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ON DETERMINING THE COEFFICIENT IN A PARABOLIC EQUATION WITH NONLOCAL BOUNDARY AND INTEGRAL CONDITION

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Abstract. We study existence and uniqueness of the inverse problem of determination of the unknown coefficient $b(t)$ multiplying u_t in a linear parabolic equation in the case of nonlocal boundary conditions containing a real parameter and integral overdetermination conditions. Under some consistency conditions on the input data the existence, uniqueness and continuously dependence upon the data of the classical solution are shown by using the generalized Fourier method.

1. Introduction

Consider the problem of finding a pair of functions $\{b(t),\ u(x,t)\}$ satisfying the following equation

$$b(t)u_t = u_{xx} + f(x,t), \quad 0 < x < 1, \quad 0 < t \leq T. \tag{1}$$

with initial conditon

$$u(x,0) = \varphi(x), \quad 0 \leq x \leq 1, \tag{2}$$

the boundary conditions

$$u(0,t) = 0, \quad u_x(0,t) = u_x(1,t) + \alpha u(1,t), \quad 0 \leq t \leq T, \tag{3}$$

and the energy condition

$$\int_0^1 u(x,t)\,dx = E(t), \quad 0 \leq t \leq T. \tag{4}$$

Where the parameter α in an arbitrary real number and $f(x,t)$, $\varphi(x)$, $E(t)$ are given functions. The nonlocal second boundary condition in (1.3) is the main specific feature of this problem; for $\alpha = 0$, it acquires the form

$$u_x(0,t) = u_x(1,t), \quad u(0,t) = 0, \quad 0 \leq t \leq T, \tag{5}$$

and was comprehensively studied in [1], are well-known as the Samarskii–Ionkin conditions, whilst (1.4) specifies an integral additional specification of the energy. The problem of finding a pair $\{b(t), u(x,t)\}$ will be called an inverse problem.

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Denote the domain Q_T by

$$Q_T = \{(x,t) : 0 < x < 1, \quad 0 < t \leq T\}.$$

The paper is organized as follows. In Section 2, the eigenvalues and eigenfunctions of the auxiliary spectral problem and some of their properties are introduced. Moreover, the existence and uniqueness of the solution of the inverse problem $(1.1)-(1.4)$ are proved in section 2. Finally the continuous dependence upon the data of the solution of the inverse problem is given in Section 4.

Definition 1 *The pair* $\{b(t), u(x,t)\}$ *from the class* $C[0,T] \times C^{2,1}(Q_T) \cap C^{1,0}\left(\overline{Q}_T\right)$ *for which conditions* $(1.1)-(1.4)$ *are satisfied, is called the classical solution of the inverse problem* $(1.1)-(1.4)$.

Additional information is given in the form of the integral observation condition (1.4) arises from many important applications in heat transfer, thermoelasticity, control theory, life sciences and plays an important role in engineering and physics [15]-[24]. For example, for heat propagation in a thin rod in which the law of variation $E(t)$ of the total quantity of heat in the rod is given in [1].

There is a wide bibliography dealing with various inverse problems for parabolic equations. Note the papers [8]-[14] etc. in which inverse problems of finding the unknown coefficient in the leading part of an equation were studied in various statements. However, inverse problems with additional conditions different from (1.4) were considered in these papers. To the best of the author's knowledge, the inverse problems of finding the leading coefficient in a parabolic equation with condition (1.4) and generalized nonlocal condition (1.3) have not been studied yet. The inverse problems in this paper are similar from the mathematical point of view that nonlocal boundary and overdetermination conditions are used, the parameter $b(t)$ needs to be determined by thermal energy $E(t)$, and the existence and uniqueness of the classical solution of the problem $(1.1)-(1.4)$ is reduced to fixed point principles by applying the Fourier method. The boundary conditions (1.3) admit the expansions by the system of eigenfunctions and associated functions corresponding to the spectral problem.

2. The auxiliary spectral problem

The use of the Fourier method for solving problem $(1.1)-(1.3)$ leads to the spectral problem for the operator L given by the differential expression and boundary conditions

$$\left\{ \begin{array}{c} LX(x) \equiv X''(x) = -\lambda X(x), \qquad 0 \leq x \leq 1 \\ X'(0) = X'(1) + \alpha X(1), \quad X(0) = 0. \end{array} \right. \tag{6}$$

The boundary conditions in (2.1) are regular, but not strongly regular ([2], pp.66-67). The system of root functions of the operator L is complete, but it does not form even a regular basis in $L^2(0,1)$[3]. However, as it was shown in [4], on the base of these eigenfunctions one can construct a basis allowing one to use the method of separation of variables for solving the initial-boundary value problem subject to the boundary condition (1.3).

In recent times a special attention has been paid to the study of direct and inverse problems for various classes of partial differential equations in particular cases of boundary conditions which are not strongly regular. Let us cite only those of them that are most close to the subject under consideration [5]-[7].

In this paper we propose to use the results described in [4] for solving the inverse problem $(1.1)-(1.4)$. So, for construction of the basis of eigenfunctions of problem (2.1), we cite the necessary results from [4].

Consider the case in which $\alpha \neq 0$. We seek eigenvalues in the set of real numbers. Note that $\lambda = 0$ is not an eigenvalue, since problem (2.1) for this value of λ has only the trivial solution. Suppose that $\lambda > 0$. Then the eigenfunction should have the form $X(x) = \sin \sqrt{\lambda} x$. By taking into account the nonlocal boundary condition, we obtain the eigenvalues and eigenfunctions of the operator L of the form

$$\lambda_k^{(1)} = (2k\pi)^2, \ k = 1, 2, ..., \quad X_k^{(1)}(x) = \sin 2k\pi x, \ k = 1, 2, ... \tag{7}$$

and the form

$$\lambda_k^{(2)} = (2\beta_k)^2, \quad X_k^{(2)}(x) = \sin 2\beta_k x, \qquad k = 0, 1, 2, ... \tag{8}$$

where β_k, satisfying the inequalities $\pi k < \beta_k < \pi k + \pi/2$, $k = 0, 1, 2, ...$, for positive α:

$$\frac{\alpha}{2k\pi}\left(1 - \frac{1}{2k\pi}\right) < \beta_k - \pi k < \frac{\alpha}{2k\pi}\left(1 + \frac{1}{2k\pi}\right),$$

for sufficiently large k. This system is almost normed, but it does not form even a regular basis in $L_2(0,1)$. The corresponding auxiliary system

$$\left\{ \begin{array}{c} X_0(x) = X_0^{(2)}(x)(2\beta_0)^{-1} \\ X_{2k}(x) = X_k^{(1)}(x), \quad k = 1, 2, \\ X_{2k-1}(x) = \left(X_k^{(2)}(x) - X_k^{(1)}(x)\right)\left(2\left(\beta_k - \pi k\right)\right)^{-1}, \quad k = 1, 2, ... \end{array} \right. . \tag{9}$$

Form a Riesz basis in $L_2(0,1)$, and to find the biorthogonal system of $\{X_k(x), \ k = 0, 1, 2, ...\}$, consider the adjoint differential operator L^* of L in the sense of the inner product of the space $L_2(0,1)$:

$$L^*Y(x) \equiv Y''(x) = -\lambda Y(x), \qquad 0 \leq x \leq 1.$$

So, for two arbitrary twice continuously differentiable functions $X(x)$ and $Y(x)$ on $[0,1]$ such that $X'(0) = X'(1) + \alpha X(1), \ X(0) = 0$, we have

$$(Lu, v) = \int_0^1 \left(-X''(x)\right) Y(x) dx = \int_0^1 X(x)\left(-Y''(x)\right) dx = (u, L^*v),$$

this implies

$$-X(1)Y(1) = -X'(1)Y(1) + X'(0)Y(0), \tag{10}$$

we obtain $\left[X'(1) - X(1)\right] Y(1) = X'(0)Y(0)$, by using $X'(0) = X'(1) + \alpha X(1)$, we have $\left[X'(1) - X(1)\right] Y(1) = \left[X'(1) + \alpha X(1)\right] Y(0)$, since α is arbitrary, we get $Y(1) = Y(0)$. Now by using (2.5) and $X'(0) = X'(1) + \alpha X(1)$, we obtain

$$-X(1)Y(1) = -X'(1)Y(1) + \left[X'(1) + \alpha X(1)\right] Y(0) \ \Rightarrow -X(1)\left[Y(1) + \alpha Y(0)\right] = -X'(1)\left[Y(1) - Y(0)\right],$$

since $Y(1) = Y(0)$, we get $-X(1)\left[Y(1) + \alpha Y(0)\right] = 0$, this implies $Y(1) + \alpha Y(0) = 0$. Then, The adjoint problem of (2.1) has the form

$$\left\{ \begin{array}{l} L^*Y(x) \equiv Y''(x) = -\lambda Y(x), \qquad 0 \leq x \leq 1. \\ Y(1) + \alpha Y(0) = 0, \quad Y(1) = Y(0). \end{array} \right. \tag{11}$$

It is easy to verify that the eigenvalues of this problem are same as for the problem (2.1). The system of eigenfunctions and associated functions of the problem (2.5) is denoted by

$$\begin{aligned} Y_k^{(2)}(x) &= C_k^{(2)} \cos\left(\beta_k (1-2x)\right), k = 0,1,2,...; \\ Y_k^{(1)}(x) &= C_k^{(1)} \cos\left(\beta_k (1-2x) + \arctan\left(\frac{\alpha}{2\pi k}\right)\right), k = 1,2,... \end{aligned}$$

where

$$\begin{aligned} C_k^{(1)} &= -2\left(\sin\left(\arctan\left(\frac{\alpha}{2\pi k}\right)\right)\right)^{-1}, \quad k = 1,2,... \\ C_k^{(2)} &= 2\left(\left(\sin\beta_k\right)\left(1 + \frac{\sin\left(2\beta_k\right)}{2\beta_k}\right)\right)^{-1}, \quad k = 1,2,... \end{aligned}$$

For the function system (2.4), there exists a biorthogonal normalized system, given by

$$\left\{ \begin{array}{c} Y_0(x) = Y_0^{(2)}(x)\left(2\beta_0\right) \\ Y_{2k}(x) = \left(Y_k^{(2)}(x) + Y_k^{(1)}(x)\right), \quad k = 1,2,... \\ Y_{2k-1}(x) = \left(2\left(\beta_k - \pi k\right)\right) Y_k^{(2)}(x), \quad k = 1,2,... \end{array} \right. \tag{12}$$

It is shown in [4] that the systems (2.4) and (2.7) form biorthogonal systems on the interval $[0,1]$, i.e.

$$(X_i, Y_j) = \int_0^1 X_i(x)\, Y_j(x) dx = \delta_{ij} = \left\{ \begin{array}{l} 0, \ i \neq j \\ 1, \ i = j \end{array} \right. .$$

3. Existence and uniqueness of the solution of the inverse problem

We have the following assumptions on the data of the problem (1.1) – (1.4):

$$(A_1): \left\{ \begin{array}{ll} (A_1)_1: & \varphi(x) \in C^4[0,1]; \\ (A_1)_2: & \varphi_x(0) = \varphi_x(1) + \alpha\varphi(1), \ \varphi(0) = 0; \\ (A_1)_3: & \varphi_0 < 0, \varphi_{2k-1} < 0, \ k = 1,2,..., \end{array} \right.$$

$$(A_2): \left\{ \begin{array}{ll} (A_2)_1: & E(t) \in C^1[0,T]; \\ (A_2)_2: & E(0) = \int_0^1 \varphi(x)\, dx; \\ (A_2)_3: & E'(t) > 0, \end{array} \right.$$

$$(A_3): \left\{ \begin{array}{ll} (A_3)_1: & f(x,t) \in C\left(\overline{Q}_T\right), \ f(\bullet,t) \in C^2[0,1]; \\ (A_3)_2: & f_x(0,t) = f_x(1,t) + \alpha f(1,t), \quad f(0,t) = 0, \ 0 \le t \le T; \\ (A_3)_3: & F_0(t) > 0, F_{2k-1}(t) > 0, \ k = 1,2,.... \end{array} \right.$$

where $\varphi_k = \int_0^1 \varphi(x)\, Y_k(x) dx$, $F_k(t) = \int_0^1 f(x,t)\, Y_k(x) dx$, $k = 1,2,....$

The main result is presented as follows.

Theorem 1 Let the assumptions $(A_1) - (A_3)$ be satisfied. Then the inverse problem $(1) - (4)$ has a unique classical solution.

Proof. By applying the standard procedure of the Fourier method, Any solution of equation (1.1) can represented as:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)\, X_k(x), \tag{13}$$

So, by replacing $u(x,t)$ in equation (1.1) by the representation (3.1), we get

$$b(t)\sum_{k=0}^{\infty} u_k^{'}(t) X_k(x) = \sum_{k=0}^{\infty} u_k(t) X_k^{''}(x) + f(x,t), \tag{14}$$

multiplying the equation (3.2) by $Y_k(x)$, and integrating over $(0,1)$, we get the following system of equations:

$$\begin{aligned} u_0^{'}(t) + \frac{\lambda_0^{(2)}}{b(t)} u_0(t) &= \frac{1}{b(t)} F_0(t), \\ u_{2k}^{'}(t) + \frac{\lambda_k^{(1)}}{b(t)} u_{2k}(t) &= \frac{1}{b(t)} F_{2k}(t), \ k = 1, 2, ... \\ u_{2k-1}^{'}(t) + \frac{\lambda_k^{(2)}}{b(t)} u_{2k-1}(t) &= \frac{1}{b(t)} F_{2k-1}(t), \ k = 1, 2, ... \end{aligned}$$

Substituting the solution of this system of equations and initial condition (1.2) in (3.1), we obtain the solution of the problem (1.1) – (1.3) in the following form:

$$\begin{aligned} u(x,t) &= \left[\varphi_0 e^{-\lambda_0^{(2)}\int_0^t \frac{1}{b(s)}ds} + \int_0^t \frac{F_0(\tau)}{b(\tau)} e^{-\lambda_0^{(2)}\int_\tau^t \frac{1}{b(s)}ds} d\tau\right] X_0(x) \\ &+ \sum_{k=1}^{\infty}\left[\varphi_{2k} e^{-\lambda_k^{(1)}\int_0^t \frac{1}{b(s)}ds} + \int_0^t \frac{F_{2k}(\tau)}{b(\tau)} e^{-\lambda_k^{(1)}\int_\tau^t \frac{1}{b(s)}ds} d\tau\right] X_{2k}(x) \\ &+ \sum_{k=1}^{\infty}\left[\varphi_{2k-1} e^{-\lambda_k^{(2)}\int_\tau^t \frac{1}{b(s)}ds} + \int_0^t \frac{F_{2k-1}(\tau)}{b(\tau)} e^{-\lambda_k^{(2)}\int_\tau^t \frac{1}{b(s)}ds} d\tau\right] X_{2k-1}(x) \end{aligned} \tag{15}$$

Under the conditions $(A_1)_1$ and $(A_2)_1$ the series (3.3) and its x-partial derivative are uniformly convergent in $\overline{Q}_T$ since their majorizing sums are absolutely convergent. Therefore, their sums $u(x,t)$ and $u_x(x,t)$ are continuous in $\overline{Q}_T$. In addition, the t-partial derivative and the xx-second-order partial derivative series are uniformly convergent in Q_T. Thus, we have $u(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}\left(\overline{Q}_T\right)$. In addition, $u_t(x,t)$ is continuous in $\overline{Q}_T$. Differentiating (1.4) under the assumption $(A_2)_1$, we obtain

$$\int_0^1 u_t(x,t)\,dx = E^{'}(t), \quad 0 \le t \le T, \tag{16}$$

using (3.3) and (3.4), yield

$$P[b(t)] = b(t), \tag{17}$$

where

$$P[b(t)] = \frac{1}{E'(t)}\left[\begin{array}{c} \left(\frac{1-\cos(2\beta_0)}{4\beta_0^2}\right)\left(-\lambda_0^{(2)}\varphi_0 e^{-\lambda_0^{(2)}\int_0^t \frac{1}{b(s)}ds} + F_0(t)\, e^{-\lambda_0^{(2)}\int_\tau^t \frac{1}{b(s)}ds}\right) \\ +\sum_{k=1}^{\infty}\left(\frac{1-\cos(2\beta_k)}{4\beta_k(\beta_k-\pi k)}\right)\left(-\lambda_k^{(2)}\varphi_{2k-1} e^{-\lambda_k^{(2)}\int_0^t \frac{1}{b(s)}ds} + F_{2k-1}(t)\, e^{-\lambda_k^{(2)}\int_\tau^t \frac{1}{b(s)}ds}\right)\end{array}\right]. \tag{18}$$

Let us denote

$$C^{+}[0,T] = \{b(t) \in C[0,T] : b(t) > 0\}.$$

It is easy to verify that under conditions $(A_1)_3$, $(A_2)_3$ and $(A_3)_3$,

$$P : C^{+}[0,T] \to C^{+}[0,T].$$

Let us show that P is a contraction mapping in $C^{+}[0,T]$. Then, we have for $a(t)$, $b(t)\in C^{+}[0,T]$, the estimates

$$\begin{aligned}\left|e^{-\lambda\int_{\tau}^{t}a(s)ds}-e^{-\lambda\int_{\tau}^{t}b(s)ds}\right| &\leqslant \xi\max_{t\in[0,T]}|a(t)-b(t)|;\\ \left|e^{-\lambda\int_{0}^{t}a(s)ds}-e^{-\lambda\int_{0}^{t}b(s)ds}\right| &\leqslant \xi\max_{t\in[0,T]}|a(t)-b(t)|,\end{aligned}$$

are true by using the mean value theorem, where $\xi=\max_{k\geqslant 1}\left(\lambda_k^{(2)}\right)\max_{t\in[0,T]}|a(t)|$. From the last inequalities, we obtain

$$|P[a(t)]-P[b(t)]|\leqslant \xi c_0(c_1+c_2+c_3+c4)\max_{t\in[0,T]}|a(t)-b(t)|, \tag{19}$$

where

$$\begin{aligned}c_0 &= \max_{t\in[0,T]}\left(\frac{1}{E'(t)}\right);\\ c_1 &= \left(\frac{1-\cos(2\beta_0)}{4\beta_0^2}\right)\lambda_0^{(2)}\varphi_0;\\ c_2 &= \left(\frac{1-\cos(2\beta_0)}{4\beta_0^2}\right)\max_{t\in[0,T]}F_0(t);\\ c_3 &= \sum_{k=1}^{\infty}\left[\left(\frac{1-\cos(2\beta_k)}{4\beta_k(\beta_k-\pi k)}\right)\lambda_k^{(2)}\varphi_{2k-1}\right];\\ c_4 &= \sum_{k=1}^{\infty}\left[\left(\frac{1-\cos(2\beta_k)}{4\beta_k(\beta_k-\pi k)}\right)\max_{t\in[0,T]}F_{2k-1}(t)\right].\end{aligned}$$

In the cas $\xi c_0(c_1+c_2+c_3+c4)<1$. Equation (3.7) has a unique solution $b(t)\in C^{+}[0,T]$, by the Banach fixed point theorem.

Now, let us show that the solution (a,u), obtained for $(1)-(4)$, is unique. Suppose that (b,v) is also a solution pair of $(1.1)-(1.4)$. Then from the representation (3.3) of the solution, we have:

$$\begin{aligned}u(x,t)-v(x,t) = &\left[\begin{array}{c}\varphi_0\left(e^{-\lambda_0^{(2)}\int_0^t\frac{1}{a(s)}ds}-e^{-\lambda_0^{(2)}\int_0^t\frac{1}{b(s)}ds}\right)\\ +\int_0^t\frac{F_0(\tau)}{b(\tau)}\left(e^{-\lambda_0^{(2)}\int_\tau^t\frac{1}{a(s)}ds}-e^{-\lambda_0^{(2)}\int_\tau^t\frac{1}{b(s)}ds}\right)d\tau\end{array}\right]X_0(x)\\ &+\sum_{k=1}^{\infty}\left[\begin{array}{c}\varphi_{2k}\left(e^{-\lambda_k^{(1)}\int_0^t\frac{1}{a(s)}ds}-e^{-\lambda_k^{(2)}\int_0^t\frac{1}{b(s)}ds}\right)\\ +\int_0^t\frac{F_{2k}(\tau)}{b(\tau)}\left(e^{-\lambda_k^{(1)}\int_\tau^t\frac{1}{a(s)}ds}-e^{-\lambda_k^{(1)}\int_\tau^t\frac{1}{b(s)}ds}\right)d\tau\end{array}\right]X_{2k}(x)\\ &+\sum_{k=1}^{\infty}\left[\begin{array}{c}\varphi_{2k-1}\left(e^{-\lambda_k^{(2)}\int_0^t\frac{1}{a(s)}ds}-e^{-\lambda_k^{(2)}\int_0^t\frac{1}{b(s)}ds}\right)\\ +\int_0^t\frac{F_{2k-1}(\tau)}{b(\tau)}\left(e^{-\lambda_k^{(2)}\int_\tau^t\frac{1}{a(s)}ds}-e^{-\lambda_k^{(2)}\int_\tau^t\frac{1}{b(s)}ds}\right)d\tau\end{array}\right]X_{2k-1}(x)\end{aligned} \tag{20}$$

From the equation (3.5), and (3.7), we obtain

$$\max_{t\in[0,T]}|a(t)-b(t)|\leqslant \xi c_0(c_1+c_2+c_3+c4)\max_{t\in[0,T]}|a(t)-b(t)|,$$

since $\xi c_0(c_1+c_2+c_3+c4)<1$, implies that $a=b$. By substituting $a=b$ into (3.11), we get $u=v$.

Theorem 1 has been proved. □

4. Continuous dependence of (b,u) upon the data

Theorem 2 Under assumption $(A_1)-(A_3)$, the solution (b,u) depends continuously upon the data.

Proof. Let $\Phi(\varphi,F,E)$ and $\overline{\Phi}\left(\overline{\varphi},\overline{F},\overline{E}\right)$ be two sets of the data, which satisfy the assumptions $(A_1)-(A_3)$. Then there exist positive constants M_i , $i=1,2,3$ such that

$$\begin{aligned} \|\varphi\|_{C^2[0,1]} &\leqslant M_1,\ \|F\|_{C^2[0,1]}\leqslant M_2,\ \|E\|_{C^1[0,T]}\leqslant M_3, && (21)\\ \|\overline{\varphi}\|_{C^2[0,1]} &\leqslant M_1,\ \left\|\overline{F}\right\|_{C^2[0,1]}\leqslant M_2,\ \left\|\overline{E}\right\|_{C^1[0,T]}\leqslant M_3. && (22)\end{aligned}$$

Let (b,u) and $(\bar{b},\overline{u})$ be solutions of the inverse problem $(1.1)-(1.4)$ corresponding to the data Φ and $\overline{\Phi}$, respectively. According to (3.5), we have

$$b(t) = \frac{1}{E'(t)}\left[\begin{array}{c}\left(\frac{1-\cos(2\beta_0)}{4\beta_0^2}\right)\left(-\lambda_0^{(2)}\varphi_0 e^{-\lambda_0^{(2)}\int_0^t\frac{1}{b(s)}ds}+F_0(t)\,e^{-\lambda_0^{(2)}\int_\tau^t\frac{1}{b(s)}ds}\right)\\ +\sum_{k=1}^{\infty}\left(\frac{1-\cos(2\beta_k)}{4\beta_k(\beta_k-\pi k)}\right)\left(-\lambda_k^{(2)}\varphi_{2k-1}e^{-\lambda_k^{(2)}\int_0^t\frac{1}{b(s)}ds}+F_{2k-1}(t)\,e^{-\lambda_k^{(2)}\int_\tau^t\frac{1}{b(s)}ds}\right)\end{array}\right],$$

$$\overline{b}(t) = \frac{1}{\overline{E'}(t)}\left[\begin{array}{c}\left(\frac{1-\cos(2\beta_0)}{4\beta_0^2}\right)\left(-\lambda_0^{(2)}\overline{\varphi}_0 e^{-\lambda_0^{(2)}\int_0^t\frac{1}{\overline{b}(s)}ds}+\overline{F}_0(t)\,e^{-\lambda_0^{(2)}\int_\tau^t\frac{1}{\overline{b}(s)}ds}\right)\\ +\sum_{k=1}^{\infty}\left(\frac{1-\cos(2\beta_k)}{4\beta_k(\beta_k-\pi k)}\right)\left(-\lambda_k^{(2)}\overline{\varphi}_{2k-1}e^{-\lambda_k^{(2)}\int_0^t\frac{1}{\overline{b}(s)}ds}+\overline{F}_{2k-1}(t)\,e^{-\lambda_k^{(2)}\int_\tau^t\frac{1}{\overline{b}(s)}ds}\right)\end{array}\right].$$

First, let us estimate the difference $b-\overline{b}$. It is easy to see that by using (4.1) and (4.2) , then

$$\left|\varphi FE-\overline{\varphi}\overline{F}\overline{E}\right|\leqslant M_2M_3\left|\varphi-\overline{\varphi}\right|+M_1M_3\left|F-\overline{F}\right|+M_1M_2\left|E-\overline{E}\right|.$$

by using the previous inequality, we obtain

$$\left\|b-\overline{b}\right\|_{C[0,T]}\leqslant M_4\left\|b-\overline{b}\right\|_{C[0,T]}+M_5\left\|\varphi-\overline{\varphi}\right\|_{C^2[0,1]}+M_6\left\|F-\overline{F}\right\|_{C^2[0,1]}+M_7\left\|E-\overline{E}\right\|_{C^1[0,T]},$$

this implies

$$\left\|b-\overline{b}\right\|_{C[0,T]}\leqslant\frac{M_8}{(1-M_4)}\left\|\Phi-\overline{\Phi}\right\|,$$

where $\left\|\Phi-\overline{\Phi}\right\|=\left\|\varphi-\overline{\varphi}\right\|_{C^2[0,1]}+\left\|F-\overline{F}\right\|_{C^2[0,1]}+\left\|E-\overline{E}\right\|_{C^1[0,T]}$.

From (3.3), a similar estimate is also obtained for the difference $u-\overline{u}$ as

$$\left\|u-\overline{u}\right\|_{C[0,T]}\leqslant M_9\left\|\Phi-\overline{\Phi}\right\|.$$

□

Future work will extend the analysis performed in this paper to inverse time dependent source problems for the heat equation and some hyperbolic type equation subject to more general nonlocal boundary conditions [25] with integral condition.

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