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SOME COMMENTS AND NOTES ON ALMOST PERIODIC FUNCTIONS AND CHANGING-PERIODIC TIME SCALES

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Abstract. In this paper, we correct some misleading statements made in the paper (Y.K. Li, Some remarks on almost periodic time scales and almost periodic functions on time scales, Electr. J. Math. Anal. Appl. 5 (2017) 42-49) on recent work concerning almost periodic time scales and changing-periodic time scales.

1. Introduction

Recently, Y.K. Li, etc. (see [1]) made some misleading comments on recent work concerning almost periodic functions and changing-periodic time scales. Unfortunately these mistakes also appear in a recent publication of Y.K. Li and P. Wang (see [2]). The purpose of this paper is to consider the mistakes in [1, 2] (which we will illustrate with examples) and try to motivate some ideas for future research . In addition we will discuss uniformly almost periodic functions on periodic time scales and changing-periodic time scales from the literature; see [3, 4, 5] and the references therein.

2. Comments

We begin by noting that the basic concept of the integral on time scales introduced in [1] is incorrect. The integral in [1] was introduced partly to claim that paper [3] is unnecessary.

For the convenience of the reader we note in [3] that the authors proposed the concept of a relatively dense set on time scales. Let $\mathbb{T}$ be a periodic time scale and $\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}$.

Definition 2.1 ([3]). *Let $A \subset \Pi$. We say that A is relatively dense in Π if there exists a positive number $l \in \Pi$ such that for all $a \in \Pi$ we have $[a, a + l]_\Pi \cap A \neq \emptyset$; here l is called the inclusion length.*

Note the ε-translation number set is relatively dense in Π, rather than in $\mathbb{T}$ because $\mathbb{T} \cap \Pi$ may be an empty set. The concept of almost periodic functions on periodic time scales is as follows:

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Definition 2.2 ([3])**.** *Let $\mathbb{T}$ be a periodic time scale. A function $f \in C(\mathbb{T}\times D, \mathbb{E}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-translation set of f*

$$E\{\varepsilon, f, S\} = \{\tau \in \Pi : |f(t+\tau, x) - f(t, x)| < \varepsilon, \ for\ all\ (t, x) \in \mathbb{T} \times S\}$$

is a relatively dense set in *Π for all $\varepsilon > 0$ and for each compact subset S of D; that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that*

$$|f(t+\tau, x) - f(t, x)| < \varepsilon, \quad for\ all\ t \in \mathbb{T} \times S.$$

Here τ is called the ε-translation number of f and and $l(\varepsilon, S)$ is called the inclusion length of $E\{\varepsilon, f, S\}$.

As mentioned above the author in [1] believes that [3] is unnecessary. In [1, 2] two proposed integral computation rules were used to justify this. We restate these two rules from the original papers:

(i)**:** From [1] (on lines 7-15, page 45), the author wrote:
"... according to our convention,

$$\frac{1}{l}\int_t^{t+l} f(s)\Delta s := \int_t^{\rho(t+l)} f(s)\Delta s \text{ if } t \in \mathbb{T},\ t+l \notin \mathbb{T},\ l \in \mathbb{R}.$$

Hence, the integral $\int_t^{t+l} f(s)\Delta s$ is well defined."

(ii)**:** From [2] (lines 4-6 from the bottom on Page 466), the author wrote
"From now on, for any $a, b \in \mathbb{R}$ and $a \le b$, denote

$$a^* = \inf\{s \in \mathbb{T},\ s \ge a\},\ b^* = \sup\{s \in \mathbb{T},\ s \le b\},$$

and for the integrable function f, denote

$$\int_a^b f(t)\Delta t = \int_{a^*}^{b^*} f(t)\Delta t. \tag{2.1}$$

Obviously, $a^*, b^* \in \mathbb{T}$. If $a \in \mathbb{T}$, then $a^* = a$, and if $b \in \mathbb{T}$, then $b^* = b$."

Note that in [2], all the results are established under rule (2.1) and all integral calculations on time scales obey rule (2.1). Note that (2.1) indicates that for any $a, b \notin \mathbb{T}$, the authors in [2] have found a way so that $\int_a^b f(s)\Delta s$ is reasonable (in fact, as we will see from our knowledge of time scales calculus, this symbolic notation of the integral is false).

First note (i) is incorrect because $t + l \notin \mathbb{T}$, and so $\rho(t + l)$ will make no sense since on a time scale the domain of ρ is $\mathbb{T}$.

The rule in (ii) is incorrect because the measure of sets on time scales is different from the measure of sets on the real line (we note as well that this justifies that the ε-translation number set of an almost periodic function f is relatively dense in Π, rather than in $\mathbb{R}$; as a result Definition 8 from [1] is incorrect). A simple example will show why (ii) is incorrect. Let $f(t) \equiv 3$ and consider the time scale

$$\mathbb{T} = [0, 1] \cup [2, 3] \cup [4, 5] \cup [6, 7].$$

From the calculation rule in [2] we have for $a = \frac{3}{2} \notin \mathbb{T}$, $b = \frac{11}{2} \notin \mathbb{T}$, (see (2.1)) that

$$\int_{\frac{3}{2}}^{\frac{11}{2}} 3\Delta t = \int_{\frac{3}{2}}^{\frac{7}{2}} 3\Delta t + \int_{\frac{7}{2}}^{\frac{11}{2}} 3\Delta t = \int_2^3 3\Delta t + \int_4^5 3\Delta t = 6.$$

Also from (2.1) directly we get

$$\int_{\frac{3}{2}}^{\frac{11}{2}} 3\Delta t = \int_{2}^{5} 3\Delta t = 9.$$

Thus $9 = 6$, a contradicition. In fact, in general the rule is incorrect because when calculating, the measure of some (left or right) scattered points of a time scale is ignored. Under such a rule, the interval additivity of the integral, i.e.,

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$$

will be false. Unfortunately the results in [2] are incorrect. In [2], all upper and lower bounds of the integrals are not in $\mathbb{T}$, i.e, $a, b \notin \mathbb{T}$, and unfortunately (2.1) is used to define $\int_a^b f(s)\Delta s$. However we feel if one uses the ideas in [3] (i.e., the ε-translation number set of various types of almost periodic functions on periodic time scales should be relatively dense in Π, rather than in $\mathbb{R}$ or $\mathbb{T}$), the concept of Stepanove-Almost periodic functions can be corrected successfully and we present this new concept here for the reader.

Let $f : \mathbb{T} \to \mathbb{X}$ be a locally p-integrable function, where $1 \le p < \infty$, and we denote the set of all such functions by $L^p_{Loc}(\mathbb{T}, \mathbb{X})$. Let $\mathbb{T}$ be a periodic time scale and

$$L^p_S(\mathbb{T}, \mathbb{X}) = \left\{ f \in L^p_{Loc}(\mathbb{T}, \mathbb{X}) : \|f\|_{S^p_l} = \sup_{t\in\mathbb{T}} \left(\frac{1}{l} \int_t^{t+l} \|f(s)\|^p \Delta s \right)^{\frac{1}{p}} < \infty, \text{ for fixed } l \in \Pi \right\}.$$

Definition 2.3. *A function $f \in L^p_S(\mathbb{T}, \mathbb{X})$ is said to be Stepanov-Almost periodic function if the ε-translation number set of f*

$$E\{\varepsilon, f\} = \left\{ \tau \in \Pi : \left\| f\big(t+\tau\big) - f(t) \right\|_{S^p_l} < \varepsilon \,\text{for all}\; t \in \mathbb{T} \right\}$$

is relatively dense in *Π, that is, for any given $\varepsilon > 0$, there exists a constant $L(\varepsilon) \in (0, \infty)_\Pi$ such that each interval of length $L(\varepsilon)$ contains a $\tau(\varepsilon) \in E\{\varepsilon, f\}$ such that*

$$\left\| f\big(t+\tau\big) - f(t) \right\|_{S^p_l} < \varepsilon \,\text{for all}\; t \in \mathbb{T},$$

where

$$\left\| f\big(t+\tau\big) - f(t) \right\|_{S^p_l} = \sup_{t\in\mathbb{T}} \left(\frac{1}{l} \int_t^{t+l} \|f(s+\tau) - f(s)\|^p \Delta s \right)^{\frac{1}{p}}.$$

Now τ is called the ε-translation number of f and $L(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, f\}$.

We note as well using Definition 2.1 when investigating almost periodic problems on periodic time scales, one is able to establish the required integrals and one is able to guarantee that they are Cauchy integrals on time scales.

Next we note in [1] that the author claims that some of the results in [4] are incorrect. This claim is established through a counter-example (the following example is from Example 1 in [1] (page 45)):

(*iii*)**:** $\mathbb{T} = \big\{ -2k, 2k+1,\, k \in \mathbb{N} \big\}$. The author claims the concept of changing-periodic time scales introduced in [4] is incorrect because there is no time scale $\hat{\mathbb{T}}$ which is periodic and $\hat{\mathbb{T}} \subset \mathbb{T}$.

Before we discuss (*iii*) we introduce the concept of changing-periodic time scales. For this, we need the following concepts.

Definition 2.4 ([4]). *We say a time scale is an infinite time scale if one of the conditions are satisfied:* $\sup \mathbb{T} = +\infty$ *and* $\inf \mathbb{T} = -\infty$ *or* $\sup \mathbb{T} = +\infty$ *or* $\inf \mathbb{T} = -\infty$.

Remark 2.1. *Under Definition 2.4, it is clear that an infinite time scale indicates it has at least an infinite boundary. For a simple example take* $\mathbb{T} = \bigcup_{k=1}^{+\infty}[2k, 2k+1]$.

Definition 2.5 ([4]). *Let* $\mathbb{T}$ *be a time scale and we say* $\mathbb{T}$ *is a zero-periodic time scale if and only if there exists no nonzero real number* ω *such that* $t + \omega \in \mathbb{T}$ *for all* $t \in \mathbb{T}$.

Remark 2.2. *Note Definition 2.5 indicates that for a zero-periodic time scale,* $t + \omega \in \mathbb{T}$ *for all* $t \in \mathbb{T}$ *if and only if* $\omega = 0$. *Hence, we one can easily see that a finite union of closed intervals is a zero-periodic time scale. For example,* $\mathbb{T} = [0, 1] \cup [2, 3] \cup [4, 5]$ *and* $\mathbb{T} = \bigcup_{k=0}^{100}[2k, 2k + 1]$, *etc.*

Definition 2.6 ([4]). *A time scale sequence* $\{\mathbb{T}_i\}_{i\in\mathbb{Z}^+}$ *is called a well connective sequence if and only if for* $i \neq j$, *one has* $\mathbb{T}_i \cap \mathbb{T}_j = \{t^k_{ij}\}_{k\in\mathbb{Z}}$, *where* $\{t^k_{ij}\}$ *is the countable points set or an empty set, and* t^k_{ij} *is called the connective point between* $\mathbb{T}_i$ *and* $\mathbb{T}_j$ *for each* $k \in \mathbb{Z}$, *and the set* $\{t^k_{ij}\}$ *is called the connective points set of this well connective sequence.*

Remark 2.3. *Under Definition 2.6, we will give some examples to help the reader understand Definition 2.6. We provide a time scale sequence* $\{\mathbb{T}_i\}_{i\in\mathbb{Z}^+}$.

(*a*): *Let* $\mathbb{T}_1 = \bigcup_{k=0}^{\infty}[4k, 4k + 1]$, $\mathbb{T}_2 = \bigcup_{k=0}^{\infty}[4k + 2, 4k + 3]$, $\mathbb{T}_i = \left\{\dfrac{4i-1}{2}\right\}$ *for* $i \geq 3$. *Then* $\mathbb{T}_i \cap \mathbb{T}_j = \emptyset$ *if* $i \neq j$. *According to Definition 2.6, one has* $\{t^k_{ij}\}_{k\in\mathbb{Z}} = \emptyset$, *and then such a time scale sequence is a well connective sequence.*

(*b*): *Let* $\mathbb{T}_1 = \bigcup_{k=0}^{\infty}[2k, 2k + 1]$, $\mathbb{T}_2 = \{1, 2\} \bigcup \left(\bigcup_{k=1}^{\infty}\left\{\dfrac{4k-1}{2}\right\}\right)$, $\mathbb{T}_i = \left\{\dfrac{4i-1}{2} + \dfrac{1}{3}\right\}$ *for* $i \geq 3$. *We can obtain* $\mathbb{T}_1 \cap \mathbb{T}_2 = \{t^k_{12}\}_{k\in\mathbb{Z}} = \{1, 2\}$ *is a countable set. Further,* $\mathbb{T}_1 \cap \mathbb{T}_i = \emptyset$ *and* $\mathbb{T}_2 \cap \mathbb{T}_i = \emptyset$ *for* $i \geq 3$, $\mathbb{T}_i \cap \mathbb{T}_j = \emptyset$ *for* $i, j \geq 3, i \neq j$, *that is,* $\{t^k_{1i}\}_{k\in\mathbb{Z}} = \{t^k_{2i}\}_{k\in\mathbb{Z}} = \{t^k_{ij}\}_{k\in\mathbb{Z}} = \emptyset$ *for* $i \geq 3$, $i \neq j$. *According to Definition 2.6, such a time scale sequence is a well connective sequence.*

(*c*): *Let* $\mathbb{T}_1 = \bigcup_{k=0}^{\infty}[k(1 + a), k(1 + a) + 1]$, *where*

$$a > 0 \text{ and } \sum_{n=1}^{\infty}\frac{1}{n^2} < a.$$

For $i \geq 2$, *let*

$$\mathbb{T}_i = \bigcup_{k=0}^{\infty}\left[k(1 + a) + 1 + \sum_{i_0=2}^{i-1}\frac{1}{i_0^2},\ k(1 + a) + 1 + \sum_{i_0=2}^{i}\frac{1}{i_0^2}\right].$$

One can obtain that

$$\mathbb{T}_1 \cap \mathbb{T}_j = \begin{cases}\{t^k_{12}\}_{k\in\mathbb{Z}} = \bigcup_{k=0}^{\infty}\{k(1 + a) + 1\}, & i = 1, j = 2,\\ \{t^k_{1j}\}_{k\in\mathbb{Z}} = \emptyset, & i = 1, j \neq 2,\end{cases}$$

for $i \neq 1$, $j \neq 2$, we have

$$\mathbb{T}_i \cap \mathbb{T}_j = \begin{cases} \{t_{ij}^k\}_{k\in\mathbb{Z}} = \mathbb{T}_i \cap \mathbb{T}_j = \bigcup_{k=0}^{\infty} \left\{k(1+a)+1+\sum_{i_0=2}^{i} \frac{1}{i_0^2}\right\}, & j = i+1, \\ \{t_{ij}^k\}_{k\in\mathbb{Z}} = \emptyset, \ \ j \neq i+1. \end{cases}$$

According to Definition 2.6, such a time scale sequence is a well connective sequence.

(d): *If $\mathbb{T}_1 = \bigcup_{k=0}^{\infty}[2k, 2k+1]$ and $\mathbb{T}_i = \bigcup_{k=0}^{\infty}[2k+\frac{1}{i}, 2k+1+\frac{1}{i}]$, then for any $i \neq j$, $\mathbb{T}_i \cap \mathbb{T}_j$ is an uncountable set, such a time scale sequence is not a well connective sequence.*

Now, the concept of changing-periodic time scales is introduced as follows:

Definition 2.7 ([4]). *Let $\mathbb{T}$ be an infinite time scale. We say $\mathbb{T}$ is a changing-periodic or a piecewise-periodic time scale if the following conditions are fulfilled:*

(a): *$\mathbb{T} = \left(\bigcup\limits_{i=1}^{\infty} \mathbb{T}_i\right) \bigcup \mathbb{T}_r$ and $\{\mathbb{T}_i\}_{i\in\mathbb{Z}^+}$ is a well connective time scale sequence, where $\mathbb{T}_r = \bigcup\limits_{i=1}^{k}[\alpha_i, \beta_i]$ and k is some finite number and $[\alpha_i, \beta_i]$ are closed intervals for $i = 1, 2, \ldots, k$ or $\mathbb{T}_r = \emptyset$;*

(b): *S_i is a nonempty subsets of $\mathbb{R}$ with $0 \notin S_i$ for each $i \in \mathbb{Z}^+$ and $\Pi = \left(\bigcup_{i=1}^{\infty} S_i\right) \bigcup R_0$, where $R_0 = \{0\}$ or $R_0 = \emptyset$;*

(c): *for all $t \in \mathbb{T}_i$ and all $\omega \in S_i$, we have $t + \omega \in \mathbb{T}_i$, i.e., $\mathbb{T}_i$ is an ω-periodic time scale;*

(d): *for $i \neq j$, for all $t \in \mathbb{T}_i \backslash \{t_{ij}^k\}$ and all $\omega \in S_j$, we have $t + \omega \notin \mathbb{T}$, where $\{t_{ij}^k\}$ is the connective points set of the time scale sequence $\{\mathbb{T}_i\}_{i\in\mathbb{Z}^+}$;*

(e): *$R_0 = \{0\}$ if and only if $\mathbb{T}_r$ is a zero-periodic time scale and $R_0 = \emptyset$ if and only if $\mathbb{T}_r = \emptyset$;*

and the set Π is called a changing-periods set of $\mathbb{T}$, $\mathbb{T}_i$ is called the periodic sub-timescale of $\mathbb{T}$ and S_i is called the periods subset of $\mathbb{T}$ or the periods set of $\mathbb{T}_i$, $\mathbb{T}_r$ is called the remaining time scale of $\mathbb{T}$ and R_0 the remaining periods set of $\mathbb{T}$.

From condition (c) in Definition 2.7, one can observe that $\mathbb{T}_i$ is the periodic time scale attached with translation direction for each $i \in \mathbb{Z}^+$. The condition (c) is $t + \omega \in \mathbb{T}_i$, rather than $t \pm \omega \in \mathbb{T}_i$. Hence, in this concept of changing-periodic time scales, $\mathbb{T}_i$ is a periodic time scale with translation direction. In [5] the authors introduced the concept of periodic time scales attached with translation direction to help a reader understand the decomposition theorem of time scales. The following is the concept of periodic time scales attached with translation direction.

Definition 2.8. *We say $\mathbb{T}$ is called a periodic time scale if*

$$\Pi_2 := \{\tau \in \mathbb{R} : \mathbb{T}^{\tau} \subseteq \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}. \tag{2.2}$$

Furthermore, we can describe it in detail as follows:

(a): *if for any $p > 0$, there exists a number $P > p$ and $P \in \Pi_2$, we say $\mathbb{T}$ is a positive-direction periodic time scale;*

(b): *if for any $q < 0$, there exists a number $Q < q$ and $Q \in \Pi_2$, we say $\mathbb{T}$ is a negative-direction periodic time scale.*

(c): *if $\pm\tau \in \Pi_2$, we say $\mathbb{T}$ is a bi-direction periodic time scale;*

(d): *we say $\mathbb{T}$ is an oriented-direction periodic time scale if $\mathbb{T}$ is a positive-direction periodic time scale or a negative-direction periodic time scale.*

Using Definition 2.8 in [5] some theorems from [4] were proved again.

Theorem 2.1. *If $\mathbb{T}$ is an infinite time scale and the graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ is bounded, then $\mathbb{T}$ is a changing-periodic time scale.*

Theorem 2.2 (Decomposition Theorem of Time Scales)**.** *Let $\mathbb{T}$ be an infinite time scale and the graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ be bounded, then $\mathbb{T}$ is a changing-periodic time scale, i.e., there exists a countable periodic decomposition such that $\mathbb{T} = \left(\bigcup_{i=1}^{\infty} \mathbb{T}_i\right) \bigcup \mathbb{T}_r$ and $\mathbb{T}_i$ is an ω-periodic sub-timescale, $\omega \in S_i$, $i \in \mathbb{Z}^+$, where $\mathbb{T}_i$, S_i, $\mathbb{T}_r$ are satisfied the conditions in Definition 2.7.*

Now let us return to the counter-example in [1], i.e., (*iii*) above. This time scale is the simplest changing-periodic time scale and can be decomposed into two periodic time scales attached with translation direction, i.e., $\mathbb{T}_1 = \{-2k, k \in \mathbb{N}\}$ and $\mathbb{T}_2 = \{2k+1, k \in \mathbb{N}\}$. One can observe that $\mathbb{T}_1$ is a negative-direction periodic time scale and $\mathbb{T}_2$ is a positive-direction periodic time scale. Hence, this counter-example is invalid (note the author in [1] neglected condition (c) in Definition 2.7).

We now provide several examples of changing-periodic time scales.

Example 2.1. *Let $k \in \mathbb{Z}$, and consider the following time scale:*

$$\mathbb{T} = \left\{ \bigcup_{k=-\infty}^{+\infty} \left[\frac{3}{2}(2k+1), \frac{3}{2}(2k+1)+\frac{1}{12}\right]\right\} \bigcup \left\{ \bigcup_{k=-\infty}^{+\infty} \left[\frac{3\sqrt{2}}{2}(2k+1), \frac{3\sqrt{2}}{2}(2k+1)+\frac{\sqrt{3}}{5}\right]\right\}.$$

We denote

$$\mathbb{T}_1 = \bigcup_{k=-\infty}^{+\infty} \left[\frac{3}{2}(2k+1), \frac{3}{2}(2k+1)+\frac{1}{12}\right] \text{ and } \mathbb{T}_2 = \bigcup_{k=-\infty}^{+\infty} \left[\frac{3\sqrt{2}}{2}(2k+1), \frac{3\sqrt{2}}{2}(2k+1)+\frac{\sqrt{3}}{5}\right].$$

Then, by a direct calculation the set Π_2 is

$$\Pi_2 = \{3n, n \in \mathbb{Z}\} \bigcup \{3\sqrt{2}n, n \in \mathbb{Z}\} := S_1 \cup S_2.$$

This time scale is a changing-periodic time scale according to Definition 2.7.

Example 2.2. *Let $k \in \mathbb{Z}$, and consider the following time scale:*

$$\mathbb{T} = \left\{ \bigcup_{k=0}^{+\infty} \left[-\frac{3}{2}(2k+1), -\frac{3}{2}(2k+1)-\frac{1}{12}\right]\right\} \bigcup \left\{ \bigcup_{k=0}^{+\infty} \left[\frac{3\sqrt{2}}{2}(2k+1), \frac{3\sqrt{2}}{2}(2k+1)+\frac{\sqrt{3}}{5}\right]\right\}.$$

We denote

$$\mathbb{T}_1 = \bigcup_{k=0}^{+\infty} \left[-\frac{3}{2}(2k+1), -\frac{3}{2}(2k+1)-\frac{1}{12}\right] \text{ and } \mathbb{T}_2 = \bigcup_{k=0}^{+\infty} \left[\frac{3\sqrt{2}}{2}(2k+1), \frac{3\sqrt{2}}{2}(2k+1)+\frac{\sqrt{3}}{5}\right].$$

Then, by direct calculation the set Π_2 is

$$\Pi_2 = \{-3n, n \in \mathbb{N}\} \bigcup \{3\sqrt{2}n, n \in \mathbb{N}\} := S_1 \cup S_2.$$

According to Definition 2.7, this time scale is a changing-periodic time scale in which $\mathbb{T}_1$ is a negative-direction periodic sub-timescale and $\mathbb{T}_2$ is a positive-direction periodic sub-timescale.

Example 2.3. *Consider $\mathbb{T} = \{-4k, 4k+3 : k \in \mathbb{N}\}$. Note that $\mathbb{T}_1 = \{-4k : k \in \mathbb{N}\}$ and $\mathbb{T}_2 = \{4k+3 : k \in \mathbb{N}\}$ are oriented-direction periodic time scales, i.e., $\mathbb{T}_1$ is a negative-direction periodic time scale and $\mathbb{T}_2$ is a positive-direction periodic time scale. Hence, $\mathbb{T}$ is a changing-periodic time scale.*

Unfortunately there are problems with the definitions proposed in [1]. For example, Definition 4, Definition 5 and Definition 6 are called “almost periodic time scales” but there is no almost periodicity at all on the time scales, and in particular almost periodicity should reflect an approximation between the time scale and its translation (however there is no approximation in these definitions). Also there are problems in [8], as we indicate below.

First we state a result from [5]. Let

$$\Pi_1 := \{\tau \in \mathbb{R} : \mathbb{T} \cap \mathbb{T}^\tau \neq \emptyset\} \neq \{0\},$$

where $\mathbb{T}^\tau := \mathbb{T} + \tau = \{t + \tau : \forall t \in \mathbb{T}\}$.

Theorem 2.3 ([5])**.** *Let $\mathbb{T}$ be an oriented-direction intersection time scale. For any given $\varepsilon > 0$, there exists a constant $l(\varepsilon)$ such that each interval of length $l(\varepsilon)$ contains a $\tau(\varepsilon) \in \Pi_1$ such that*

$$d(\mathbb{T}, \mathbb{T}_\tau) < \varepsilon, \tag{2.3}$$

that is, for any $\varepsilon > 0$, the following set

$$E\{\mathbb{T}, \varepsilon\} := \{\tau \in \Pi_1 : d(\mathbb{T}, \mathbb{T}_\tau) < \varepsilon\} \tag{2.4}$$

is relatively dense in Π_1, where $d(\cdot,\cdot)$ denote a Hausdorff distance and $\mathbb{T}_\tau = \mathbb{T} \cap \mathbb{T}^\tau$. Then $\mathbb{T}$ is an oriented-direction periodic time scale.

In [7], the concept of almost periodic time scales was introduced to study the approximation of time scales. In [1] the author introduced a similar concept of almost periodic time scales (see Definition 9 in [8]), but according to Theorem 2.3 above, Definition 9 in [8] is in fact a oriented periodic time scale. Also some definitions and concepts are also incorrect in [8]. For example, Definition 16 from [8] is actually a periodic function on a periodic time scale because the time scale and the function have a common ε. Thus, if $\varepsilon = 0$, then $d(\mathbb{T}, \mathbb{T}^\tau) = 0 \Leftrightarrow \mathbb{T} = \mathbb{T}^\tau$, and $f(t + \tau, x) = f(t, x)$. In the next section we present some incorrect remarks fron [1, 2, 8] and we indicate some corrections.

3. Some incorrect remarks and corrections

In [1, 2, 8], the authors presented some incorrect comments and in this section we discuss and correct them.

Remark 3.1 (**Incorrect Remark 3 from** [1])**.** *From the above, we see that if we adopt Definition 8 from* [1] *as the definition of almost periodic functions on time scales, all the results of* [6] *remain true.*

Definition 3.1 (**Incorrect Definition 8 from** [1])**.** *Let $\mathbb{T}$ be a periodic time scale. A function $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-translation set of f*

$$E\{\varepsilon, f, S\} = \{\tau \in \Pi : |f(t + \tau, x) - f(t, x)| < \varepsilon,\ for\ all\ (t, x) \in \mathbb{T} \times S\}$$

is a relatively dense set in *$\mathbb{R}$ for all $\varepsilon > 0$ and for each compact subset S of D; that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that*

$$|f(t + \tau, x) - f(t, x)| < \varepsilon, \quad for\ all\ t \in \mathbb{T} \times S.$$

τ is called the ε-translation number of f and and $l(\varepsilon, S)$ is called the inclusion length of $E\{\varepsilon, f, S\}$.

Remark 3.2. *Definition 3.1 arose from the incorrect computation rules* (*i*) *and* (*ii*) *in Section 2. The correct definition, in our opinion, should be Definition 2.2 above.*

Theorem 3.1 (**Incorrect Theorem 3.21 from** [6])**.** *If* $f(t,x)$ *is almost periodic in* t *uniformly for* $x \in D$*, then, for any* $\varepsilon > 0$*, there exists a positive constant* $L = L(\varepsilon, S)$ *and for any* $a \in \mathbb{R}$*, there exist a constant* $\eta > 0$ *and* $\alpha \in \mathbb{R}$*, such that* $([\alpha, \alpha+\eta] \cap \Pi) \subset [a, a+L]$ *and* $([\alpha, \alpha+\eta] \cap \Pi]) \subset E(\varepsilon, f, S)$*.*

Remark 3.3. *For Theorem 3.1, because of the interval* $[a, a+L]$*, where* $a, L \in \mathbb{R}$*, the integral*

$$\int_{t+a_1}^{t+a_2} f(s,x)\Delta s \quad \text{makes no sense for } \forall t \in \mathbb{T} \text{ and } \forall a_1, a_2 \in [a,\, a+L].$$

In fact, by adopting Definition 2.1 and Definition 2.2, we can provide the following correct theorem:

Theorem 3.2 (**Correction of Theorem 3.21 from** [6])**.** *If* $f(t,x)$ *is almost periodic in* t *uniformly for* $x \in D$*, then, for any* $\varepsilon > 0$*, there exists a positive constant* $L \in \Pi$ *such that for any* $a \in \Pi$*, there exist* $\eta \in (0, +\infty)_\Pi$ *and* $\alpha \in \Pi$*, such that* $([\alpha, \alpha+\eta]_\Pi) \subset [a, a+L]_\Pi$ *and* $([\alpha, \alpha+\eta]_\Pi]) \subset E(\varepsilon, f, S)$*.*

Remark 3.4. *Now, for all* $a_1, a_2 \in [a, a+L]_\Pi$*, the integral* $\int_{t+a_1}^{t+a_2} f(s,x)\Delta s$ *is a Cauchy integral on time scales with* $t+a_1, t+a_2 \in \mathbb{T}$*. The proof of Theorem 3.2 is easy using Definition 2.1.*

Remark 3.5. *Theorem 3.22 from* [6] *is based on Theorem 3.21 and Theorem 3.22 which refers to the product and quotient of two almost periodic functions. If you do not use the corrections in* [3]*, then some mean-value integrals for almost periodic functions may make no sense on periodic time scales. These mistakes continued in* [2] *because they say* $E\{\varepsilon, f\}$ *is relatively dense in* $\mathbb{R}$*, rather than in* Π*. For example there are mistakes on page 467, in the proof of Lemma 2.11,*

$$\int_{t_0-\delta}^{t_0-\delta+l} \ldots \Delta s,\, t_0 \in \mathbb{T},\, \delta, l \in \mathbb{R},\, t_0-\delta, t_0-\delta+l \in \mathbb{R},$$

and on page 469, in the proof of Lemma 3.4,

$$\int_{t-k}^{t-k+l} \ldots \Delta s,\;\; t \in \mathbb{T},\, k \in \mathbb{Z},\, l \in \mathbb{R},\, t-k, t-k+l \in \mathbb{R},$$

etc. Note some authors use the results in [6] *(note corrections are needed as pointed out in* [3]*) which lead to mistakes.*

Remark 3.6 (**Incorrect Remark 4 from** [1])**.** *Example 1 shows that there exists a time scale that satisfies all the conditions of Theorem 1, but it contains no sub time scale that is an invariant under a translation time scale. Therefore, Theorem 1 is incorrect.*

Remark 3.7. *Note the time scale in Example 1 from* [1] *is a changing-periodic time scale and can be decomposed into two periodic time scales attached with translation direction, i.e.,* $\mathbb{T}_1 = \{-2k, k \in \mathbb{N}\}$ *and* $\mathbb{T}_2 = \{2k+1, k \in \mathbb{N}\}$*. Note* $\mathbb{T}_1$ *is a negative-direction periodic time scale and* $\mathbb{T}_2$ *is a positive-direction periodic time scale. Hence, this counter-example is invalid. Therefore, Remark 5, Remark 6, Remark 7, Remark 8 from* [1] *are all incorrect.*

From [5], we recall the following theorem:

Theorem 3.3. *Let $\mathbb{T}$ be an arbitrary time scale with $\sup \mathbb{T} = +\infty$, $\inf \mathbb{T} = -\infty$. If $\mu : \mathbb{T} \to \mathbb{R}^+$ is bounded, then $\mathbb{T}$ contains at least one oriented-direction periodic time scale.*

Remark 3.8. *It is easy to see that $\{-2k, 2k+1, k \in \mathbb{N}\}$ contains two periodic time scales attached with translation direction, i.e., $\mathbb{T}_1 = \{-2k, k \in \mathbb{N}\}$ and $\mathbb{T}_2 = \{2k+1, k \in \mathbb{N}\}$.*

Remark 3.9 (**Incorrect Remark 9 from** [1])**.** *Since the fact that $\mathbb{T}$ is an almost periodic time scale under Definition 15 may do not guarantee that the set $\{\tau \in \Pi_\varepsilon : \mathbb{T} \cap \mathbb{T}^\tau \neq \emptyset\}$, is relatively dense. Therefore, Definition 16 is not well defined. A correction for this, we refer to* [8].

To discuss this we recall a result from [5]. Let

$$\Pi_1 := \{\tau \in \mathbb{R} : \mathbb{T} \cap \mathbb{T}^\tau \neq \emptyset\} \neq \{0\},$$

where $\mathbb{T}^\tau := \mathbb{T} + \tau = \{t + \tau : \forall t \in \mathbb{T}\}$.

Definition 3.2 ([5])**.** *Let X and Y be two non-empty subsets of a metric space (M, d). We define their Hausdorff distance $d(X, Y)$ by*

$$d(X,Y) = \max\left\{ \sup_{x\in X} \inf_{y\in Y} \tilde{d}(x,y), \sup_{y\in Y} \inf_{x\in X} \tilde{d}(x,y) \right\}, \tag{3.5}$$

where $\tilde{d}(\cdot,\cdot)$ denotes the distance between two points.

Hence, from Definition 3.2, if we assume that $X = \mathbb{T}_1$ and $Y = \mathbb{T}_2$, then

$$d(\mathbb{T}_1, \mathbb{T}_2) = \max\left\{ \sup_{t\in \mathbb{T}_1} \inf_{s\in \mathbb{T}_2} \tilde{d}(t,s), \sup_{s\in \mathbb{T}_2} \inf_{t\in \mathbb{T}_1} \tilde{d}(t,s) \right\}.$$

In [7], the authors let τ be a number and set the time scales:

$$\mathbb{T} := \bigcup_{i=-\infty}^{+\infty} [\alpha_i, \beta_i], \qquad \mathbb{T}^\tau := \mathbb{T} + \tau = \{t + \tau : \forall t \in \mathbb{T}\} := \bigcup_{i=-\infty}^{+\infty} [\alpha_i^\tau, \beta_i^\tau]. \tag{3.6}$$

Define the distance between two time scales, $\mathbb{T}$ and $\mathbb{T}^\tau$ by

$$d(\mathbb{T}, \mathbb{T}^\tau) = \max\left\{ \sup_{i\in\mathbb{Z}} |\alpha_i - \alpha_i^\tau|, \sup_{i\in\mathbb{Z}} |\beta_i - \beta_i^\tau| \right\}, \tag{3.7}$$

where

$$\alpha_i^\tau := \inf\left\{\alpha \in \mathbb{T}^\tau : |\alpha_i - \alpha|\right\} \text{ and } \beta_i^\tau := \inf\left\{\beta \in \mathbb{T}^\tau : |\beta_i - \beta|\right\}. \tag{3.8}$$

Note if we let $X = \mathbb{T}$ and $Y = \mathbb{T}^\tau$ in Definition 3.2, then (3.5) immediately turns into (3.7) and we can calculate the distance between $\mathbb{T}$ and $\mathbb{T}^\tau$ from the distance of their intervals, i.e, from formula (3.7).

Definition 3.3 ([5])**.** *Let $\mathbb{T}$ be an oriented-direction intersection time scale. We say $\mathbb{T}$ is an almost periodic time scale if for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains a $\tau(\varepsilon) \in \Pi_1$ such that*

$$d(\mathbb{T}, \mathbb{T}^\tau) < \varepsilon,$$

i.e., for any $\varepsilon > 0$, the following set

$$E\{\mathbb{T}, \varepsilon\} = \{\tau \in \Pi_1 : d(\mathbb{T}^\tau, \mathbb{T}) < \varepsilon\}$$

is relatively dense in Π_1. *Here* τ *is called the* ε*-translation number of* $\mathbb{T}$ *and* $l(\varepsilon)$ *is called the inclusion length of* $E\{\mathbb{T},\varepsilon\}$, $E\{\mathbb{T},\varepsilon\}$ *is called the* ε*-translation set of* $\mathbb{T}$, *and for simplicity, we use the notation* $E\{\mathbb{T},\varepsilon\} := \Pi_\varepsilon$.

Remark 3.10. *In* [5], *the authors discuss almost periodic time scales when* $\mathbb{T}\cap\mathbb{T}^\tau \neq \emptyset$ *can be guaranteed. Definition 8 from* [7] *is also correct. Some comments and some results in* [8] *are incorrect.*

To discuss Remark 3.9, we need to consider some comments and definitions from [8].

Definition 3.4 (**Incorrect Definition 9 from** [8])**.** *A time scale* $\mathbb{T}$ *is called an almost periodic time scale if for every* $\varepsilon > 0$, *there exists a constant* $l(\varepsilon) > 0$ *such that each interval of length* $l(\varepsilon)$ *contains a* $\tau(\varepsilon)$ *such that* $\mathbb{T}_\tau \neq$ *and* $\operatorname{dist}(\mathbb{T},\mathbb{T}_\tau) < \varepsilon$; *that is, for any* $\varepsilon > 0$, *the following set* $\Pi(\mathbb{T},\varepsilon) = \{\tau \in \mathbb{R} : \operatorname{dist}(\mathbb{T},\mathbb{T}_\tau) < \varepsilon\}$ *is relatively dense.*

Remark 3.11. *Note Definition 3.4 is a slight modification of Definition 8 in* [7], *i.e, the authors changed* $\mathbb{T}^\tau$ *from Definition 8 in* [7] *to* $\mathbb{T}\cap\mathbb{T}^\tau := \mathbb{T}_\tau$ *and added the condition* $\mathbb{T}_\tau \neq \emptyset$. *From Theorem 2.3 above we see that Definition 3.4 is actually a oriented periodic time scales, i.e., it is with no almost periodicity. When discussing almost periodic time scales, one should take the translation* $\mathbb{T}^\tau$ *of* $\mathbb{T}$, *rather than* $\mathbb{T}_\tau$, *because* $\mathbb{T}_\tau$ *will lead to the periodicity of the time scale (see* [5] *for details). Lemma 10, Lemma 11, Theorem 12, Theorem 13, Theorem 14 from* [8] *are incorrect.*

Remark 3.12. *Based on Definition 3.4, all theorems related to it from* [8] *are incorrect.*

Definition 3.5 (**Incorrect Definition 16 from** [8])**.** *Let* $\mathbb{T}$ *be an almost periodic time scale. A function* $f \in C(\mathbb{T}\times D,\mathbb{E}^n)$ *is called an almost periodic function in* $t\in\mathbb{T}$ *uniformly for* $x \in D$ *if the* ε*-translation set of* f

$$E\{\varepsilon,f,S\} = \{\tau \in \Pi(\mathbb{T},\varepsilon) : |f(t+\tau,x) - f(t,x)| < \varepsilon,\ \forall (t,x) \in \mathbb{T}_\tau \times S\}$$

is relatively dense for all $\varepsilon > 0$ *and for each compact subset* S *of* D.

Remark 3.13. *Note that there are two* ε *in Definition 3.5. One is in* $\Pi(\mathbb{T},\varepsilon)$, *the other is in* $|f(t+\tau,x) - f(t,x)| < \varepsilon$. *The two* ε*'s are equivalent in this definition. Hence, time scales defined in Definition 3.4 are periodic time scales, and then* $\varepsilon = 0$. *Thus, one has* $f(t+\tau,x) = f(t,x)$, *which implies* $f(t,x)$ *is periodic in* t. *This is incorrect. Therefore, Theorem 17, Theorem 19, Theorem 20, Definition 21 from* [8] *are incorrect.*

Definition 3.6 ([5, 9])**.** *Let* $\mathbb{T}$ *be an almost periodic time scale under Definition 3.3, i.e.,* $\mathbb{T}$ *satisfies Definition 3.3. A function* $f \in C(\mathbb{T}\times D,\mathbb{E}^n)$ *is called an almost periodic function in* $t\in\mathbb{T}$ *uniformly for* $x\in D$ *if the* ε_2*-translation set of* f

$$E\{\varepsilon_2,f,S\} = \{\tau \in \Pi_{\varepsilon_1} : |f(t+\tau,x) - f(t,x)| < \varepsilon_2,\ \text{for all } (t,x) \in (\mathbb{T}\cap\mathbb{T}^{-\tau}) \times S\}$$

is a relatively dense set in Π_{ε_1} *for all* $\varepsilon_2 > \varepsilon_1 > 0$ *and for each compact subset* S *of* D; *that is, for any given* $\varepsilon_2 > \varepsilon_1 > 0$ *and each compact subset* S *of* D, *there exists a constant* $l(\varepsilon_2,S) > 0$ *such that each interval of length* $l(\varepsilon_2,S)$ *contains a* $\tau(\varepsilon_2,S) \in E\{\varepsilon_2,f,S\}$ *such that*

$$|f(t+\tau,x) - f(t,x)| < \varepsilon_2, \quad \text{for all } (t,x) \in (\mathbb{T}\cap\mathbb{T}^{-\tau}) \times S.$$

This τ *is called the* ε_2*-translation number of* f *and* $l(\varepsilon_2,S)$ *is called the inclusion length of* $E\{\varepsilon_2,f,S\}$.

Remark 3.14. *In* [9] *the authors discussed the approximation between an almost periodic function and its translation function on almost periodic time scales through the analysis of the relationship between ε_1 and ε_2, where ε_1 is for the time scale and ε_2 is for the function (note $\varepsilon_1, \varepsilon_2$ may be different). Unfortunately the author in* [1] *presented some incorrect comments in relation to this. Remark 3.9 is incorrect.*

Definition 3.7 (**Incorrect Definition 22 from** [8])**.** *Let $\mathbb{T}$ be an almost periodic time scale. For any $t \in \mathbb{T}$, $\tau \in \Pi(\mathbb{T}, \varepsilon)$, we define*

$$t\tilde{+}\tau = \begin{cases} t+\tau, & \text{if } t+\tau \in \mathbb{T}, \\ t^*+\tau, & \text{if } t+\tau \notin \mathbb{T}, \end{cases}$$

where $t^* \in \mathbb{T}_\tau$ satisfies that $\operatorname{dist}(t, \mathbb{T}_\tau) = |t - t^*| < \varepsilon$ and $(t - t^*)\operatorname{sign}(\tau) > 0$.

Remark 3.15. *Because $\mathbb{T}$ is oriented periodic time scale, Definition 3.7 is incorrect. Even if $\mathbb{T}$ is an arbitrary time scale with $\mathbb{T} \cap \mathbb{T}^\tau \neq \emptyset$, it is also incorrect because t^* from Definition 3.7 is not unique. Hence, $|f(t^* + \tau) - f(t)|$ makes no sense and $t^* + \tau$ is not the translation of t. Thus, Definition 22 is incorrect.*

Definition 3.8 (**Incorrect Definition 25 from** [8])**.** *A time scale $\mathbb{T}$ called an almost periodic time scale if $\Pi := \{\tau \in \mathbb{R} : \mathbb{T}_\tau \neq \emptyset\}$ is relatively dense in $\mathbb{R}$, where $\mathbb{T}_\tau = \mathbb{T} \cap \{\mathbb{T} - \tau\}$ or $\mathbb{T}_\tau = \mathbb{T} \cap \{\mathbb{T} \pm \tau\}$.*

Remark 3.16. *Definition 3.8 is incorrect. First, there is no almost periodicity of the time scale in this definition. Second, $\mathbb{T}_\tau$ depends on τ, i.e., $\mathbb{T}_\tau$ is different if τ is different. For such a time scale, there is no translation invariance and almost translation invariance or some special property that can support any function and its Δ-derivatives on $\mathbb{T}_\tau$, which indicates that Definition 3.8 has major problems.*

Remark 3.17 (**Incorrect Remark 10 from** [1])**.** *If we take $\mathbb{T} = \mathbb{Z}$, then $\mathbb{T} \cap \mathbb{T}^\tau = \emptyset$ for $\tau \in \{\tau \in \mathbb{R} : \varepsilon^* < d(\mathbb{T}, \mathbb{T}^\tau) < \varepsilon_1 < 1\}$. Therefore, Definition 17 is not well defined.*

Remark 3.18. *$\mathbb{Z}$ is a periodic time scale so Remark 3.17 is incorrect since $\varepsilon_1 = 0$. The condition $\mathbb{T} \cap \mathbb{T}^\tau \neq \emptyset$ was emphasized in Remark 6.1 in* [9] *(see the following remark).*

Remark 3.19 (Remark 6.1 from [9])**.** *Because $\mathbb{T}$ is an almost periodic time scale and not periodic, we emphasize that in Definition 6.2, there is a $\tau \in \Pi_{\varepsilon_1}$ such that $\mathbb{T} \cap \mathbb{T}^{-\tau} \neq \emptyset$. From these concepts of almost periodic functions on almost periodic time scales, we notice that there is a condition that τ must satisfy $\mathbb{T} \cap \mathbb{T}^{-\tau} \neq \emptyset$.*

Remark 3.20. *The author in* [1] *seems to have ignored Remark 3.19 and made an incorrect comment (i.e., Remark 3.17) .*

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