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# ON OSCILLATION OF THIRD ORDER FUNCTIONAL DIFFERENTIAL EQUATION

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Abstract. The objective of this work is to study oscillatory and asymptotic properties of the third-order nonlinear delay differential equation

$$(p(t)((r(t)x'(t))')^{\gamma})' + f(t, x(\tau(t))) = 0$$

Applying suitable comparison theorems we present new criteria for oscillation or certain asymptotic behaviors of nonoscillatory solutions.

## 1. Introduction

We are concerned with oscillatory behaviors of the third-order functional differential equations of the form

$$(p(t)((r(t)x'(t))')^{\gamma})' + f(t, x(\tau(t))) = 0 \tag{1}$$

In the sequel we will assume:

(H1) $p(t), r(t) \in C([t_0, \mathbb{R}^+))$, $r'(t) > 0$, $\int_{t_0}^{\infty} 1/r(s)ds = \infty$, $\int_{t_0}^{\infty} p^{-1/\gamma}(s)ds = \infty$ and $\tau(t) \in C([t_0, \infty))$, $\tau(t) \leq t$, $lim_{t\to\infty}\tau(t) = \infty$,

(H2) $\gamma$ is a quotient of odd positive integers,

(H3) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function , $uf(t, u) > 0$ for $u \neq 0$, $f(t, u) \geq q(t)f(u)$ , $f'(u) \geq 0$ and $-f(-uv) \geq f(uv) \geq f(u)f(v)$ for $uv > 0$, $q(t) \in C([t_0, \mathbb{R}^+))$ .

By a solution of Eq. (1) we mean a function $x(t) \in C^3[T_x, \infty)$, $T_x \geq t_0$, which has the property $p(t)((r(t)x'(t))')^{\gamma} \in C^1[T_x, \infty)$ and satisfies Eq. (1) on $[T_x, \infty)$. We assume that (1) possesses such a solution. A solution of (1) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$ and otherwise it is called to be nonoscillatory. Eq. (1) is said to be oscillatory if all its solutions are oscillatory. In recent years , there has been much research activity concerning third-order nonlinear differential equations, e.g., see [2-6,8-11]. In particular Baculíková and Dzurina [7] considered the third-order nonlinear delay equation

$$\left[a(t)\left[x''(t)\right]^{\gamma}\right]' + q(t)f(x[\tau(t)]) = 0,$$

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where $\gamma > 0$ is a quotient of odd positive integers. We conclude that if the gap between t and $\tau(t)$ is small then there exists nonoscillatory solution of (1) and so in this case our goal is to prove that every nonoscillatory solution of (1) tends to zero as $t \to \infty$, while if the difference $t - \tau(t)$ is large enough then we can study the oscillatory character of (1). So our aim in this work is to provide a general classification of oscillatory and asymptotic behaviors of the equation studied. We say that a nontrivial solution $x(t)$ of (1) is strongly decreasing if it satisfies

$$x(t)x'(t) < 0 \tag{2}$$

for all sufficiently large t and it is said to be strongly increasing if

$$x(t)x'(t) > 0 \tag{3}$$

## 2. Main Results

We start our main results with the classification of the possible nonoscillatory solutions of (1).

**Lemma 1.** Let $x(t)$ be a nonoscillatory solution of Eq. (1). Then $x(t)$ is either strongly increasing or strongly decreasing.

*Proof.* Let x(t) be a nonoscillatory solution of Eq. (1). We may assume that $x(t) > 0$, eventually ,if it is an eventually negative, the proof is similar. From (H3), we obtain

$$(p(t)((r(t)x'(t))')^{\gamma})' = -f(t, x(\tau(t))) \leq q(t)f(x(\tau(t))) < 0,$$

eventually. Then, $p(t)((r(t)x'(t))')^{\gamma}$ is decreasing and of one sign and it follows from hypotheses (H1) and (H2) that there exists a $t_1 \geq t_0$ such that $(r(t)x'(t))'$ is of fixed sign for $t \geq t_1$. If we admit $(r(t)x'(t))' < 0$, then there exists a constant $K < 0$ such that

$$p(t)((r(t)x'(t))')^{\gamma} \leq K < 0, \ t \geq t_1.$$

Integrating from $t_1$ to t, we obtain

$$r(t)x'(t) \leq r(t_1)x'(t_1) + K^{1/\gamma} \int_{t_1}^{t} \frac{1}{p^{1/\gamma}(s)} ds.$$

And using (H1), we get $r(t)x'(t) \to -\infty$ for $t \to \infty$. Then, we can write

$$r(t)x'(t) \leq -c_1 < 0.$$

From (H1),

$$x(t) \leq x(t_1) - c_1 \int_{t_1}^{t} \frac{1}{r(s)} ds$$

implies that $x(t) \to -\infty$ for $t \to \infty$. This contradiction shows that $(r(t)x'(t))' > 0$. Therefore $r(t)x'(t)$ is increasing and thus either (2) or (3) holds, eventually. The proof is complete. □

**Theorem 1.** If the first-order delay equation

$$z'(t) + q(t)f\left(\int_{t_0}^{\tau(t)} \frac{1}{r(s)} \int_{t_0}^{s} p^{-1/\gamma}(u)duds\right) f\left(z^{1/\gamma}(\tau(t))\right) = 0 \tag{4}$$

is oscillatory, then every solution of Eq. (1) is either oscillatory or strongly decreasing.

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (1). We may assume that $x(t) > 0$ for $t \geq t_0$. From Lemma 1 we see that $(r(t)x'(t))' > 0$ and $x(t)$ is either strongly increasing or strongly decreasing. Assume that $x(t)$ is strongly increasing, that is $x'(t) > 0$, eventually. Using that $p(t)((r(t)x'(t))')^\gamma$ is decreasing, we are led to

$$\begin{aligned} r(t)x'(t) &\geq \int_{t_1}^{t} (r(u)x'(u))'du = \int_{t_1}^{t} p^{-1/\gamma}(u)(p(u)((r(u)x'(u))')^\gamma)^{1/\gamma}du \\ &\geq (p(t)((r(t)x'(t))')^\gamma)^{1/\gamma} \int_{t_1}^{t} p^{-1/\gamma}(u)du. \end{aligned} \tag{5}$$

Integrating (5) from $t_1$ to $t$, we have

$$\begin{aligned} x(t) &\geq \int_{t_1}^{t} \frac{(p(s)((r(s)x'(s))')^\gamma)^{1/\gamma}}{r(s)} \int_{t_1}^{s} p^{-1/\gamma}(u)duds \\ &\geq (p(t)((r(t)x'(t))')^\gamma)^{1/\gamma} \int_{t_1}^{t} \frac{1}{r(s)} \int_{t_1}^{s} p^{-1/\gamma}(u)duds. \end{aligned}$$

There exists a $t_2 \geq t_1$ such that for all $t \geq t_2$, one gets

$$x(\tau(t)) \geq (z(\tau(t)))^{1/\gamma} \int_{t_2}^{\tau(t)} \frac{1}{r(s)} \int_{t_2}^{s} p^{-1/\gamma}(u)duds, \tag{6}$$

where $z(t) = p(t)((r(t)x'(t))')^\gamma$. Combining (6) together with (1), we see that

$$\begin{aligned} -z'(t) = f(t, x(\tau(t))) &\geq q(t)f(x(\tau(t))) \\ &\geq q(t)f\left((z(\tau(t)))^{1/\gamma} \int_{t_2}^{\tau(t)} \frac{1}{r(s)} \int_{t_2}^{s} p^{-1/\gamma}(u)duds\right) \\ &\geq q(t)f\left(\int_{t_2}^{\tau(t)} \frac{1}{r(s)} \int_{t_2}^{s} p^{-1/\gamma}(u)duds\right) f\left((z(\tau(t)))^{1/\gamma}\right), \end{aligned}$$

where we have used (H3). Thus function $z(t)$ is a positive and decreasing solution of the differential inequality

$$z'(t) + q(t)f\left(\int_{t_2}^{\tau(t)} \frac{1}{r(s)} \int_{t_2}^{s} p^{-1/\gamma}(u)duds\right) f\left((z(\tau(t)))^{1/\gamma}\right) \leq 0.$$

Hence, by Theorem 1 in [8] we conclude that the corresponding differential equation (4) also has a positive solution, which contradicts the oscillation of (4). Therefore $x(t)$ is strongly decreasing. □

**Lemma 2.** Assume that $x(t)$ is a positive decreasing solution of Eq. (1). If

$$\int_{t_0}^{\infty} \frac{1}{r(v)} \left[\int_{v}^{\infty} \frac{1}{p^{1/\gamma}(u)} \left(\int_{u}^{\infty} q(s)ds\right)^{1/\gamma} du\right] dv = \infty, \tag{7}$$

then $x(t)$ tends to zero as $t \to \infty$.

*Proof.* It is clear that there exists a finite $\lim_{t\to\infty} x(t) = b$. We shall prove that $b = 0$. Assume that $b > 0$. Integrating Eq. (1) from t to $\infty$ and using $x(\tau(t)) > b$ and (H3), we obtain

$$\begin{aligned} p(t)((r(t)x'(t))')^\gamma \geq \int_t^\infty f(s, x(\tau(s))ds &\geq \int_t^\infty q(s)f(x(\tau(s)))ds \\ &\geq f(b)\int_t^\infty q(s)ds, \end{aligned}$$

whichs implies

$$(r(t)x'(t))' \geq \frac{f^{1/\gamma}(b)}{p^{1/\gamma}(t)}\left[\int_t^\infty q(s)ds\right]^{1/\gamma}.$$

Integrating the last inequality from $t$ to $\infty$, we get

$$-r(t)x'(t) \geq f^{1/\gamma}(b)\int_t^\infty \frac{1}{p^{1/\gamma}(u)}\left[\int_u^\infty q(s)ds\right]^{1/\gamma} du.$$

Now integrating from $t_1$ to $t$, we arrive that

$$x(t_1) \geq f^{1/\gamma}(b)\int_{t_1}^t \frac{1}{r(v)}\left[\int_v^\infty \frac{1}{p^{1/\gamma}(u)}\left[\int_u^\infty q(s)ds\right]^{1/\gamma}\right] dudv.$$

Letting $t \to \infty$ we have a contradiction with (7) and so we have verified that $\lim_{t\to\infty} x(t) = 0$. □

Combining Theorem 1 and Lemma 2 we get:

**Theorem 2.** Assume that (7) holds. If Eq. (4) is oscillatory, then every solution of (1) is oscillatory or tends to zero as $t \to \infty$.

In Theorems 1 and 2 , I have established new comparison principles that enable us to reduce properties of third order nonlinear differential equation (1) from oscillation of the first order nonlinear delay Eq. (4).

**Example** Let us consider the third-order functional differential equation

$$\left[t\left(\frac{1}{t}x'(t)\right)'\right]' + \frac{1}{t^2}x(\frac{t}{2}) = 0, \ t \geq 1. \tag{8}$$

Now (7) holds and Eq. (4) reduces to

$$z'(t) + \frac{1}{8}\left(ln(t) - ln2 - \frac{1}{2} + \frac{4}{t^2}\right)z(\frac{t}{2}) = 0. \tag{9}$$

On the other hand, Theorem 2.1.1 in [1] quarantees oscillation of Eq. (9) provided that

$$\lim_{t\to\infty}\int_{\frac{t}{2}}^t \frac{1}{8}\left(ln(t) - ln2 - \frac{1}{2} + \frac{4}{t^2}\right)ds > \frac{1}{e},$$

which is evidently fulfilled and according to Theorem 2 every positive solution of Eq. (9) tends to zero as $t \to \infty$.

**Theorem 3.** Let $\tau'(t) > 0$. Assume that there exists a function $\zeta(t) \in C^1([t_0, \infty))$ such that

$$\zeta'(t) \geq 0,\ \zeta(t) > 0 \textit{ and } \sigma(t) = \tau(\zeta(\zeta(t))) < t.$$

If both the first-order delay Eq. (4) and

$$z'(t) + \Theta(t) f^{1/\gamma}(z(\sigma(t))) = 0 \tag{10}$$

where

$$\Theta(t) = \frac{1}{r(t)} \int_t^{\zeta(t)} \frac{1}{p^{1/\gamma}(v_2)} \left( \int_{v_2}^{\zeta(v_2)} q(v_1) dv_1 \right)^{1/\gamma} dv_2$$

,are oscillatory , then Eq. (1) is oscillatory.

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (1). We assume that $x(t) > 0$. From Theorem 1, we see that $x(t)$ is strongly decreasing. Integrating of (1) from $t$ to $\zeta(t)$ yields

$$p(t) \left( (r(t)x'(t))' \right)^{\gamma} \geq \int_t^{\zeta(t)} q(v_1) f(x(\tau(v_1)) dv_1 \geq f(x(\tau(\zeta(t)))) \int_t^{\zeta(t)} q(v_1) dv_1.$$

Thus

$$(r'(t)x'(t))' \geq \frac{f^{1/\gamma}(x(\tau(\zeta(t))))}{p^{1/\gamma}(t)} \left( \int_t^{\zeta(t)} q(v_1) dv_1 \right)^{1/\gamma}.$$

Integrating from $t$ to $\zeta(t)$ once more, we get

$$\begin{aligned} -r(t)x'(t) &\geq \int_t^{\zeta(t)} \frac{f^{1/\gamma}(x(\tau(\zeta(v_2))))}{p^{1/\gamma}(v_2)} \left( \int_{v_2}^{\zeta(v_2)} q(v_1) dv_1 \right)^{1/\gamma} dv_2 \\ &\geq f^{1/\gamma}(x(\sigma(t)) \int_t^{\zeta(t)} \frac{1}{p^{1/\gamma}(v_2)} \left( \int_{v_2}^{\zeta(v_2)} q(v_1) dv_1 \right)^{1/\gamma} dv_2. \end{aligned}$$

Integrating from $t$ to $\infty$, we gets

$$x(t) \geq \int_t^{\infty} \frac{f^{1/\gamma}(x(\sigma(v_3))}{r(v_3)} \int_{v_3}^{\zeta(v_3)} \frac{1}{p^{1/\gamma}(v_2)} \left( \int_{v_2}^{\zeta(v_2)} q(v_1) dv_1 \right)^{1/\gamma} dv_2 dv_3. \tag{11}$$

Let us denote the right hand side of (11) by $z(t)$. Then $z(t) > 0$ one can easily verify that $z(t)$ is a solution of the differential inequality

$$z'(t) + \Theta(t) f^{1/\gamma}(z(\sigma(t))) \leq 0,$$

where

$$\Theta(t) = \frac{1}{r(t)} \int_t^{\zeta(t)} \frac{1}{p^{1/\gamma}(v_2)} \left( \int_{v_2}^{\zeta(v_2)} q(v_1) dv_1 \right)^{1/\gamma} dv_2.$$

Then, show that the corresponding differential equation (10) also has a positive solution. This contradiction finishes the proof. □

## References


[1] Ladde, G. S., Lakshmikantham, S., Zhang, V.,Oscillation Theory of Differential Equations with Deviating Arguments, Marcel Dekker, New York, 1987.

[2] Graces, R., Agarwal, R. P., Pavani, R., Thandapani, R., On the oscillation of certain third order nonlinear functional differential equations, Appl. Math. Comput., 202 102-112 2008.

[3] Baculíková, B., Dzurina, J., Oscillation of third-order neutral differential equations, Math. Comput. Modelling, 52, 215-226, 2010.

[4] Baculíková, B., Elebbasy, E., Saker, S., Dzurina, J., Oscillation criteria for third order nonlinear differential equations, Math. Slovaca, 58, 201-220, 2008.

[5] Dzurina, J. Asymptotic properties of third order delay differential equations, Czechoslovak Math. J., 45, (120), 443-448, 1995.

[6] Tiryaki, A., Aktas, M. F.,Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping, J. Math. Anal. Appl., 325, 54-68, 2007.

[7] Baculíková, B., Dzurina, J., Oscillation of third-order nonlinear differential equations, Appl. Math. Let., 24, 466-470, 2011.

[8] Ch.G. Philos, On the existence of nonoscillatory solutions tending to zero at $\infty$ for differential equations with positive delay, Arch. Math., 36, no. 2, 168-178, 1981.

[9] Tunç, C., On the non-oscillation of solutions of some nonlinear differential equations of third order. Nonlinear Dyn. Syst. Theory 7 , no. 4, 419-430, 2007.

[10] Tunç, C., On the non-oscillation of solutions of nonhomogeneous third order differential equations. Soochow J. Math. 23, , no. 1, 1-7, 1997.

[11] Tunç, C., Non-oscillation criteria for a class of nonlinear differential equations of third order. Bull. Greek Math. Soc., 39,, 131-137, 1997.



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