MODIFIED BANACH FIXED POINT RESULTS FOR LOCALLY CONTRACTIVE MAPPINGS IN COMPLETE G_d-METRIC LIKE SPACE

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ABSTRACT. In this paper we discuss unique fixed point of mappings satisfying a locally contractive condition on a closed ball in a complete G_d-metric like space. Examples have been given to show the novelty of our work. Our results improve/generalize several well known recent and classical results.

1. INTRODUCTION

The notion of metric spaces is one of the useful topic in Analysis. The study of metric spaces expressed the most important role to many fields both in pure and applied science such as biology, physics and computer science. Some generalizations of the notion of a metric space have been proposed by some authors, such as rectangular metric spaces, metric Like Space, quasi metric spaces. Mustafa et. al. [27] introduced the notion of a G-metric space as generalization of the metric space.

A point $a \in X$ is said to be a fixed point of mapping $\Gamma : X \to X$, if $a = \Gamma a$. Karanpınar et. al. [17] proved fixed point theorems for globally contractive mappings in G-metric spaces. Recently, many results appeared related to fixed point theorem for mappings satisfying different contractive conditions in complete G-metric spaces and metric like spaces/dislocated metric spaces (see [1]-[38]). Arshad et. al. [5] proved a result concerning the existence of fixed points of a mapping satisfying a contractive conditions on closed ball in a complete dislocated metric space. For further results on closed ball (see [6, 9, 10, 11, 31, 32, 33]).

In this paper we have obtained fixed point theorems for a locally contractive self mapping in a complete G-metric like space on a closed ball to generalize, extend and improve some classical fixed point results. We have used weaker contractive condition and weaker restrictions to obtain unique fixed point. We give the following definitions which will be needed in the sequel.

Definition 1 [27] Let X be a nonempty set, and let $G : X \times X \times X \to [0, \infty)$, be a function satisfying the following properties:

(G_1) $G(a, b, c) = 0$ if and only if $a = b = c$;

(G_2) $0 < G(a, a, b)$, for all $a, b \in X$, with $a \neq b$;

(G_3) $G(a, a, b) \leq G(a, b, c)$, for all $a, b, c \in X$ with $b \neq c$;

(G_4) $G(a, b, c) = G(a, c, b) = G(b, a, c) = G(b, c, a) = G(c, a, b) = G(c, b, a)$, (symmetry in all three variables);

(G_5) $G(a, b, c) \leq G(a, d, d) + G(d, b, c)$, for all $a, b, c, d \in X$, (rectangle inequality).

Then the function G is called a Generalized Metric, more specifically a G-metric on X and the pair (X, G) is called a G-metric space. It is known that the function $G_d(x, y, z)$ on G-metric space X is jointly continuous in all three of its variables.

Definition 2 [27] Let X be a nonempty set and let $G_d : X \times X \times X \to [0, \infty)$ be a function satisfying the following axioms:

(i) If $G_d(a, b, c) = G_d(a, c, b) = G_d(b, a, c) = G_d(b, c, a,) = G_d(c, a, b) = G_d(c, b, a) = 0$, then $a = b = c$;

(ii) $G_d(a, b, c) \leq G_d(a, a, d) + G_d(d, b, c)$, for all $a, b, c, d \in X$, (rectangle inequality).

Then the pair (X, G_d) is called the quasi G_d-metric like space. It is clear that if $G_d(a, b, c) = G_d(b, c, a) = G_d(c, a, b) = \cdots = 0$ then from (i) $a = b = c$. But if $a = b = c$ then $G_d(a, b, c)$ may not be 0. It is observed that if $G_d(a, b, c) = G_d(a, c, b) = G_d(b, a, c) = G_d(b, c, a) = G_d(c, a, b) = G_d(c, b, a)$ for all $a, b, c \in X$, then (X, G_d) becomes a G_d-metric like space.

Example 1 Let $X = [0, \infty)$ be a non empty set and $G_d : X \times X \times X \to [0, \infty)$ be a function defined by

$$G_d(a, b, c) = a + b + c, \text{ for all } a, b, c \in X.$$

Then clearly $G_d : X \times X \times X \to [0, \infty)$ is G_d-metric like space.

Proposition 1 Let (X, G_d) be a G_d-metric like space. Then the function $G_d(a, b, c)$ is jointly continuous in all three variables.

Definition 3 Let (X, G_d) be a G_d-metric like space, and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G_d(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G_d-convergent to x. Thus, if $x_n \to x$ in a G_d-metric like space (X, G_d), then for any $\epsilon > 0$, there exists $n, m \in N$ such that $G_d(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.

Definition 4 Let (X, G_d) be a G_d-metric like space. A sequence $\{x_n\}$ is called G_d-Cauchy sequence if, for $\epsilon > 0$ there exists a positive integer $n^\star \in N$ such that $G_d(x_n, x_m, x_l) < \epsilon$ for all $n, l, m \geq n^\star$; i.e. if $G_d(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 5 A G_d-metric like space (X, G_d) is said to be G_d-complete if every G_d-Cauchy sequence in (X, G_d) is G_d-convergent in X.

Proposition 2 Let (X, G_d) be a G_d-metric like space, then the following are equivalent:

(1) $\{x_n\}$ is G_d convergent to x.

(2) $G_d(x_n, x_n, x) \to 0$ as $n \to \infty$.

(3) $G_d(x_n, x, x) \to 0$ as $n \to \infty$.

(4) $G_d(x_n, x_m, x) \to 0$ as $m \; n \to \infty$.

Definition 6 Let (X, G_d) be a G_d-metric like space then for $x_0 \in X$, $r > 0$, the

G_d-closed ball with centre x_0 and radius r is,

$$\overline{B_{G_d}(x_0, r)} = \{y \in X : G_d(x_0, y, y) \leq r\}.$$

2. Main Result

Theorem 1 Suppose for a complete G_d-metric like space (X, G_d), if a mapping $\Gamma : X \longrightarrow X$ satisfies,

$$G_d(\Gamma a, \Gamma b, \Gamma c) \leq \xi W(a, b, c) \tag{1}$$

for all $a, a_0, b, c \in \overline{B_{G_d}(a_0, r)} \subseteq X$ and $r > 0$, where $\xi \in [0, \frac{1}{2})$ and

$$
\begin{aligned}
W(a, b, c) &= \max\{G_d(b, \Gamma^2 a, \Gamma b), G_d(\Gamma a, \Gamma^2 a, \Gamma b), G_d(a, \Gamma a, b), G_d(a, \Gamma a, c),\\
&\quad G_d(c, \Gamma^2 a, \Gamma c), G_d(b, \Gamma a, \Gamma b), G_d(\Gamma a, \Gamma^2 a, \Gamma c), G_d(c, \Gamma a, \Gamma b),\\
&\quad G_d(a, b, c), G_d(a, \Gamma a, \Gamma a), G_d(b, \Gamma b, \Gamma b), G_d(c, \Gamma c, \Gamma c),\\
&\quad G_d(a, \Gamma b, \Gamma b), G_d(b, \Gamma c, \Gamma c)\}.
\end{aligned} \tag{2}
$$

Also

$$G_d(a_0, a_1, a_1) \leq (1 - \rho)r, \tag{3}$$

where $\rho \in \{\xi, \Upsilon = \frac{\xi}{1-\xi}\}$ and $\rho \in [0, 1)$. Then, there exists a unique $a \in \overline{B_{G_d}(a_0, r)}$ such that $\Gamma a = a$.

Proof. Consider a picard sequence $\{a_n\}$ with initial guess $a_0 \in X$ such that

$$a_{n+1} = \Gamma a_n, \text{ for all } n \in N.$$

Suppose $a_{n+1} \neq a_n$, for all $n \in N \cup \{0\}$, for otherwise, if such n exists, then a_n is the fixed point of Γ. From (3), it is clear that

$$G_d(a_0, a_1, a_1) \leq (1 - \rho)r \leq r.$$

Then, $a_1 \in \overline{B_{G_d}(a_0, r)}$. Now, consider the relation

$$
\begin{aligned}
G_d(a_1, a_2, a_2) &= G_d(\Gamma a_0, \Gamma a_1, \Gamma a_1)\\
&\leq \xi W(a_0, a_1, a_1).
\end{aligned} \tag{4}
$$

From (2),

$$
\begin{aligned}
W(a_0, a_1, a_1) &= \max\{G_d(a_0, a_1, a_1), G_d(a_1, a_2, a_2), G_d(a_1, a_1, a_2),\\
&\quad G_d(a_0, a_2, a_2)\}.
\end{aligned}
$$

In first case, if $W(a_0, a_1, a_1) = G_d(a_1, a_2, a_2)$, then, inequality (4) implies

$$
\begin{aligned}
G_d(a_1, a_2, a_2) &\leq \xi G_d(a_1, a_2, a_2)\\
(1 - \xi)G_d(a_1, a_2, a_2) &\leq 0
\end{aligned}
$$

It is contradiction because $a_1 \neq a_2$. In second case, if $W(a_0, a_1, a_1) = G_d(a_1, a_1, a_2)$, then, we have

$$
\begin{aligned}
G_d(a_1, a_2, a_2) &\leq \xi G_d(a_1, a_1, a_2)\\
&\leq 2\xi G_d(a_1, a_2, a_2)\\
(1 - 2\xi)G_d(a_1, a_2, a_2) &\leq 0.
\end{aligned}
$$

It is again a contradiction because $a_1 \neq a_2$. In third case, if $W(a_0, a_1, a_1) = G_d(a_0, a_1, a_1)$, then, we have

$$G_d(a_1, a_2, a_2) \leq \xi G_d(a_0, a_1, a_1). \tag{5}$$

In fourth case, if $W(a_0, a_1, a_1) = G_d(a_0, a_2, a_2)$ then,

$$
\begin{aligned}
G_d(a_1, a_2, a_2) &\leq \xi G_d(a_0, a_2, a_2) \\
&\leq \xi G_d(a_0, a_1, a_1) + \xi G_d(a_1, a_2, a_2) \\
(1 - \xi) G_d(a_1, a_2, a_2) &\leq \xi G_d(a_0, a_1, a_1) \\
G_d(a_1, a_2, a_2) &\leq \Upsilon G_d(a_0, a_1, a_1).
\end{aligned} \tag{6}
$$

Hence, by combining (5) and (6), for $\rho \in \{\xi, \Upsilon\}$, we have

$$
G_d(a_1, a_2, a_2) \leq \rho G_d(a_0, a_1, a_1).
$$

Now, by rectangle property, we have

$$
\begin{aligned}
G_d(a_0, a_2, a_2) &\leq G_d(a_0, a_1, a_1) + G_d(a_1, a_2, a_2) \\
&\leq (1 + \rho) G_d(a_0, a_1, a_1) \\
&\leq (1 + \rho)(1 - \rho)r = (1 - \rho^2)r \leq r
\end{aligned}
$$

Hence, $a_2 \in \overline{B_{G_d}(a_0, r)}$. Now, let $a_3, a_{4,\ldots\ldots}, a_i \in \overline{B_{G_d}(a_0, r)}$, for some $i \in N$. Now, from (1), we have

$$
G_d(a_i, a_{i+1}, a_{i+1}) = G_d(\Gamma a_{i-1}, \Gamma a_{i,} \Gamma a_i) \leq \xi W(a_{i-1}, a_{i,} a_i).
$$

From (2),

$$
\begin{aligned}
W(a_{i-1}, a_{i,} a_i) &= \max\{G_d(a_i, a_{i+1}, a_{i+1}), G_d(a_{i-1}, a_{i+1}, a_{i+1}), G_d(a_{i-1}, a_{i,} a_i), \\
&\quad G_d(a_i, a_{i,} a_{i+1})\}.
\end{aligned}
$$

In first case, if $W(a_{i-1}, a_{i,} a_i) = G_d(a_i, a_{i+1}, a_{i+1})$ then, we have

$$
\begin{aligned}
G_d(a_i, a_{i+1}, a_{i+1}) &\leq \xi G_d(a_i, a_{i+1}, a_{i+1}) \\
(1 - \xi) G_d(a_i, a_{i+1}, a_{i+1}) &\leq 0.
\end{aligned}
$$

It is contradiction because $a_i \neq a_{i+1}$. In second case if $W(a_{i-1}, a_{i,} a_i) = G_d(a_i, a_{i,} a_{i+1})$ then,

$$
\begin{aligned}
G_d(a_i, a_{i+1}, a_{i+1}) &\leq \xi G_d(a_i, a_{i,} a_{i+1}) \\
&\leq \xi G_d(a_i, a_{i+1}, a_{i+1}) + \xi G_d(a_{i+1}, a_i, a_{i+1}).
\end{aligned}
$$

By symmetry property of G_d-metric like space, we have $G_d(a_i, a_{i+1}, a_{i+1}) = G_d(a_{i+1}, a_i, a_{i+1})$. Therefore

$$
\begin{aligned}
G_d(a_i, a_{i+1}, a_{i+1}) &\leq 2\xi G_d(a_i, a_{i+1}, a_{i+1}) \\
(1 - 2\xi) G_d(a_i, a_{i+1}, a_{i+1}) &\leq 0.
\end{aligned}
$$

It is again a contradiction because $a_i \neq a_{i+1}$. In third case, if $W(a_{i-1}, a_{i,} a_i) = G_d(a_{i-1}, a_{i+1}, a_{i+1})$, then

$$
\begin{aligned}
G_d(a_i, a_{i+1}, a_{i+1}) &\leq \xi G_d(a_{i-1}, a_{i+1}, a_{i+1}) \\
&\leq \xi G_d(a_{i-1}, a_{i,} a_i) + \xi G_d(a_i, a_{i+1}, a_{i+1}) \\
(1 - \xi) G_d(a_i, a_{i+1}, a_{i+1}) &\leq \xi G_d(a_{i-1}, a_{i,} a_i) \\
G_d(a_i, a_{i+1}, a_{i+1}) &\leq \Upsilon G_d(a_{i-1}, a_{i,} a_i).
\end{aligned} \tag{7}
$$

In fourth case if $W(a_{i-1}, a_{i,} a_i) = G_d(a_{i-1}, a_{i,} a_i)$, then

$$
G_d(a_i, a_{i+1}, a_{i+1}) \leq \xi G_d(a_{i-1}, a_{i,} a_i) \tag{8}
$$

Hence, from relations (7) and (8), we have

$$
\begin{aligned}
G_d(a_i, a_{i+1}, a_{i+1}) &\leq \rho G_d(a_{i-1}, a_i, a_i) \\
&\leq \rho^2 G_d(a_{i-2}, a_{i-1}, a_{i-1}) \\
&\vdots \\
G_d(a_i, a_{i+1}, a_{i+1}) &\leq \rho^i G_d(a_0, a_1, a_1),
\end{aligned}
\tag{9}
$$

where $\rho \in \{\xi, \Upsilon = \frac{\xi}{1-\xi}\}$. Now from rectangle property, we have

$$
\begin{aligned}
G_d(a_0, a_{i+1}, a_{i+1}) &\leq G_d(a_0, a_1, a_1) + G_d(a_1, a_2, a_2) + ... + G_d(a_i, a_{i+1}, a_{i+1}) \\
G_d(a_0, a_{i+1}, a_{i+1}) &\leq G_d(a_0, a_1, a_1) + \rho G_d(a_0, a_1, a_1) + \rho^2 G_d(a_0, a_1, a_1) \\
&\quad +..... + \rho^i G_d(a_0, a_1, a_1) \\
&\leq (1 + \rho + \rho^2 + ... + \rho^i) G_d(a_0, a_1, a_1) \\
&\leq (\frac{1 - \rho^{i+1}}{1 - \rho})(1 - \rho)r \leq r.
\end{aligned}
$$

Hence, $a_{i+1} \in \overline{B_{G_d}(a_0, r)}$. Therefore, picard sequence $\{a_n\} \in \overline{B_{G_d}(a_0, r)}$, for all $n \in N \cup \{0\}$. Now, to show that picard sequence $\{a_n\}$ is Cauchy sequence, consider for $m, n \in N$, such that for $n < m$,

$$
\begin{aligned}
G_d(a_n, a_m, a_m) &\leq G_d(a_n, a_{n+1}, a_{n+1}) + G_d(a_{n+1}, a_{n+2}, a_{n+2}) + ... \\
&\quad + G_d(a_{m-2}, a_{m-1}, a_{m-1}) + G_d(a_{m-1}, a_m, a_m) \\
&\leq \rho^n G_d(a_0, a_1, a_1) + \rho^{n+1} G_d(a_0, a_1, a_1) \\
&\quad + \rho^{n+2} G_d(a_0, a_1, a_1) + ... + \rho^{m-1} G_d(a_0, a_1, a_1) \\
&\leq (1 + \rho + \rho^2 + ... + \rho^{m-n-1})\rho^n G_d(a_0, a_1, a_1) \\
&\leq (\frac{1 - \rho^{m-n}}{1 - \rho})\rho^n G_d(a_0, a_1, a_1) \\
&\leq (1 - \rho^{m-n})\rho^n r \leq \rho^n r
\end{aligned}
$$

As $\rho \in [0, 1)$, then $\rho^n \to 0$, if $n \to \infty$. Hence $\rho^n r \to 0$, if $n \to \infty$. So, we have

$$
G_d(a_n, a_m, a_m) \to 0, \text{ as } n \to \infty.
$$

Therefore picard sequence $\{a_n\}$ is a Cauchy sequence in closed ball $\overline{B_{G_d}(a_0, r)}$. As closed ball $\overline{B_{G_d}(a_0, r)}$ is closed subset of set X, then the sequence $\{a_n\}$ is convergent in closed ball $\overline{B_{G_d}(a_0, r)}$ and the point of convergence is $a \in \overline{B_{G_d}(a_0, r)}$. Hence $a_n \to a$ as $n \to \infty$. In general it is clear that,

$$
\underset{n \to \infty}{Lim} G_d(a_n, a, a) = \underset{n \to \infty}{Lim} G_d(a, a_n, a_n) = 0
\tag{10}
$$

To check either $a \in \overline{B_{G_d}(a_0, r)}$ is a fixed point of $\Gamma : X \to X$ or not, consider

$$
\begin{aligned}
G_d(a, \Gamma a, \Gamma a) &\leq G_d(a, a_{n+1}, a_{n+1}) + G_d(a_{n+1}, \Gamma a, \Gamma a) \\
&\leq G_d(a, a_{n+1}, a_{n+1}) + \xi W(a_n, a, a)
\end{aligned}
\tag{11}
$$

From (2),

$$
\begin{aligned}
W(a_n, a, a) \;=\; \max\{ & G_d(a, \Gamma^2 a_n, \Gamma a), G_d(\Gamma a_n, \Gamma^2 a_n, \Gamma a), G_d(a_n, \Gamma a_n, a), \\
& G_d(a_n, \Gamma a_n, a), G_d(a, \Gamma^2 a_n, \Gamma a), G_d(a, \Gamma a_n, \Gamma a), \\
& G_d(\Gamma a_n, \Gamma^2 a_n, \Gamma a), G_d(a, \Gamma a_n, \Gamma a), G_d(a_n, a, a), \\
& G_d(a_n, \Gamma a_n, \Gamma a_n), G_d(a, \Gamma a, \Gamma a), G_d(a, \Gamma a, \Gamma a), \\
& G_d(a_n, \Gamma a, \Gamma a), G_d(a, \Gamma a, \Gamma a)\}
\end{aligned}
$$

$$
\begin{aligned}
W(a_n, a, a) \;=\; \max\{ & G_d(a, a_{n+2}, \Gamma a), G_d(a_{n+1}, a_{n+2}, \Gamma a), G_d(a_n, a_{n+1}, a), \\
& G_d(a_n, a_{n+1}, a), G_d(a, a_{n+2}, \Gamma a), G_d(a, a_{n+1}, \Gamma a), \\
& G_d(a_{n+1}, a_{n+2}, \Gamma a), G_d(a, a_{n+1}, \Gamma a), G_d(a_n, a, a), \\
& G_d(a_n, a_{n+1}, a_{n+1}), G_d(a, \Gamma a, \Gamma a), G_d(a, \Gamma a, \Gamma a), \\
& G_d(a_n, \Gamma a, \Gamma a), G_d(a, \Gamma a, \Gamma a)\}
\end{aligned}
$$

$$
\begin{aligned}
W(a_n, a, a) \;=\; \max\{ & G_d(a, a_{n+2}, \Gamma a), G_d(a_{n+1}, a_{n+2}, \Gamma a), G_d(a_n, a_{n+1}, a), \\
& G_d(a, a_{n+1}, \Gamma a), G_d(a_n, a, a), G_d(a_n, a_{n+1}, a_{n+1}), \\
& G_d(a, \Gamma a, \Gamma a), G_d(a_n, \Gamma a, \Gamma a)\}. \tag{12}
\end{aligned}
$$

After applying limit $n \to \infty$ on (11), for every selection of $W(a_n, a, a)$ from (12), we get

$$
G_d(a, \Gamma a, \Gamma a) \leq 0.
$$

Hence, $\Gamma a = a$ or $a \in \overline{B_{G_d}(a_0, r)}$ is a fixed point of Γ. For uniqueness of fixed point, consider $a, b \in \overline{B_{G_d}(a_0, r)}$ are two distinct fixed points of $\Gamma : X \to X$. So consider the relation,

$$
G_d(a, b, b) = G_d(\Gamma a, \Gamma b, \Gamma b) \leq \xi W(a, b, b). \tag{13}
$$

Where

$$
W(a, b, b) = \max\{G_d(a, b, b), G_d(a, a, b), G_d(a, a, a), G_d(b, b, b)\}. \tag{14}
$$

Now,

$$
G_d(a, a, a) = G_d(\Gamma a, \Gamma a, \Gamma a) \leq \xi W(a, a, a) = \xi G_d(a, a, a).
$$

So $G_d(a, a, a) = 0$. Similarly $G_d(b, b, b)$. If $W(a, b, b) = G_d(a, b, b)$ in (14), relation (13) gives, $G_d(a, b, b) = 0$. If $W(a, b, b) = G_d(a, b, b)$ in (14), relation (13) gives,

$$
\begin{aligned}
G_d(a, b, b) &\leq \xi G_d(a, a, b) \\
&\leq \xi G_d(a, b, b) + \xi G_d(b, a, b) \\
(1 - 2\xi) G_d(a, b, b) &\leq 0.
\end{aligned}
$$

Hence in each case $G_d(a, b, b) = 0$. It is contradiction to our assumption, that is $a \neq b$. So our supposition is wrong. Hence fixed point of $\Gamma : X \to X$ is unique.

Example 2 If for a set $X = [0, 2]$, a mapping $G_d : X \times X \times X \longrightarrow X$, for all $a, b, c \in X$ defined by,

$$
G_d(a, b, c) = a + b + c
$$

then (X, G_d) is complete G_d-metric like space. Let mapping $\Gamma : X \to X$ are defined by,

$$
\Gamma a = \begin{cases} \frac{a}{8} & \text{if } a \in [0, 1] \\ a + \frac{1}{8} & \text{if } a \in (1, 2]. \end{cases}
$$

Let $a_0 = \frac{2}{3}$ and $r = \frac{8}{3}$ such that $\overline{B_{G_d}(a_0, r)} = [0, 1]$. Also let $\rho \in \{\xi = \frac{1}{3}, \Upsilon = \frac{\xi}{1-\xi} = \frac{1}{2}\} \subseteq [0, 1)$ such that,

$$\text{for } \rho = \frac{1}{3}, \ (1 - \rho)r = \frac{16}{9},$$

and

$$\text{for } \rho = \frac{1}{2}, \ (1 - \rho)r = \frac{4}{3}.$$

Also as

$$G_d(a_0, a_1, a_1) = \frac{2}{3} + 2\Gamma(\frac{2}{3}) = \frac{5}{6}.$$

Clearly

$$G_d(a_0, a_1, a_1) \le (1 - \rho)r, \ \text{for every } \rho \in \{\frac{1}{3}, \frac{1}{2}\}.$$

To show contractive condition is locally contractive, for first case let $a, b, c \in [0, 1]$ then,

$$G_d(\Gamma a, \Gamma b, \Gamma c) = G_d(\frac{a}{8}, \frac{b}{8}, \frac{c}{8})$$

$$G_d(\Gamma a, \Gamma b, \Gamma c) = \frac{1}{8}(a + b + c)$$

Also let

$$\begin{aligned} W(a, b, c) = \ &\max\{\frac{9a + 8b}{8}, \frac{a + 72b}{64}, \frac{9a + 8b}{64}, \frac{a + 9b}{8}, \\ &\frac{9a + 8c}{8}, \frac{a + 72c}{64}, \frac{9a + 8c}{64}, \frac{8c + a + b}{8}, a + b + c, \\ &\frac{5a}{4}, \frac{5b}{4}, \frac{5c}{4}, \frac{4a + b}{4}, \frac{4b + c}{4}\} \end{aligned}$$

If $a, b, c \in [0, 1]$, then

$$0 \le \frac{9a + 8b}{64}, \frac{9a + 8c}{64} \le \frac{17}{64}, 0 \le \frac{a + 72b}{64}, \frac{a + 72c}{64} \le \frac{73}{64}$$

$$0 \le \frac{a + 9b}{8}, \frac{8c + a + b}{8}, \frac{5a}{4}, \frac{5b}{4}, \frac{5c}{4}, \frac{4a + b}{4}, \frac{4b + c}{4} \le \frac{5}{4}$$

$$0 \le \frac{9a + 8b}{8}, \frac{9a + 8c}{8} \le \frac{17}{8}, 0 \le a + b + c \le 3$$

Clearly above inequalities shows that maximum value for $W(a, b, c)$ is

$$W(a, b, c) = a + b + c$$

As,

$$\frac{1}{8}(a + b + c) \le \frac{1}{3}(a + b + c)$$

So,

$$G_d(\Gamma a, \Gamma b, \Gamma c) \le \xi W(a, b, c)$$

Hence contractive condition is locally satisfied on $\overline{B_{G_d}(a_0, r)} = [0, 1]$. For the second case if $a, b, c \in (1, 2]$ then,

$$G_d(\Gamma a, \Gamma b, \Gamma c) = G_d(a + \frac{1}{8}, b + \frac{1}{8}, c + \frac{1}{8})$$

$$G_d(\Gamma a, \Gamma b, \Gamma c) = (a + b + c) + \frac{3}{8}$$

Also, let

$$
\begin{aligned}
W(a,b,c) \quad = \quad & \max\{a+2b+\frac{3}{8}, 2a+b+\frac{1}{8}, a+2b+\frac{1}{4}, 2a+b+\frac{1}{2} \\
& 2a+c+\frac{1}{8}, a+2c+\frac{3}{8}, 2a+c+\frac{1}{2}, a+b+c+\frac{1}{4}, \\
& a+b+c, 3a+\frac{1}{4}, 3b+\frac{1}{4}, 3c+\frac{1}{4}, a+2b+\frac{1}{4}, \\
& b+2c+\frac{1}{4}\}
\end{aligned}
$$

If for all $a,b,c \in (1,2]$ then

$$
\frac{25}{8} \quad < \quad 2a+b+\frac{1}{8}, 2a+c+\frac{1}{8} \leq \frac{49}{8}, \ 3 < a+b+c \leq 6, \ \frac{13}{4} < a+b+c+\frac{1}{4} \leq \frac{25}{4}
$$

$$
\frac{13}{4} \quad < \quad a+2b+\frac{1}{4}, 3a+\frac{1}{4}, 3b+\frac{1}{4}, 3c+\frac{1}{4}, a+2b+\frac{1}{4}, b+2c+\frac{1}{4} \leq \frac{25}{4}
$$

$$
\frac{27}{8} \quad < \quad a+2b+\frac{3}{8}, a+2c+\frac{3}{8} \leq \frac{51}{8}, \ \frac{7}{2} < 2a+b+\frac{1}{2}, 2a+c+\frac{1}{2} \leq \frac{13}{2}
$$

Clearly above inequalities shows that maximum values for $W(a,b,c)$ are,

$$
W(a,b,c) = 2a+b+\frac{1}{2} \ \text{and} \ W(a,b,c) = 2a+c+\frac{1}{2}
$$

Now as

$$
\begin{aligned}
(a+b+c)+\frac{3}{8} \quad &\geq \quad \frac{1}{3}(2a+b+\frac{1}{2}) \\
G_d(\Gamma a, \Gamma b, \Gamma c) \quad &\geq \quad \xi W(a,b,c)
\end{aligned}
$$

or

$$
\begin{aligned}
(a+b+c)+\frac{3}{8} \quad &\geq \quad \frac{1}{3}(2a+c+\frac{1}{2}) \\
G_d(\Gamma a, \Gamma b, \Gamma c) \quad &\geq \quad \xi W(a,b,c)
\end{aligned}
$$

Hence, contractive condition is failed outside of $\overline{B_{G_d}(a_0,r)} = [0,1]$. Therefore fixed point of $\Gamma : X \to X$ exists and is $0 \in \overline{B_{G_d}(a_0,r)}$ such that $\Gamma 0 = 0$

In above theorem, interval for contractive condition can be extended to $[0,1)$ as shown by following corollary.

Corollary 1 Suppose for a G_d-metric like space (X, G_d) if a defined mapping $\Gamma : X \longrightarrow X$ satisfies,

$$
G_d(\Gamma a, \Gamma b, \Gamma c) \leq \xi W(a,b,c),
$$

for all $a_0, a, b, c \in \overline{B_{G_d}(a_0,r)} \subseteq X$ and $r > 0$, where $\xi \in [0,1)$ and

$$
\begin{aligned}
W(a,b,c) \quad = \quad & \max\{G_d(b, \Gamma^2 a, \Gamma b), G_d(\Gamma a, \Gamma^2 a, \Gamma b), G_d(a, \Gamma a, b), G_d(c, \Gamma^2 a, \Gamma c), \\
& G_d(\Gamma a, \Gamma^2 a, \Gamma c), G_d(a, \Gamma a, c), G_d(a, b, c), G_d(a, \Gamma a, \Gamma a), \\
& G_d(b, \Gamma b, \Gamma b), G_d(c, \Gamma c, \Gamma c), G_d(b, \Gamma c, \Gamma c)\}.
\end{aligned}
$$

And

$$
G_d(a_0, a_1, a_1) \leq (1-\xi)r.
$$

Then, there exists a unique $a \in \overline{B_{G_d}(a_0,r)}$ such that $\Gamma a = a$.

The prior bound can be used at the beginning of the calculation for estimating the required number of iterations to obtain the assumed accuracy. While posterior estimate can be used at intermediate stegosaur at the end of the calculation. Posterior estimate is at least as accurate as prior estimate. Now, we discuss error approximations and their related example.

Corollary 2 (Iteration, Error Bounds) From Theorem 1, picard iterative sequence, with arbitrary initial guess $a_0 \in \overline{B_{G_d}(a_0, r)} \subseteq X$, converges to unique fixed point $a \in \overline{B_{G_d}(a_0, r)}$ of mapping $\Gamma : X \to X$. Then, the error estimates are the prior estimate

$$G_d(a_n, a, a) \leq \frac{\rho^n}{1 - \rho} G_d(a_0, a_1, a_1), \tag{15}$$

and the posterior estimate

$$G_d(a_n, a, a) \leq \frac{\rho}{1 - \rho} G_d(a_{n-1}, a_n, a_n). \tag{16}$$

Proof. As from Theorem 1,

$$G_d(a_n, a_m, a_m) \leq \rho^n \left(\frac{1 - \rho^{m-n}}{1 - \rho} \right) G_d(a_0, a_1, a_1)$$

As sequence $\{a_m\}$ is convergent to $a \in \overline{B_{G_d}(a_0, r)} \subseteq X$, then by taking $m \to \infty$ gives $a_m \to \infty$ and $\rho^{m-n} \to 0$. Therefore, above relation leads to the *prior estimate* i.e.,

$$G_d(a_n, a, a) \leq \frac{\rho^n}{1 - \rho} G_d(a_0, a_1, a_1).$$

Setting $n = 1$ and write b_o for a_0 and b_1 for a_1 in (15) gives,

$$G_d(b_1, a, a) \leq \frac{\rho}{1 - \rho} G_d(b_0, b_1, b_1).$$

Letting $b_o = a_{n-1}$ then $b_1 = \Gamma b_o = \Gamma a_{n-1} = a_n$ in above relation leads to the *posterior estimate* i.e.,

$$G_d(a_n, a, a) \leq \frac{\rho}{1 - \rho} G_d(a_{n-1}, a_n, a_n).$$

Example 3 If for a set $X = [0, 2]$, a mapping $G_d : X \times X \times X \longrightarrow X$, for all $a, b, c \in X$ defined by,

$$G_d(a, b, c) = a + b + c$$

then (X, G_d) is complete G_d-metric like space. Let mapping $\Gamma : X \to X$ are defined by,

$$\Gamma a = \begin{cases} \frac{a}{8} & \text{if } a \in [0, 1] \\ a + \frac{1}{8} & \text{if } a \in (1, 2] \end{cases}$$

Let $a_0 = \frac{2}{3}$ and $r = \frac{8}{3}$ such that $\overline{B_{G_d}(a_0, r)} = [0, 1]$. Also let $\rho \in \{\xi = \frac{1}{3}, \Upsilon = \frac{\xi}{1 - \xi} = \frac{1}{2}\} \subseteq [0, 1)$. Construct the picard iterative sequence taking $a_0 = \frac{2}{3} \in [0, 1]$ as initial guess as,

$$a_n = \Gamma a_{n-1} = \frac{a_0}{8^n}, \text{ for all } n \in N \cup \{0\}$$

Also as

$$G_d(a_0, a_1, a_1) = \frac{2}{3} + 2\Gamma\left(\frac{2}{3}\right) = \frac{5}{6}$$

and

$$G_d(a_n, a, a) = a_n + 2a$$

As picard sequence $\{a_n\}$ satisfies all the conditions of Theorem 1 as in Example 2, then if $n \to \infty$, we have $a_n \to a$ i.e. $a_n \approx a$. Then

$$G_d(a_n, a, a) = 3a_n = \frac{3a_0}{8^n} = \frac{2}{8^n}$$

As from prior estimate

$$G_d(a_n, a, a) \leq \frac{\rho^n}{1-\rho} G_d(a_0, a_1, a_1)$$

If $\rho = \frac{1}{3}$ then, we have

$$
\begin{aligned}
\frac{2}{8^n} &\leq \frac{3}{2.3^n} \cdot \frac{5}{6} \\
\frac{\ln(\frac{8}{5})}{\ln(\frac{8}{3})} &\leq n \Longrightarrow 0.47919 \leq n \\
n &= 1, 2, 3, ... \text{ being integer}
\end{aligned}
$$

If $\rho = \frac{1}{2}$ then, we have

$$
\begin{aligned}
\frac{2}{8^n} &\leq \frac{2}{2^n} \cdot \frac{5}{6} \\
\frac{\ln(\frac{6}{5})}{\ln(\frac{8}{2})} &\leq n \Longrightarrow 0.1315172 \leq n \\
n &= 1, 2, 3, ... \text{ being integer}
\end{aligned}
$$

In either case if $\rho \in \{\frac{1}{3}, \frac{1}{2}\}$, picard sequence $\{a_n\}$ converges for $n = 1, 2, 3, ...$ If $n = 2$

$$a_2 = \Gamma a_1 = \frac{a_0}{8^2} = 0.0104166667$$

If $n = 3$, then

$$a_3 = \Gamma a_2 = \frac{a_0}{8^3} = 0.0013020833$$

Therefore,

$$0.0013020833 \approx \Gamma 0.0104166667$$

This suggests, when integer $n \geq 1$ goes on increasing, picard sequence moves towards fixed point of Γ which is $a = 0 \in [0, 1]$, i.e., $\Gamma 0 = 0$.

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