DHAGE ITERATION METHOD FOR INITIAL VALUE PROBLEMS OF NONLINEAR SECOND ORDER HYBRID FUNCTIONAL DIFFERENTIAL EQUATIONS

SHYAM B. DHAGE AND ASHOK D. KADAM

ABSTRACT. In this paper we prove the existence and uniqueness results for approximate solution of an initial value problem of second order nonlinear functional differential equations via construction of an algorithm. The main results rely on the Dhage iteration method embodied in a recent hybrid fixed point principle of Dhage (2014) in a partially ordered normed linear space. Examples are also furnished to illustrate the hypotheses and the abstract results of this paper.

1. STATEMENT OF THE PROBLEM

Given the real numbers $r > 0$ and $T > 0$, consider the closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, T]$ in $\mathbb{R}$ and let $J = [-r, T]$. By $\mathcal{C} = C(I_0, \mathbb{R})$ we denote the space of continuous real-valued functions defined on I_0. We equip the space $\mathcal{C}$ with he norm $\| \cdot \|_{\mathcal{C}}$ defined by

$$\|x\|_{\mathcal{C}} = \sup_{-r \leq \theta \leq 0} |x(\theta)|. \tag{1}$$

Clearly, $\mathcal{C}$ is a Banach space with this supremum norm and it is called the history space of the functional differential equation in question.

For any continuous function $x : J \to \mathbb{R}$ and for any $t \in I$, we denote by x_t the element of the space $\mathcal{C}$ defined by

$$x_t(\theta) = x(t + \theta), \; -r \leq \theta \leq 0. \tag{2}$$

Differential equations involving the history of the dynamic systems are called functional differential equations and it has been recognized long back the importance of such problems in the theory of differential equations. Since then, several classes of nonlinear functional differential equations have been discussed in the literature for different qualitative properties of the solutions. A special class of functional differential equations has been discussed in Dhage [8, 9, 12], Dhage and Dhage [14] and Dhage and Dhage [15] for the existence and approximation of solutions via a new Dhage iteration method. Very recently the Dhage iteration method

is successfully applied to first order hybrid functional differential equation of delay type by Dhage [12, 13]. Therefore, it is desirable to extend this method to other functional differential equations involving delay. The present paper is also an attempt in this direction.

In this paper, we consider the nonlinear second order functional differential equation (in short FDE)

$$\left. \begin{array}{l} x''(t) = f(t, x_t), \ t \in I, \\ x_0 = \phi, \ x'(0) = \eta, \end{array} \right\} \qquad (3)$$

where $\phi \in \mathcal{C}$ and $f : I \times \mathcal{C} \to \mathbb{R}$ is a continuous function.

Definition 1.1. *A function $x \in C^2(J, \mathbb{R})$ is said to be a solution* solution *of the FDE (3) on J if*

(i) $x_0 = \phi, \ x'(0) = \eta,$

(ii) $x_t \in \mathcal{C}$ *for each $t \in I$, and*

(iii) x *is twice continuously differentiable on I and satisfies the equation in (3),*

where $C^2(J, \mathbb{R})$ is the space of twice continuously differentiable real-valued functions defined on J.

The FDE (3) is well-known and extensively discussed in the literature for different aspects of the solutions. See Hale [18], Ntouyas [20, 21] and the references therein. There is a vast literature on nonlinear functional differential equations for different aspects of the solutions via different approaches and methods. The method of upper and lower solution or monotone method is interesting and well-known, however it requires the existence of both the lower as well as upper solutions as well as certain inequality involving monotonicity of the nonlinearity. In this paper we prove the existence of solution for FDE (3) via Dhage iteration method which does not require the existence of both upper and lower solution as well as the related monotonic inequality and also obtain the algorithm for the solutions. The novelty of the present paper lies in its method which is completely new in the field of functional differential equations and yields the monotonic successive approximations for the solutions under some well-known natural conditions.

The rest of the paper is organized as follows. Section 2 deals with the preliminary definitions and auxiliary results that will be used in subsequent sections of the paper. The main results are given in Sections 3 and 4. Illustrative examples are also furnished at the end of each section.

2. Auxiliary Results

Throughout this paper, unless otherwise mentioned, let $(E, \preceq, \| \cdot \|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Guo and Lakshmikatham [17] and the references therein. Similarly a few details of a partially ordered normed linear space are given in Dhage [4] while orderings defined

by different order cones are given in Deimling [1], Guo and Lakshmikantham [17], and the references therein.

We need the following definitions (see Dhage [4, 5, 6] and the references therein) in what follows.

Definition 2.1. *A mapping $\mathcal{T} : E \to E$ is called* **isotone** *or* **nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called* **nonincreasing** *if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called* **monotonic** *or simply* **monotone** *if it is either nondecreasing or nonincreasing on E.*

Definition 2.2. *A mapping $\mathcal{T} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice-versa.*

Definition 2.3. *A non-empty subset S of the partially ordered Banach space E is called* **partially bounded** *if every chain C in S is bounded. An operator $\mathcal{T}$ on a partially normed linear space E into itself is called* **partially bounded** *if $\mathcal{T}(E)$ is a partially bounded subset of E. $\mathcal{T}$ is called* **uniformly partially bounded** *if all chains C in $\mathcal{T}(E)$ are bounded by a unique constant.*

Definition 2.4. *A non-empty subset S of the partially ordered Banach space E is called* **partially compact** *if every chain C in S is a relatively compact subset of E. A mapping $\mathcal{T} : E \to E$ is called* **partially compact** *if $\mathcal{T}(E)$ is a partially relatively compact subset of E. $\mathcal{T}$ is called* **uniformly partially compact** *if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called* **partially totally bounded** *if for any bounded subset S of E, $\mathcal{T}(S)$ is a partially relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Remark 2.1. *Suppose that $\mathcal{T}$ is a nondecreasing operator on E into itself. Then $\mathcal{T}$ is a partially bounded or partially compact if $\mathcal{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.*

Definition 2.5. *The order relation $\preceq$ and the metric d on a non-empty set E are said to be $\mathcal{D}$-***compatible** *if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be $\mathcal{D}$-compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are $\mathcal{D}$-compatible. A subset S of E is called* **Janhavi** *if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are $\mathcal{D}$-compatible in it. In particular, if $S = E$, then E is called a* **Janhavi metric** *or* **Janhavi Banach space***.*

Definition 2.6. *An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. An operator $\mathcal{T} : E \to E$ is called partially nonlinear $\mathcal{D}$-contraction if there exists a $\mathcal{D}$-function ψ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \le \psi\big(\|x - y\|\big) \tag{4}$$

for all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a partial Lipschitz operator with a Lischitz constant

k and moreover, if $0 < k < 1$, $\mathcal{T}$ is called a *partial linear contraction* on E with a contraction constant k.

Remark 2.2. *Note that every partial nonlinear contraction mapping $\mathcal{T}$ on a partially ordered normed linear space E into itself is partially continuous but the converse may not be true.*

The **Dhage iteration method** embodied in the following applicable hybrid fixed point theorem of Dhage [5] in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of other hybrid fixed point theorems involving the **Dhage iteration principle** and method are given in Dhage [5, 6, 7], Dhage *et.al.* [16] and the references therein.

Theorem 2.1 (Dhage [5, 6]). *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that every compact chain C in E is Janhavi. Let $\mathcal{T} : E \rightarrow E$ be a partially continuous, nondecreasing and partially compact operator. If there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $\mathcal{T}x_0 \preceq x_0$, then the operator equation $\mathcal{T}x = x$ has a solution x^* in E and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Theorem 2.2 (Dhage [5, 6]). *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered Banach space and let $\mathcal{T} : E \rightarrow E$ be a nondecreasing and partially nonlinear $\mathcal{D}$-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$. If $\mathcal{T}$ is continuous or E is regular, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower and an upper bound.*

Remark 2.3. *The regularity of E in above Theorem 2.1 may be replaced with a stronger continuity condition of the operator $\mathcal{T}$ on E which is a result proved in Dhage [4].*

Remark 2.4. *The condition that every compact chain of E is Janhavi holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in it. This simple fact is used to prove the main existence results of this paper.*

3. Main Results

In this section, we prove existence and approximation results for the FDE (3) on a closed and bounded interval $J = [-r, T]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the FDE (3) in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{5}$$

and

$$x \leq y \iff x(t) \leq y(t) \quad \text{for all} \ \ t \in J. \tag{6}$$

Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it. See Dhage [4, 5, 6] and references therein. The following useful lemma concerning the Janhavi subsets of $C(J, \mathbb{R})$ follows immediately from the Arzelá-Ascoli theorem for compactness.

Lemma 3.1. *Let $\big(C(J,\mathbb{R}), \leq, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (5) and (6) respectively. Then every partially compact subset of $C(J,\mathbb{R})$ is Janhavi.*

Proof. The proof of the lemma is well-known and appears in the papers of Dhage [6], Dhage and Dhage [14] and so we omit the details. □

We introduce an order relation $\leq_{\mathcal{C}}$ in $\mathcal{C}$ induced by the order relation $\leq$ defined in $C(J,\mathbb{R})$. This will avoid the confusion of comparison between the elements of two Banach spaces $\mathcal{C}$ and $\mathbb{C}(J,\mathbb{R})$. Thus, for any $x, y \in \mathcal{C}$, $x \leq_{\mathcal{C}} y$ implies $x(\theta) \leq y(\theta)$ for all $\theta \in I_0$. Note that if $x, y \in C(J,\mathbb{R})$ and $x \leq y$, then $x_t \leq_{\mathcal{C}} y_t$ for all $t \in I$.

We need the following definition in what follows.

Definition 3.1. *A twice differentiable function $u \in C^2(J,\mathbb{R})$ is said to be a lower solution of the FDE (3) if u is twice continuously differentiable on I and satisfies the inequalities*

$$\left. \begin{aligned} u''(t) &\leq & f(t, u_t), \ \ t \in I, \\ u_0 &\leq_{\mathcal{C}} & \phi, \ u'(0) \leq \eta. \end{aligned} \right\} \tag{$*$}$$

Similarly, a twice differentiable function $v \in C^2(J,\mathbb{R})$ is called an upper solution of the FDE (3) if the above inequalities are satisfied with reverse sign.

We consider the following set of assumptions in what follows:

(H$_1$) There exists a constant $M_f > 0$ such that $|f(t,x)| \leq M_f$ for all $t \in I$ and $x \in \mathcal{C}$.

(H$_2$) $f(t,x)$ is nondecreasing in x for each $t \in I$.

(H$_3$) There exists $\mathcal{D}$-function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$0 \leq f(t,x) - f(t,y) \leq \varphi(\|x - y\|_{\mathcal{C}})$$

for all $t \in I$ and $x, y \in \mathcal{C}$, $x \geq_{\mathcal{C}} y$.

(H$_4$) FDE (3) has a lower solution $u \in C^2(J,\mathbb{R})$.

Lemma 3.2. *A function $x \in C(J,\mathbb{R})$ is a solution of the FDE (3) if and only if it is a solution of the nonlinear integral equation*

$$x(t) = \begin{cases} \phi(0) + \eta t + \displaystyle\int_0^t (t - s) f(s, x_s)\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{7}$$

Theorem 3.1. *Suppose that hypotheses (H$_1$), (H$_2$) and (H$_4$) hold. Then the FDE (3) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_0 = u,$$

$$x_{n+1}(t) = \begin{cases} \phi(0) + \eta t + \displaystyle\int_0^t (t - s) f(s, x_s^n)\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \tag{8}$$

where $x_s^n(\theta) = x_n(s + \theta)$, $\theta \in I_0$, converges monotonically to x^*.

Proof. Set $E = C(J, \mathbb{R})$. Then, in view of Lemma 3.1, every compact chain C in E possesses the compatibility property with respect to the norm $\| \cdot \|$ and the order relation $\leq$ so that every compact chain C is Janhavi in E.

Define an operator $\mathcal{T}$ on E by

$$\mathcal{T}x(t) = \begin{cases} \phi(0) + \eta t + \int_0^t (t-s) f(s, x_s) \, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{9}$$

From the continuity of the integral, it follows that $\mathcal{T}$ defines the operator $\mathcal{T} : E \to E$. Applying Lemma 3.2, the FDE (3) is equivalent to the operator equation

$$\mathcal{T}x(t) = x(t), \ t \in J.$$

Now, we show that the operators $\mathcal{T}$ satisfies all the conditions of Theorem 2.1 in a series of following steps.

Step I: $\mathcal{T}$ *is nondecreasing on* E.

Let $x, y \in E$ be such that $x \geq y$. Then $x_t \geq_C y_t$ for all $t \in I$ and by hypothesis (H_2), we get

$$\mathcal{T}x(t) = \begin{cases} \phi(0) + t\eta + \int_0^t (t-s) f(s, x_s) \, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

$$\geq \begin{cases} \phi(0) + t\eta + \int_0^t (t-s) f(s, y_s) \, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

$$= \mathcal{T}y(t),$$

for all $t \in J$. This shows that the operator $\mathcal{T}$ is also nondecreasing on E.

Step II: $\mathcal{T}$ *is partially continuous on* E.

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$ as $n \to \infty$. Then $x_s^n \to x_s$ as $n \to \infty$. Since f is continuous, we have

$$\lim_{n \to \infty} \mathcal{T}x_n(t) = \begin{cases} \phi(0) + t\eta + \int_0^t \left[\lim_{n \to \infty} (t-s) f(s, x_s^n) \right] ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

$$= \begin{cases} \phi(0) + t\eta + \int_0^t (t-s) f(s, x_s) \, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

$$= \mathcal{T}x(t),$$

for all $t \in J$. This shows that $\mathcal{T}x_n$ converges to $\mathcal{T}x$ pointwise on J.

Now we show that $\{\mathcal{T}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. There are three cases:

Case a): Let $t_1, t_2 \in J$ with $t_1 > t_2 \geq 0$. Then we have

$$|\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)|$$

$$= \left| (t_2 - t_1)\eta + \int_0^{t_2} (t_2 - s)f\left(s, x_s^n\right) ds - \int_0^{t_1} (t_1 - s)f\left(s, x_s^n\right) ds \right|$$

$$= \left| (t_2 - t_1)\eta + \int_0^{t_2} (t_2 - t_1)f\left(s, x_s^n\right) ds - \int_{t_2}^{t_1} (t_1 - s)f\left(s, x_s^n\right) ds \right|$$

$$\leq |t_2 - t_1|\,|\eta| + M_f \int_0^{t_2} ||t_2 - t_1| ds + M_f \int_{t_2}^{t_1} |t_1 - s| ds$$

$$\to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$.

Case b): Let $t_1, t_2 \in J$ with $t_2 < t_1 \leq 0$. Then we have

$$|\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| = |\phi(t_2) - \phi(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$.

Case c): Let $t_1, t_2 \in J$ with $t_2 < 0 < t_1$. Then we have

$$|\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1.$$

Thus in all three cases, we obtain

$$|\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{T}x_n \to \mathcal{T}x$ is uniform and that $\mathcal{T}$ is a partially continuous operator on E into itself in view of Remark 2.1.

Step III: $\mathcal{T}$ *is partially compact operator on* E.

Let C be an arbitrary chain in E. We show that $\mathcal{T}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathcal{T}(C)$ is uniformly bounded. Let $y \in \mathcal{T}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{T}x$. By hypothesis (H_1)

$$|y(t)| = |\mathcal{T}x(t)|$$

$$\leq \begin{cases} |\phi(0)| + T|\eta| + T\displaystyle\int_0^t |f(s, x_s)|\, ds, & \text{if } t \in I, \\[2mm] |\phi(t)|, & \text{if } t \in I_0. \end{cases}$$

$$\leq \|\phi\| + T|\eta| + M_f T^2 = r,$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\| \leq \|\mathcal{T}x\| \leq r$ for all $y \in \mathcal{T}(C)$. Hence $\mathcal{T}(C)$ is a uniformly bounded subset of E. Next we show that $\mathcal{T}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$, with $t_1 < t_2$. Then proceeding with the arguments that given in Step II it can be shown that

$$\left| y(t_2) - y(t_1) \right| = |\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $y \in \mathcal{T}(C)$. This shows that $\mathcal{T}(C)$ is an equicontinuous subset of E. Now, $\mathcal{T}(C)$ is a uniformly bounded and equicontinuous subset of functions in E and

hence it is compact in view of Arzelá-Ascoli theorem. Consequently $\mathcal{T} : E \to E$ is a partially compact operator on E into itself.

Step IV: u *satisfies the inequality* $u \leq \mathcal{T}u$.

By hypothesis (H_4), the FDE (3) has a lower solution u defined on J. Then we have

$$\begin{cases} u''(t) & \leq & f(t, u_t), \;\; t \in I, \\ u_0 & \leq_{\mathcal{C}} & \phi, \;\; u'(0) \leq \eta. \end{cases}$$

Integrating the above inequality from 0 to t, we get

$$u(t) \leq \begin{cases} \phi(0) + \eta t + \displaystyle\int_0^t (t-s) f(s, u_s)\, ds, & \text{if} \;\; t \in I, \\[2mm] \phi(t), & \text{if} \;\; t \in I_0. \end{cases}$$

$$= \mathcal{T}u(t)$$

for all $t \in J$. As a result we have that $u \leq \mathcal{T}u$.

Thus, $\mathcal{T}$ satisfies all the conditions of Theorem 2.1 and so the operator equation $\mathcal{T}x = x$ has a solution. Consequently the integral equation and the equation (3) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (9) converges monotonically to x^*. This completes the proof. □

Remark 3.1. The conclusion of Theorems 3.1 also remains true if we replace the hypothesis (H_4) with the following ones:

(H_4') The FDE (3) has an upper solution $v \in C^2(J, \mathbb{R})$.

The proof of Theorem 3.1 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Example 3.1. Given the closed and bounded intervals $I_0 = \left[-\frac{\pi}{2}, 0\right]$ and $I = [0, 1]$, consider the FDE

$$\left.\begin{array}{r} x''(t) = f_1(t, x_t), \;\; t \in I, \\ x_0 = \phi, \;\; x'(0) = 1, \end{array}\right\} \tag{10}$$

where $\phi \in \mathcal{C}$ and $f_1 : I \times \mathcal{C} \to \mathbb{R}$ is a continuous functions given by

$$\phi(\theta) = \sin\theta, \quad \theta \in \left[-\frac{\pi}{2}, 0\right],$$

and

$$f_1(t, x) = \begin{cases} \tanh(\|x\|_{\mathcal{C}}) + 1, & \text{if} \;\; x \geq_{\mathcal{C}} 0, \; x \neq 0, \\ 1, & \text{if} \;\; x \leq_{\mathcal{C}} 0, \end{cases}$$

for all $t \in I$.

Clearly, f_1 is bounded on $I \times \mathcal{C}$ with $M_{f_1} = 2$. Again, let $x, y \in \mathcal{C}$ be such that $x \geq_{\mathcal{C}} y \geq_{\mathcal{C}} 0$. Then $\|x\|_{\mathcal{C}} \geq \|y\|_{\mathcal{C}} \geq 0$ and therefore, we have

$$f_1(t, x) = \tanh(\|x\|_{\mathcal{C}}) + 1 \geq \tanh(\|y\|_{\mathcal{C}}) + 1 = f_1(t, y)$$

for all $t \in I$. Again, if $x, y \in \mathcal{C}$ be such that $x \leq_{\mathcal{C}} y \leq_{\mathcal{C}} 0$, then we obtain

$$f_1(t, x) = 1 = f_1(t, y)$$

for all $t \in I$. This shows that the function $f_1(t,x)$ is nondecreasing in x for each $t \in I$. Finally,

$$u(t) = \begin{cases} t(t+1), & \text{if } t \in I, \\ \sin t, & \text{if } t \in I_0, \end{cases}$$

is a lower solution of the FDE (10) defined on J. Thus, f_1 satisfies the hypotheses (H_1), (H_2) and (H_4). Hence we apply Theorem 3.1 and conclude that the FDE (10) has a solution x^* on J and the sequence $\{x_n\}$ of successive approximation defined by

$$x_0(t) = \begin{cases} t(t+1), & \text{if } t \in I, \\ \sin t, & \text{if } t \in I_0, \end{cases}$$

$$x_{n+1}(t) = \begin{cases} t + \int_0^t (t-s) f_1(s, x_s^n)\, ds, & \text{if } t \in I, \\ \sin t, & \text{if } t \in I_0, \end{cases}$$

converges monotonically to x^*.

Remark 3.2. *The conclusion in Example 3.1 is also true if we replace the lower solution u with the upper solution v given by*

$$v(t) = \begin{cases} t(2t+1), & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0 \right]. \end{cases}$$

Theorem 3.2. *Suppose that hypotheses (H_3) and (H_4) hold. Then the FDE (3) has a unique solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by (8) converges monotonically to x^*.*

Proof. Set $E = C(J, \mathbb{R})$. Clearly, E is a lattice w.r.t. the order relation $\leq$ and so the lower and the upper bound exist for every pair of elements in E. Define the operator $\mathcal{T}$ by (9). Then, the FDE (3) is equivalent to the operator equation (9). We shall show that $\mathcal{T}$ satisfies all the conditions of Theorem 2.2 in E.

Clearly, $\mathcal{T}$ is a nondecreasing operator on E into itself. We shall simply show that the operator $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-contraction on E. Let $x, y \in E$ be any two elements such that $x \geq y$. Then, by hypothesis (H_3),

$$\begin{aligned} |\mathcal{T}x(t) - \mathcal{T}y(t)| &\leq \int_0^t (t-s)|f(s, x_s) - f(s, y_s)|\, ds \\ &\leq T \int_0^t \varphi(\|x_s - y_s\|_C)\, ds \\ &\leq T^2 \varphi(\|x - y\|) \end{aligned} \tag{11}$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|)$$

for all $x, y \in E$, $x \geq y$, where $\psi(r) = T^2 \varphi(r) < r$ for $r > 0$. As a result $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-contraction on E in view of Remark 2.3. Furthermore, it can be shown as in the proof of Theorem 3.1 that the function u given in hypothesis (H_3) satisfies the the operator inequality $u \leq \mathcal{T}u$ on J. Now a direct application of

Theorem 2.2 yields that the FDE (3) has a unique solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by (9) converges monotonically to x^*. $\square$

Remark 3.3. The conclusion of Theorems 3.2 also remains true if we replace the hypothesis (H$_4$) with the following ones:

(H$'_4$) The FDE (3) has an upper solution $v \in C^2(J, \mathbb{R})$.

The proof of Theorem 3.2 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Example 3.2. Given the closed and bounded intervals $I_0 = \left[-\frac{\pi}{2}, 0\right]$ and $I = [0, 1]$, consider the FDE

$$\left.\begin{array}{r} x'(t) = f_2(t, x_t), \ t \in [0, 1], \\ x_0 = \phi, \ x'(0) = 1 \end{array}\right\} \tag{12}$$

where $\phi \in \mathcal{C}$ and $f_2 : I \times \mathcal{C} \to \mathbb{R}$ is a continuous functions given by

$$\phi(\theta) = \sin\theta, \quad \theta \in \left[-\frac{\pi}{2}, 0\right],$$

and

$$f_2(t, x) = \begin{cases} \dfrac{\|x\|_{\mathcal{C}}}{1 + \|x\|_{\mathcal{C}}} + 1, & \text{if } x \geq_{\mathcal{C}} 0, \ x \neq 0, \\[3mm] 1, & \text{if } x \leq_{\mathcal{C}} 0, \end{cases}$$

for all $t \in I$.

Clearly, f_2 is continuous on $I \times \mathcal{C}$. We show that f_2 satisfies the hypotheses (H$_3$) and (H$_4$). Let $x, y \in \mathcal{C}$ be such that $x \geq_{\mathcal{C}} y \geq_{\mathcal{C}} 0$. Then $\|x\|_{\mathcal{C}} \geq \|y\|_{\mathcal{C}} \geq 0$ and therefore, we have

$$0 \leq f_2(t, x) - f_2(t, x) = \frac{\|x\|_{\mathcal{C}}}{1 + \|x\|_{\mathcal{C}}} - \frac{\|y\|_{\mathcal{C}}}{1 + \|y\|_{\mathcal{C}}} \leq \varphi(\|x - y\|_{\mathcal{C}})$$

for all $t \in I$, where $\psi(r) = \dfrac{r}{1 + r} < r$, $r > 0$. Again, if $x, y \in \mathcal{C}$ be such that $x \leq_{\mathcal{C}} y \leq_{\mathcal{C}} 0$, then we obtain

$$0 \leq f_2(t, x) - f_2(t, x) \leq \varphi(\|x - y\|_{\mathcal{C}})$$

for all $t \in I$. This shows that the function $f(t, x)$ is nondecreasing in x for each $t \in I$. Finally,

$$u(t) = \begin{cases} t(t + 1), & \text{if } t \in [0, 1], \\[2mm] \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

is a lower solution of the FDE (12) defined on J. Thus, f satisfies the hypotheses (H$_3$) and (H$_4$). Hence we apply Theorem 3.2 and conclude that the FDE (12) has a solution x^* on J and the sequence $\{x_n\}$ of successive approximation defined by

$$x_0(t) = \begin{cases} t(t + 1), & \text{if } t \in [0, 1], \\[2mm] \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

$$x_{n+1}(t) = \begin{cases} t + \displaystyle\int_0^t (t - s) f_2(s, x_s^n) \, ds, & \text{if } t \in [0, 1], \\[4mm] \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

converges monotonically to x^*.

Remark 3.4. *The conclusion in Example 3.2 is also true if we replace the lower solution u with the upper solution v given by*

$$v(t) = \begin{cases} t(2t+1), & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right]. \end{cases}$$

4. Linear Perturbation of First Type

Now, consider the hybrid functional differential equation with a linear perturbation of first type, namely,

$$\left.\begin{aligned} x''(t) &= f(t, x_t) + g(t, x_t), \ t \in I, \\ x_0 &= \phi, \ x'(0) = \eta, \end{aligned}\right\} \tag{13}$$

where $\phi \in \mathcal{C}$ and $f, g : I \times \mathcal{C} \to \mathbb{R}$ are continuous functions.

By a *solution* of the FDE (13) we mean a twice differentiable function $x \in C^2(J, \mathbb{R})$ that satisfies the equations in (13).

The FDE (13) is well-known in the literature and studied via different methods for existence of solution. The FDE (13) is a liner perturbation of first type of the FDE (1) and different types of perturbations are given in Dhage [3] which can be handled with the hybrid fixed point theorems involving the sum of two operators. See Dhage [2] and the references therein. The novelty of present study lies in its study of new Dhage iteration method for proving the existence as well as approximation of the solution. As a result of our new approach we obtain algorithm for the solutions of FDE (13) on J. We use the Dhage iteration method embodied in the following hybrid fixed point principle of Dhage [5]. See also Dhage [6] for the related results.

Theorem 4.1. *Let $\left(E, \preceq, \|\cdot\|\right)$ be a regular partially ordered complete normed linear space such that every compact chain C of E is Janhavi. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two nondecreasing operators such that*

    (a) *$\mathcal{A}$ is a partially bounded and partially nonlinear $\mathcal{D}$-contraction,*
    (b) *$\mathcal{B}$ is partially continuous and partially compact,*
    (c) *there exists an element $\alpha_0 \in X$ such that $\alpha_0 \preceq \mathcal{A}\alpha_0 + \mathcal{B}\alpha_0$ or $\alpha_0 \succeq \mathcal{A}\alpha_0 + \mathcal{B}\alpha_0$.*

Then the operator equation

$$\mathcal{A}x + \mathcal{B}x = x \tag{14}$$

has a solution x^* and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, $n = 0, 1, \ldots$; converges monotonically to x^*.

We need the following definition in what follows.

Definition 4.1. *A twice differentiable function $u \in C^2(J, \mathbb{R})$ is said to be a lower solution of the equation (13) if*

    (i) *$u_t \in \mathcal{C}$ for each $t \in I$, and*
    (ii) *u is twice continuously differentiable on I and satisfies the inequalities*

$$\left.\begin{aligned} u''(t) &\leq f(t, u_t) + g(t, u_t), \ t \in I, \\ u_0 &\leq_{\mathcal{C}} \phi, \ u'(0) \leq \eta. \end{aligned}\right\} \tag{$*$}$$

Similarly, a differentiable function $v \in C^2(J, \mathbb{R})$ is called an upper solution of the FDE (13) if the above inequalities are satisfied with reverse sign.

We consider the following set of hypotheses in what follows.

(H$_5$) There exists a constant $M_g > 0$ such that $|g(t, x)| \leq M_g$ for all $t \in I$ and $x \in \mathcal{C}$.

(H$_6$) $g(t, x)$ is nondecreasing in x for each $t \in I$.

(H$_7$) FDE (13) has a lower solution $u \in C^2(J, \mathbb{R})$.

Our main existence and approximation result for the FDE (13) is as follows.

Theorem 4.2. *Suppose that hypotheses* $(H_1) - (H_3)$ *and* $(H_5) - (H_7)$ *hold. Then the FDE (13) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_0 = u,$$

$$x_{n+1}(t) = \begin{cases} \phi(0) + t\eta + \displaystyle\int_0^t (t-s)f(s, x_s^n)\, ds \\ \quad + \displaystyle\int_0^t (t-s)g(s, x_s^n)\, ds, \quad if \ \ t \in I, \\ \phi(t), \quad if \ \ t \in I_0, \end{cases} \tag{15}$$

where $x_s^n(\theta) = x_n(s + \theta)$, $\theta \in I_0$, converges monotonically to x^*.

Proof. Set $E = C(J, \mathbb{R})$. Then, in view of Lemma 3.1, every partially compact subset S of E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ so that every compact chain C in E is Janhavi.

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = \begin{cases} \displaystyle\int_0^t (t-s)f(s, x_s)\, ds, \quad if \ \ t \in I, \\ 0, \quad if \ \ t \in I_0. \end{cases} \tag{16}$$

and

$$\mathcal{B}x(t) = \begin{cases} \phi(0) + t\eta + \displaystyle\int_0^t (t-s)g(s, x_s)\, ds, \quad if \ \ t \in I, \\ \phi(t), \quad if \ \ t \in I_0. \end{cases} \tag{17}$$

From the continuity of the integral, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operator $\mathcal{A}, \mathcal{B} : E \to E$. Applying Lemma 3.2, the FDE (13) is equivalent to the operator equation

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \ t \in J. \tag{18}$$

Now, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.1. Proceeding with the arguments that given in Theorem 3.1, it can be shown that $\mathcal{B}$ is a partially continuous and compact operator on E into itself. By hypothesis (H$_1$), $\mathcal{A}$ is a bounded operator on E. Again following the arguments that given in theorem 3.2 is is shown that $\mathcal{A}$ is a partial nonlinear contraction on E into itself. Now a direct application of Theorem 2.2 yields that the FDE (13) has a solution x^* and the sequence $\{x_n\}$ of successive approximations defined by (17) converges to x^*. This completes the proof. $\square$

Remark 4.1. The conclusion of Theorem 4.1 also remains true if we replace the hypothesis (H_7) with the following one:

(H_7') The FDE (13) has an upper solution $v \in C^2(J, \mathbb{R})$.

The proof of Theorem 3.1 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Remark 4.2. We note that if the FDEs (3) and (13) have a lower solution u as well as an upper solution v such that $u \leq v$, then under the given conditions of Theorems 3.1 and 4.1 it has corresponding solutions x_* and x^* and these solutions satisfy $x_* \leq x^*$. Hence they are the minimal and maximal solutions of the FDEs (3) and (13) respectively in the vector segment $[u, v]$ of the Banach space $E = C(J, \mathbb{R})$, where the vector segment $[u, v]$ is a set in $C^1(J, \mathbb{R})$ defined by

$$[u, v] = \{x \in C(J, \mathbb{R}) \mid u \leq x \leq v\}.$$

This is because the order relation $\leq$ defined by (6) is equivalent to the order relation defined by the order cone $\mathcal{K} = \{x \in C(J, \mathbb{R}) \mid x \geq \theta\}$ which is a closed set in $C(J, \mathbb{R})$. The existence results concerning the maximal and minimal solutions for the HFDE (3) may be obtained via generalized iteration method under weaker Caratheódory condition of the nonlinearity f as did in Guo and Lakshmikaham [17] but in that case we do not get any algorithm for approximating the extremal solutions. Similarly, we can not apply monotone iterative technique given in Ladde *et.al* [19] for the problems (3) and (13) for proving the existence theorem, because we do not assume the existence of both lower as well as upper solutions of the HFDEs (3) and (13).

Example 4.1. Given the closed and bounded intervals $I_0 = \left[-\frac{\pi}{2}, 0\right]$ and $I = [0, 1]$ and given a function $\phi \in \mathcal{C}(I_0, \mathbb{R})$, consider the FDE

$$\left.\begin{array}{l} x''(t) = f_1(t, x_t) + f_2(t, x_t), \ t \in I, \\[2mm] x_0 = \phi, \ x'(0) = 1, \end{array}\right\} \tag{19}$$

where $\phi \in \mathcal{C}$ and $f_1, f_2 : I \times \mathcal{C} \to \mathbb{R}$ are continuous functions given by

$$\phi(\theta) = \sin\theta, \quad \theta \in \left[-\frac{\pi}{2}, 0\right],$$

$$f_1(t, x) = \begin{cases} \dfrac{\|x\|_{\mathcal{C}}}{1 + \|x\|_{\mathcal{C}}} + 1, & \text{if } x \geq_{\mathcal{C}} 0, \ x \neq 0, \\[3mm] 1, & \text{if } x \leq_{\mathcal{C}} 0, \end{cases}$$

and

$$f_2(t, x) = \begin{cases} \tanh(\|x\|_{\mathcal{C}}) + 1, & \text{if } x \geq_{\mathcal{C}} 0, \ x \neq 0, \\[2mm] 1, & \text{if } x \leq_{\mathcal{C}} 0, \end{cases}$$

for all $t \in I$.

Clearly the functions f_1 and f_2 satisfy the hypotheses (H_1) and (H_5) with $M_{f_1} = 2 = M_{f_2}$. Next, it can be show as in Theorem 3.1 the nonlinearity f_1 satisfies the hypothesis (H_3). Similarly, the nonlinearity f_2 satisfies the hypothesis (H_6). Again,

it can be verified that

$$
u(t) = \begin{cases} 2(t+1), & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}
$$

is a lower solution for the FDE (13 defined on $J = \left[-\frac{\pi}{2}, 1\right]$. Thus, f_1 and f_2 satisfy all the hypotheses of Theorem 4.1. Hence the FDE (13) has a solution x^* and the sequence $\{x_n\}$ of successive approximations defined by

$$
x_0(t) = \begin{cases} 2(t+1), & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}
$$

$$
x_{n+1}(t) = \begin{cases} t + \displaystyle\int_0^t (t-s) f_1(s, x_s^n)\,ds \\ \quad + \displaystyle\int_0^t (t-s) f_2(s, x_s^n)\,ds, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}
$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in \left[-\frac{\pi}{2}, 0\right]$, converges monotonically to x^*.

Remark 4.3. *The conclusion in Example 4.1 is also true if we replace the lower solution u with the upper solution v given by*

$$
v(t) = \begin{cases} t(4t+1), & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right]. \end{cases}
$$

5. Conclusion

In this paper we have discussed a very simple nonlinear second order ordinary functional differential equation via Dhage iteration method by constructing an algorithm for the solutions. However, other several nonlinear functional differential equations could also be studied for existence and approximation of the solutions using Dhage iteration method in an analogous way with appropriate modifications. Again, here our discussion is limited to proving the existence theorem for the functional differential equation under consideration, but other qualitative aspects such as maximal and minimal solutions and comparison principle etc. could also be studied by constructing the algorithm via Dhage iteration method on the lines of Dhage [10] and the references therein. It is known that the comparison principle is very much useful in the theory of nonlinear functional differential equations for proving the qualitative properties of the solutions. Therefore, we claim that the Dhage iteration method is a powerful method in the theory of nonlinear differential and integral equations. Finally, while concluding this paper we mention that the use of Dhage iteration method in the qualitative study of nonlinear functional differential equations is interesting and some of the results in this directions will be reported elsewhere.

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Shyam B. Dhage

Kasubai, Gurukul Colony, Ahmedpur-413515, Dist. Latur, Maharashtra, India
  E-mail address: `sbdhage4791@gmail.com`

Ashok D. Kadam

Department of Mathematics, A.E.S. Arts, Com.& Sci. College, Hingoli-431513, Maharashtra, India
  E-mail address: `dradk-15862@rediffmail.com`