3 - \mu x_2 - C_2E_2x_2 - r_2x_2 - \gamma_2x_2 - \delta x_2 + \theta\beta x_1x_3 - \mu x_3 - C_2E_3x_3 - r_3x_3 - \gamma_3x_3 + \delta x_2$$

$$\frac{dM}{dt} \geq r_1x_1 - \frac{r_1x_1^2}{L} - CM - DM - \beta x_1x_3 - \mu(x_2 + x_3) - r_2x_2 - r_3x_3$$

where $C = max(C_1E_1, C_2E_2, C_2E_3)$ and $D = max(\gamma_1, \gamma_2, \gamma_3)$

$$\frac{dM}{dt} \geq r_1\frac{P}{(\frac{r_1F}{L} - r_1)} - r_1L - (C+D)M - \beta(\frac{r_1L}{A-(\alpha+\theta\beta L)})^2 - 2\mu\frac{r_1L}{A-(\alpha+\theta\beta L)} - r_2\frac{r_1L}{A-(\alpha+\theta\beta L)} - r_3\frac{r_1L}{A-(\alpha+\theta\beta L)}$$

$$\frac{dM}{dt} \geq H - (C+D)M$$

where $H = r_1\frac{P}{(\frac{r_1F}{L}-r_1)} - r_1L - \beta(\frac{r_1L}{A-(\alpha+\theta\beta L)})^2 - 2\mu\frac{r_1L}{A-(\alpha+\theta\beta L)} - r_2\frac{r_1L}{A-(\alpha+\theta\beta L)} - r_3\frac{r_1L}{A-(\alpha+\theta\beta L)}$

$$\frac{dM}{dt} + (C+D)M \geq H$$

$M \to \frac{H}{(C+D)}$ as $t \to \infty$

Hence, the system is bounded. □

In the next section, we will discuss the existence of the equilibrium points.

4. Existence of Equilibrium Points

In this section, we obtain the existence of four equilibrium points. **theorem** The system has four equilibrium points.

The steady state solutions are obtained from the following system of equation:

$$r_1x_1(1 - \frac{x_1}{L}) - C_1E_1x_1 - \gamma_1{x_1}^2 - \beta x_1x_3 = 0 \quad (7)$$

$$\alpha x_3 - \mu x_2 - C_2E_2x_2 - r_2x_2 - \gamma_2x_2 - \delta x_2 = 0 \quad (8)$$

$$\theta\beta x_1x_3 - \mu x_3 - C_2E_3x_3 - r_3x_3 - \gamma_3x_3 + \delta x_2 = 0 \quad (9)$$

The four possible equilibrium points are $E_0(0,0,0)$, $E_1(\bar{x_1},0,0)$ in which the predator species extinct,$E_2(x_1^{'},0,x_3^{'})$ in which the immature predator extinct,the interior equilibrium point $E_3(x_1{}^*,x_2{}^*,x_3{}^*)$. The state in which the prey species exist and predator species extinct is given by:

$$\bar{x_1} = \frac{r_1 - C_1E_1}{\frac{r_1}{L} + \gamma_1} \tag{10}$$

which is positive if $\frac{r_1}{C_1} > E_1$
i.e.,the ratio of growth rate to the catching coefficient is greater than the the harvesting rate of prey.
The state in which both prey and mature predator species exist and immature predator extinct is given by:

$$x_1^{'} = \frac{\mu + C_2E_3 + r_3 + \gamma_3}{\theta\beta} \tag{11}$$

$$x_3^{'} = \frac{r_1(1 - \frac{x_)^{'}}{L}) - C_1E_1 - \gamma_1 x_1^{'}}{\beta} \tag{12}$$

which is positive if $r_1(1 - \frac{x_1^{'}}{L}) > C_1E_1 + \gamma_1 x_1^{'}$ The state in which prey species, mature and immature predator species exist is obtained by solving the equation (7-9) and we get

$$x_1{}^* = \frac{[\mu + C_2E_3 + r_3 + \gamma_3][\mu + C_2E_2 + r_2 + \gamma_2 + \delta] - \delta\alpha}{[\theta\beta][\mu + C_2E_2 + r_2 + \gamma_2 + \delta]} \tag{13}$$

which exists if $[\mu + C_2E_3 + r_3 + \gamma_3][\mu + C_2E_2 + r_2 + \gamma_2 + \delta] > \delta\alpha$

$$x_3{}^* = \frac{(1 - \frac{x_1}{L}) - C_1E_1 - \gamma_1 x_1}{\beta} \tag{14}$$

exists if $(1 - \frac{x_1}{L}) > C_1E_1 + \gamma_1 x_1$

$$x_2{}^* = \frac{\alpha x_3^*}{\mu + C_2E_2 + r_2 + \gamma_2 + \delta} \tag{15}$$

5. Local Stability of Equilibrium Points

In this section, we discuss the local stability of the equilibrium points: The general jacobian matrix for the system of equation (4-6) is:

$$\begin{bmatrix} -(\frac{r_1}{L} + \gamma_1)(x_1) & 0 & -\beta x_1 \\ 0 & -[\mu + C_2E_2 + r_2 + \gamma_2 + \delta] & \alpha \\ \theta\beta x_3 & \delta & [\theta\beta x_1 - \mu - C_2E_3 - r_3 - \gamma_3] \end{bmatrix} \tag{16}$$

The eigen values corresponding to $E_0(0,0,0)$ are $-(\mu + C_2E_2 + r_2 + \gamma_2 + \delta)$, $(\mu + C_2E_3 + r_3 + \gamma_3)$ and 0. Hence, the equilibrium point $E_0(0,0,0)$ is stable.

The eigen values corresponding to $E_1(\bar{x_1},0,0)$ are $-(\frac{r_1}{L} + \gamma)\bar{x_1}$, $-(\mu + C_2E_2 + C_2E_3 + r_2 + r_3 + \gamma_2 + \gamma_2 + \delta)$ and $(\theta\beta\bar{x_1} - \mu - C_2E_3 - r_3)$. Thus $E_1(\bar{x_1},0,0)$ is stable if $\bar{x_1} < \frac{\mu+C_2E_3+r_3+\gamma_3}{\theta\beta}$ and unstable if $\bar{x_1} > \frac{\mu+C_2E_3+r_3+\gamma_3}{\theta\beta}$.

The characteristic equation corresponding to $E_1(x_1^{'}, 0, x_3^{'})$ is given by
$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$

where $a_1 = [(\mu + C_2E_2 + r_2 + \gamma_2 + \delta) + (\frac{r_1}{L} + \gamma_1)x_1^{'}]$,
$a_2 = -[\alpha\delta + (\mu + C_2E_2 + r_2 + \gamma_2 + \delta)(\frac{r_1}{L} + \gamma_1)x_1^{'} + \theta\beta^2 x_1^{'} x_3^{'}]$,
$a_3 = -[\alpha\delta(\frac{r_1}{L} + \gamma_1)x_1^{'} + \theta\beta^2 x_1^{'} x_3^{*}(\mu + C_2E_2 + r_2 + \gamma_2 + \delta)]$

Since, $a_1 > 0, a_2 < 0, a_1a_2 - a_3 < 0$. By Routh Hurwitz Stability Criterion, the equilibrium point $E_1(x_1^{'}, 0, x_3^{'})$ is not stable.

The characteristic equation corresponding to $E_3(x_1{}^*, x_2{}^*, x_3{}^*)$ is given by
$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$
where $a_1 = [\alpha\frac{x_3^*}{x_2^*} + \delta\frac{x_2^*}{x_3^*} - (\frac{r_1}{L} + \gamma_1)x_1^*]$
$a_2 = [-\theta\beta_2 x_1^* x_3^* - \frac{\alpha(\frac{r_1}{L} + \gamma_1)x_1^* x_3^*}{x_2^*} + \alpha\delta - \frac{\delta(\frac{r_1}{L} + \gamma_1)x_1^* x_2^*}{x_3^*}]$
$a_3 = [\frac{\theta\beta_2 x_1^*(x_3^*)^2}{x_2^*} + (\frac{r_1}{L} + \gamma_1)x_1^* + \alpha\delta(\frac{r_1}{L} + \gamma_1)x_1^*]$
$a_1 > 0$ if $\alpha\frac{x_3^*}{x_2^*} + \delta\frac{x_2^*}{x_3^*} > (\frac{r_1}{L} + \gamma_1)x_1^*$,
$a_2 > 0$ if $\alpha\delta > \theta\beta_2 x_1^* x_3^* + \frac{\alpha(\frac{r_1}{L} + \gamma_1)x_1^* x_3^*}{x_2^*} + \frac{\delta(\frac{r_1}{L} + \gamma_1)x_1^* x_2^*}{x_3^*}, a_1a_2 - a_3 > 0$.

Thus, by Routh Hurwitz Stability Criterion, the equilibrium point $E_3(x_1{}^*, x_2{}^*, x_3{}^*)$ is stable if it satisfies the above condition. Hence, $E_3(x_1{}^*, x_2{}^*, x_3{}^*)$ is locally asymptotically stable. Thus, we can state the following theorem. **theorem**

(1) The boundary equilibrium point $E_0(0, 0, 0)$ is stable but not asymptotically stable.
(2) The boundary equilibrium point $E_1(x_1, 0, 0)$ is locally asymptotically stable if $\bar{x_1} < \frac{\mu+C_2E_3+r_3+\gamma_3}{\theta\beta}$ and unstable if $\bar{x_1} > \frac{\mu+C_2E_3+r_3+\gamma_3}{\theta\beta}$.
(3) The boundary equilibrium point $E_2(x_1, 0, x_3)$ is not stable as $a_1 > 0, a_2 < 0, a_1a_2 - a_3 < 0$.
(4) The interior equilibrium point $E_3(x_1, x_2, x_3)$ is stable as $a_1 > 0, a_2 > 0, a_1a_2 - a_3 > 0$, where,
$a_1 = [\alpha\frac{x_3^*}{x_2^*} + \delta\frac{x_2^*}{x_3^*} - (\frac{r_1}{L} + \gamma_1)x_1^*]$
$a_2 = [-\theta\beta_2 x_1^* x_3^* - \frac{\alpha(\frac{r_1}{L} + \gamma_1)x_1^* x_3^*}{x_2^*} + \alpha\delta - \frac{\delta(\frac{r_1}{L} + \gamma_1)x_1^* x_2^*}{x_3^*}]$
$a_3 = [\frac{\theta\beta_2 x_1^*(x_3^*)^2}{x_2^*} + (\frac{r_1}{L} + \gamma_1)x_1^* + \alpha\delta(\frac{r_1}{L} + \gamma_1)x_1^*]$

6. Global Stability

In this section, we discuss global stability of $E_3(x_1{}^*, x_2{}^*, x_3{}^*)$.
We construct a Lyapunov function:

$$R(x_1,x_2,x_3) = [(x_1-x_1^*)-x_1^*\log\frac{x_1}{x_1^*}]+s_1[(x_2-x_2^*)-x_2^*\log\frac{x_2}{x_2^*}]+s_2[(x_3-x_3^*)-x_3^*\log\frac{x_3}{x_3^*}] \tag{17}$$

where, s_1 and s_2 are constants.

$$\frac{dR}{dt} = (1-\frac{x_1^*}{x_1})\frac{dx_1}{dt} + s_1(1-\frac{x_2^*}{x_2})\frac{dx_2}{dt} + s_2(1-\frac{x_3^*}{x_3})\frac{dx_3}{dt}$$
$$= (x_1-x_1^*)[r_1(1-\frac{x_1}{L}) - C_1E_1 - \gamma_1 - \beta x_3] + s_1(x_2-x_2^*)[\alpha\frac{x_3}{x_2} - \mu - C_2E_2 - r_2 - \gamma_2 - \delta] + s_2(x_3-x_3^*)[\theta\beta x_1 - \mu - C_2E_3 - r_3 - \gamma_3 + \delta\frac{x_2}{x_3}]$$
$$= -\frac{r_1}{L}(x_1-x_1^*)^2 - \gamma_1(x_1-x_1^*)^2 - (x_1-x_1^*)(x_3-x_3^*)(\beta-\theta\beta s_2) - \alpha s_1(x_2-x_2^*)(\frac{x_3^*}{x_2^*} - \frac{x_3}{x_2}) - \delta s_2(x_3-x_3^*)(\frac{x_2^*}{x_3^*} - \frac{x_2}{x_3})$$

We choose $s_2 = \frac{1}{\theta}$

$$= -\frac{r_1}{L}(x_1-x_1^*)^2 - \gamma_1(x_1-x_1^*)^2 - \alpha s_1(x_2-x_2^*)(\frac{x_3^*}{x_2^*} - \frac{x_3}{x_2}) - \frac{\delta}{\theta}(x_3-x_3^*)(\frac{x_2^*}{x_3^*} - \frac{x_2}{x_3})$$
$$= -\frac{r_1}{L}(x_1-x_1^*)^2 - \gamma_1(x_1-x_1^*)^2 - \alpha s_1(x_2-x_2^*)(\frac{x_3^*}{x_2^*}) + \alpha s_1(x_2-x_2^*)(\frac{x_3}{x_2}) - \frac{\delta}{\theta}(x_3-x_3^*)(\frac{x_2^*}{x_3^*}) - \frac{\delta}{\theta}(x_3-x_3^*)\frac{x_2}{x_3} - \alpha s_1(x_2-x_2^*)\frac{x_3^*}{x_2} - \frac{\delta}{\theta}(x_3-x_3^*)\frac{x_2^*}{x_3} + \alpha s_1(x_2-x_2^*)\frac{x_3^*}{x_2} + \frac{\delta}{\theta}(x_3-x_3^*)\frac{x_2^*}{x_3}$$
$$= -[\frac{r_1}{L}(x_1-x_1^*)^2 + \gamma_1(x_1-x_1^*)^2 + \alpha s_1(x_2-x_2^*)(\frac{x_3^*}{x_2^*}) + \frac{\delta}{\theta}(x_3-x_3^*)(\frac{x_2^*}{x_3^*}) + \alpha s_1(x_2-x_2^*)(\frac{x_3^*}{x_2} - \frac{x_3}{x_2}) + \frac{\delta}{\theta}(x_3-x_3^*)(\frac{x_2^*}{x_3} - \frac{x_2}{x_3}) - \alpha s_1(x_2-x_2^*)(\frac{x_3^*}{x_2}) - \frac{\delta}{\theta}(x_3-x_3^*)(\frac{x_2^*}{x_3})]$$
$$\leq -[\frac{r_1}{L}(x_1-x_1^*)^2 + \gamma_1(x_1-x_1^*)^2 + \alpha s_1(x_2-x_2^*)(\frac{x_3^*}{x_2^*}) + \frac{\delta}{\theta}(x_3-x_3^*)(\frac{x_2^*}{x_3^*}) + \alpha s_1(x_2-x_2^*)(\frac{x_3^*}{x_2} - \frac{x_3}{x_2}) + \frac{\delta}{\theta}(x_3-x_3^*)(\frac{x_2^*}{x_3} - \frac{x_2}{x_3}) - \alpha s_1(x_2-x_2^*)\frac{x_3^*}{\frac{H}{(C+D)}} - \frac{\delta}{\theta}(x_3-x_3^*)(\frac{x_2^*}{\frac{H}{(C+D)}})]$$

Thus, $\frac{dR}{dt} < 0$.

Hence, equilibrium point $E_3(x_1{}^*, x_2{}^*, x_3{}^*)$ is globally asymptotically stable.

Now, in the next section, we will be discussing about bionomic equilibrium point and will discuss different cases.

7. Bionomic Equilibrium

Let q_1 be the constant harvesting cost per unit effort for prey species, q_2 be the constant harvesting cost per unit effort for immature predator and q_3 be the constant harvesting cost per unit effort for mature predator. Also, let p_1, p_2 and p_3 be the price per unit biomass of the prey, immature predator and mature predator respectively.

So, the economic rent or net revenue at any time is given by

$$\pi(x_1, x_2, x_3, E_1, E_2, E_3) = (p_1C_1x_1 - q_1)E_1 + (p_2C_2x_2 - q_2)E_2 + (p_3C_2x_3 - q_3)E_3 \tag{18}$$

The Bionomic equilibrium $((x_1)_\infty, (x_2)_\infty, (x_3)_\infty, (E_1)_\infty, (E_2)_\infty, (E_3)_\infty)$ is obtained by solving the following equations:

$$r_1x_1(1 - \frac{x_1}{L}) - C_1E_1x_1 - \gamma_1{x_1}^2 - \beta x_1x_3 = 0 \tag{19}$$

$$\alpha x_3 - \mu x_2 - C_2E_2x_2 - r_2x_2 - \gamma_2x_2 - \delta x_2 = 0 \tag{20}$$

$$\theta\beta x_1x_3 - \mu x_3 - C_2E_3x_3 - r_3x_3 - \gamma_3x_3 + \delta x_2 = 0 \tag{21}$$

$$\pi(x_1, x_2, x_3, E_1, E_2, E_3) = (p_1C_1x_1 - q_1)E_1 + (p_2C_2x_2 - q_2)E_2 + (p_3C_2x_3 - q_3)E_3 = 0 \tag{22}$$

Here, equation (22) refer as Zero Profit Line. Now, in order to find the bionomic equilibrium $((x_1)_\infty, (x_2)_\infty, (x_3)_\infty, (E_1)_\infty, (E_2)_\infty, (E_3)_\infty)$, we have to consider the following case:

Case 1: $E_1 > 0$, $E_2 = 0$ and $E_3 = 0$
The harvesting cost of the both mature and immature predator are greater than the revenue (i.e., $q_2 > p_2C_2x_2$ and $q_3 > p_3C_2x_3$). So, the harvesting of both mature and immature predator will not be beneficial and hence, it will be terminated. Therefore, harvesting of prey population remains functional (i.e., $q_1 < p_1C_1x_1$). From equation (22) , we get

$$(x_1)_\infty = \frac{q_1}{p_1C_1} \tag{23}$$

Substituting the value of $(x_1)_\infty$ in equation(11), we get

$$(x_3)_\infty = \frac{r_1 - r_1\frac{(x_1)_\infty}{L} - C_1E_1 - \gamma_1(x_1)_\infty}{\beta} \tag{24}$$

$$(x_3)_\infty = \frac{r_1 - (\frac{r_1}{L})(\frac{q_1}{p_1C_1}) - C_1E_1 - (\frac{q_1}{p_1C_1})\gamma_1}{\beta} \tag{25}$$

which exists if $r_1 > (\frac{r_1}{L})(\frac{q_1}{p_1C_1}) + C_1E_1 + (\frac{q_1}{p_1C_1})\gamma_1$
Now, put the value of $(x_3)_\infty$ in equation (12), we get

$$(x_2)_\infty = \frac{\alpha(x_3)_\infty}{\mu + r_2 + \gamma_2 + \delta} \tag{26}$$

Now, put the value of $(x_1)_\infty$ and $(x_3)_\infty$ in equation (11), we get

$$(E_1)_\infty = \frac{r_1(1 - \frac{(x_1)_\infty}{L}) - \gamma_1(x_1)_\infty - \beta(x_1)_\infty(x_3)_\infty}{C_1} \tag{27}$$

exists if $r_1(1 - \frac{(x_1)_\infty}{L}) > \gamma_1(x_1)_\infty + \beta(x_1)_\infty(x_3)_\infty$

Case 2: $E_1 = 0$, $E_2 > 0$ and $E_3 = 0$
The harvesting cost of the both prey and mature predator population are greater than the revenue (i.e., $q_1 > p_1C_1x_1$ and $q_3 > p_3C_2x_3$. So, the harvesting of both prey and mature predator population will not be beneficial and hence, it will be

terminated. Therefore, harvesting of immature predator population remains functional(i.e., $q_2 < p_2C_2x_2$).
from equation (22), we get

$$(x_2)_\infty = \frac{q_2}{p_2C_2} \tag{28}$$

Substituting the value of $(x_2)_\infty$ in equation(12), we get

$$(x_3)_\infty = \frac{(\mu + C_2E_2 + r_2 + \gamma_2 + \delta)(x_2)_\infty}{\alpha} \tag{29}$$

Now, put the value of $(x_2)_\infty$ and $(x_3)_\infty$ in equation (13), we get

$$(x_1)_\infty = \frac{\mu(x_3)_\infty + r_3(x_3)_\infty + \gamma_3(x_3)_\infty - \delta(x_2)_\infty}{\theta\beta(x_3)_\infty} \tag{30}$$

exists if $\mu(x_3)_\infty + (r_3)_\infty + \gamma_3(x_3)_\infty > \delta(x_2)_\infty$
Put the value of $(x_2)_\infty$ and $(x_3)_\infty$ in the equation (12), we get

$$(E_2)_\infty = \frac{\alpha(x_3)_\infty - \mu(x_2)_\infty - r_2(x_2)_\infty - \gamma_2(x_2)_\infty - \delta(x_2)_\infty}{C_2(x_2)_\infty} \tag{31}$$

exists if $\alpha(x_3)_\infty > \mu(x_2)_\infty + r_2(x_2)_\infty + \gamma_2(x_2)_\infty + \delta(x_2)_\infty$

Case 3: $E_1 = 0$, $E_2 = 0$ and $E_3 > 0$
The harvesting cost of the both prey and immature predator population are greater than the revenue (i.e., $q_1 > p_1C_1x_1$ and $q_2 > p_2C_2x_2$. So, the harvesting of both prey and immature predator population will not be beneficial and hence, it will be terminated. Therefore, harvesting of mature predator population remains functional(i.e.,$q_3 < p_3C_2x_3$). from equation (22), we get

$$(x_3)_\infty = \frac{q_3}{p_3C_2} \tag{32}$$

Substituting the value of $(x_3)_\infty$ in equation (12), we get

$$(x_2)_\infty = \frac{\alpha(x_3)_\infty}{\mu + r_2 + \gamma_2 + \delta} \tag{33}$$

Now, put the value of $(x_2)_\infty$ and $x_3)_\infty$ in equation (13), we get

$$(x_1)_\infty = \frac{(\mu + C_2E_3 + r_3 + \gamma_3)(x_3)_\infty - \delta(x_2)_\infty}{\theta\beta(x_3)_\infty} \tag{34}$$

exists if $(\mu + C_2E_3 + r_3 + \gamma_3)(x_3)_\infty > \delta(x_2)_\infty$
Put the value of $(x_1)_\infty$, $(x_2)_\infty$ and $(x_3)_\infty$ in equation (13), we get

$$(E_3)_\infty = \frac{\theta\beta(x_1)_\infty(x_3)_\infty - \mu(x_3)_\infty - r_3(x_3)_\infty - \gamma_3(x_3)_\infty + \delta(x_2)_\infty}{C_2(x_3)_\infty} \tag{35}$$

exists if $\theta\beta(x_1)_\infty(x_3)_\infty + \delta(x_2)_\infty > \mu(x_3)_\infty + r_3(x_3)_\infty + \gamma_3(x_3)_\infty$

Case 4: $E_1 = 0$, $E_2 = 0$ and $E_3 = 0$
If the harvesting cost of the prey, mature predator and immature predator population are greater than the revenue (i.e., $q_1 > p_1C_1x_1$, $q_2 > p_2C_2x_2$ and $q_3 > p_3C_2x_3$), then the harvesting of prey, mature predator and immature predator population will not be beneficial and hence, it will be terminated.

Case 5: $E_1 = 0$, $E_2 > 0$ and $E_3 > 0$
If the harvesting cost of the prey is greater than the revenue (i.e., $q_1 > p_1C_1x_1$), then the harvesting of prey will not be beneficial and hence, it will be terminated. Therefore, the harvesting of mature and immature predator population remains functional(i.e., $q_2 < p_2C_2x_2$ and $q_3 < p_3C_2x_3$).
from equation (22), we get

$$(x_2)_\infty = \frac{q_2}{p_2C_2} \tag{36}$$

and

$$(x_3)_\infty = \frac{q_3}{p_3C_2} \tag{37}$$

Now, put the value of $(x_2)_\infty$ and $(x_3)_\infty$ in equation (13), we get

$$(x_1)_\infty = (\frac{\mu + C_2E_3 + r_3 + \gamma_3}{\theta\beta})(\frac{p_3C_2}{q_3}) - \frac{\delta}{\theta\beta}(\frac{q_3p_2}{q_2p_3}) \tag{38}$$

exists if $(\frac{\mu + C_2E_3 + r_3 + \gamma_3}{\theta\beta})(\frac{p_3C_2}{q_3}) > \frac{\delta}{\theta\beta}(\frac{q_3p_2}{q_2p_3})$
Put the value of $(x_1)_\infty$, $(x_2)_\infty$ and $(x_3)_\infty$ in equation (13), we get

$$(E_3)_\infty = \frac{\theta\beta(x_1)_\infty - \mu - r_3 - \gamma_3}{C_2} + (\frac{\delta}{C_2})(\frac{q_2p_3}{q_3p_2}) \tag{39}$$

exists if $\frac{\theta\beta(x_1)_\infty - \mu - r_3 - \gamma_3}{C_2} + (\frac{\delta}{C_2})(\frac{q_2p_3}{q_3p_2}) > 0$
Put the value of $(x_2)_\infty$ and $(x_3)_\infty$ in the equation (12), we get

$$(E_2)_\infty = (\frac{\alpha}{C_2})(\frac{q_3p_2}{q_2p_3}) - (\frac{\mu + r_2 + \gamma_2 + \delta}{C_2}) \tag{40}$$

exists if $(\frac{\alpha}{C_2})(\frac{q_3p_2}{q_2p_3}) > (\frac{\mu + r_2 + \gamma_2 + \delta}{C_2})$

Case 6: $E_1 > 0$, $E_2 = 0$ and $E_3 > 0$
If the harvesting cost of the immature predator is greater than the revenue (i.e., $q_2 > p_2C_2x_2$), then the harvesting of immature predator will not be beneficial and hence, it will be terminated. Therefore, the harvesting of prey and mature predator population remains functional(i.e., $q_1 < p_1C_1x_1$ and $q_3 < p_3C_2x_3$).
from equation (22), we get

$$(x_1)_\infty = \frac{q_1}{p_1C_1} \tag{41}$$

and

$$(x_3)_\infty = \frac{q_3}{p_3C_2} \tag{42}$$

Substituting the value of equation $(x_3)_\infty$ in equation (12), we get

$$(x_2)_\infty = \frac{\alpha q_3}{(\mu + r_2 + \gamma_2 + \delta)(p_3C_2)} \tag{43}$$

Now, put the value of $(x_1)_\infty$ and $(x_3)_\infty$ in equation (11), we get

$$(E_1)_\infty = r_1(1 - \frac{q_1}{p_1C_1L}) - \gamma_1(\frac{q_1}{p_1C_1}) - \beta(\frac{q_1q_3}{p_1p_3C_1^2C_2}) \tag{44}$$

exists if $r_1(1-\frac{q_1}{p_1C_1L}) > \gamma_1(\frac{q_1}{p_1C_1}) + \beta(\frac{q_1q_3}{p_1p_3C_1^2C_2})$
Put the value of $(x_1)_\infty$, $(x_2)_\infty$ and $(x_3)_\infty$ in equation (13), we get

$$(E_3)_\infty = \frac{\theta\beta q_1}{p_1C_1C_2} - \frac{\mu}{C_2} - \frac{r_3}{C_2} - \frac{\gamma_3}{C_2} + \delta\frac{p_3C_2(x_2)_\infty}{q_3} \tag{45}$$

exists if $\frac{\theta\beta q_1}{p_1C_1C_2} + \delta\frac{p_3C_2(x_2)_\infty}{q_3} > \frac{\mu}{C_2} + \frac{r_3}{C_2} + \frac{\gamma_3}{C_2}$

Case 7: $E_1 > 0$, $E_2 > 0$ and $E_3 = 0$
If the harvesting cost of the mature predator is greater than the revenue (i.e., $q_3 > p_3C_2x_3$), then the harvesting of mature predator will not be beneficial and hence, it will be terminated. Therefore, the harvesting of prey and immature predator population remains functional(i.e., $q_1 < p_1C_1x_1$ and $q_2 < p_2C_2x_2$).
from equation (22), we get

$$(x_1)_\infty = \frac{q_1}{p_1C_1} \tag{46}$$

and

$$(x_2)_\infty = \frac{q_2}{p_2C_2} \tag{47}$$

Substituting the value of equation $(x_2)_\infty$ in equation(12), we get

$$(x_3)_\infty = \frac{(\mu + C_2E_2 + r_2 + \gamma_2 + \delta)(q_2)}{\alpha p_2C_2} \tag{48}$$

Now, put the value of $(x_1)_\infty$ and $(x_3)_\infty$ in equation (11), we get

$$(E_1)_\infty = r_1(1-\frac{q_1}{p_1C_1L}) - \gamma_1(\frac{q_1}{p_1C_1}) - \beta(x_3)_\infty(\frac{q_1}{p_1C_1^2}) \tag{49}$$

exists if $r_1(1-\frac{q_1}{p_1C_1L}) > \gamma_1(\frac{q_1}{p_1C_1}) + \beta(x_3)_\infty(\frac{q_1}{p_1C_1^2})$
Put the value of $(x_2)_\infty$ and $(x_3)_\infty$ in the equation (12), we get

$$(E_2)_\infty = \frac{(\alpha p_2C_2)(x_3)_\infty}{C_2q_2} - \frac{\mu}{C_2} - \frac{r_2}{C_2} - \frac{\gamma_2}{C_2} - \frac{\delta}{C_2} \tag{50}$$

exists if $\frac{(\alpha p_2C_2)(x_3)_\infty}{C_2q_2} > \frac{\mu}{C_2} + \frac{r_2}{C_2} + \frac{\gamma_2}{C_2} + \frac{\delta}{C_2}$

Case 8: $E_1 > 0$, $E_2 > 0$ and $E_3 > 0$
The harvesting of prey , mature predator and immature predator population remains functional(i.e., $q_1 < p_1C_1x_1$, $q_2 < p_2C_2x_2$ and $q_3 < p_3C_3x_3$)
From equation (22), we get

$$(x_1)_\infty = \frac{q_1}{p_1C_1} \tag{51}$$

$$(x_2)_\infty = \frac{q_2}{p_2C_2} \tag{52}$$

and

$$(x_3)_\infty = \frac{q_3}{p_3C_2} \tag{53}$$

Now, put the value of $(x_1)_\infty$ and $(x_3)_\infty$ in equation (11), we get

$$(E_1)_\infty = r_1(1-\frac{q_1}{p_1C_1L}) - \gamma_1(\frac{q_1}{p_1C_1}) - \beta(\frac{q_1q_3}{p_1p_3C_1^2C_2}) \tag{54}$$

exists if $r_1(1-\frac{q_1}{p_1C_1L}) > \gamma_1(\frac{q_1}{p_1C_1}) + \beta(\frac{q_1q_3}{p_1p_3C_1^2C_2})$
Put the value of $(x_2)_\infty$ and $(x_3)_\infty$ in the equation (12), we get

$$(E_2)_\infty = \frac{\alpha q_3 p_2}{q_2 p_3 C_2} - \frac{\mu}{C_2} - \frac{r_2}{C_2} - \frac{\gamma_2}{C_2} - \frac{\delta}{C_2} \tag{55}$$

exists if $\frac{\alpha q_3 p_2}{q_2 p_3 C_2} > \frac{\mu}{C_2} + \frac{r_2}{C_2} + \frac{\gamma_2}{C_2} + \frac{\delta}{C_2}$
Put the value of $(x_1)_\infty$, $(x_2)_\infty$ and $(x_3)_\infty$ in equation (13), we get

$$(E_3)_\infty = \frac{\theta\beta q_1}{p_1C_1C_2} - \frac{\mu}{C_2} - \frac{r_3}{C_2} - \frac{\gamma_3}{C_2} + \frac{\delta p_3 q_2}{p_2 C_2 q_3} \tag{56}$$

exists if $\frac{\theta\beta q_1}{p_1C_1C_2} + \frac{\delta p_3 q_2}{p_2 C_2 q_3} > \frac{\mu}{C_2} + \frac{r_3}{C_2} + \frac{\gamma_3}{C_2}$

8. Conclusion

In this paper, we have studied the combined effect of harvesting and toxicity on stage structured predator-prey system.We have obtained the boundedness of the system and existence of boundary and interior equilibrium points. We have also obtained the local stability of boundary and interior equilibrium points .It is observed that E_0 is always stable, E_1 is stable if $x_1 < x_1^*$ and unstable if $x_1 > x_1^*$, E_2 is always unstable and E^* is locally asymptotically stavle.We have proved the global behavior of the system constructing a suitable Lyapunov function. Further, we have also obtained bionomic equilibrium points and discussed different cases when the prey and predator fish population should be harvested. The harvesting is functional or not functional has been discussed as follows:

- **Case 1:** The harvesting of prey population remains functional.
- **Case 2:** The harvesting of immature predator population remains functional.
- **Case 3:** The harvesting of mature predator population remains functional.
- **Case 4:** The harvesting of prey, mature predator and immature predator population will not be beneficial and hence, it will be terminated.
- **Case 5:** The harvesting of mature and immature predator population remains functional.
- **Case 6:** The harvesting of prey and mature predator population remains functional.
- **Case 7:** The harvesting of prey and immature predator population remains functional.
- **Case 8:** The harvesting of prey , mature predator and immature predator population remains functional.

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