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ON SOME DOUBLY NONLINEAR SYSTEM IN INHOMOGENOUS ORLICZ SPACES

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Abstract. Our aim in this paper is to discuss the existence of renormalized solutions of the following systems:

$$\frac{\partial b_i(x,u_i)}{\partial t} - div(a(x,t,u_i,\nabla u_i)) - \phi_i(x,t,u_i)) + f_i(x,u_1,u_2) = 0 \qquad \text{i=1,2.}$$

where the function $b_i(x,u_i)$ verifies some regularity conditions, the term $\Big(a(x,t,u_i,\nabla u_i)\Big)$ is a generalized Leray-Lions operator and ϕ_i is a Carathéodory function assumed satisfy only a growth condition. The source term $f_i(t,u_1,u_2)$ belongs to $L^1(\Omega\times(0,T))$.

1. Introduction

Let Ω be a bounded open subset of R^N, $(N \geq 1)$ with the segment property. Fixing a final time $T > 0$ and let $Q_T := (0,T)\times\Omega$. We prove the existence of a renormalized solutions for the nonlinear parabolic systems:

$$(b_i(x,u_i))_t - \operatorname{div}\Big(a(x,t,u_i,\nabla u_i) - \Phi_i(x,t,u_i)\Big) + f_i(x,u_1,u_2) = 0 \quad \text{in } Q, \qquad (1)$$

$$u_i = 0 \quad \text{on } \Gamma := (0,T)\times\partial\Omega, \qquad (2)$$

$$b_i(x,u_i)(t=0) = b_i(x,u_{i,0}) \quad \text{in } \Omega, \qquad (3)$$

where $i = 1,2$. Here, the vector field

$$a : \Omega\times(0,T)\times R\times R^N \to R^N \text{ is a Carathéodory function} \qquad (4)$$

where $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$ is a Leary-Lions operator defined on the inhomogeneous Orlicz-Sobolev space $W_0^{1,x}L_M(Q_T)$, M is a N-function related to the growth of $A(u)$ (see assumptions (8)-(10)), and to the growth of the lower order Carathéodory function $\phi(x,t,u)$ (see assumption (11)). $b : \Omega\times R \longrightarrow R$ is a Carathéodory function such that for every $x \in \Omega$, $b(x,.)$ is a strictly increasing C^1-function, the source term f_i is a Carathéodory function.

In the first time, on the Classical Sobolev space, The existence of renormalized solution has been proved by R.-Di Nardo et al. in [9] in the case $b(x,u) = u$, by

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H. Redwane in [12] where $b(u) = b(x, u)$, by A. Aberqi, J. Bennouna and H. Redwane, in [2], where $|\phi(x, t, s)| \leq c(x, t)|s|^\gamma$ and by L. Aharouch, J. Bennouna and A. Touzani in [3] and by A. Benkirane and J. Bennouna [7] in the Orlicz spaces and degenerated spaces.

In the second time, the existence of a renormalized solution to a class of doubly nonlinear parabolic systems, in the classical Sobolev space $b_i(u_i) = u_i$ and $\phi_i = \phi$, i = 1, 2 has been studied by H. Redwane [12] and for the parabolic version of (1.1)-(1.3), existence and uniqueness results are already proved in [8] (see also [13]) in the case $f_i(x, u1, u2)$ is replaced by $f - div(g)$, by Azroul et al. in [6] has studied the Problem (1), where the term ϕ is continuous function, who allows to eliminate it by using the Stockes formula. Recently Aberqi et al. in [2] has treated the same problem, where the right-side is $f - div(g)$ where $f \in L^1(Q)$, $g \in (L^{p'}(Q))^N$ and the term ϕ satisfy the following growth condition $\phi(x, t, s) \leq c(x, t)|s|^\gamma$.

It is our purpose in this paper to generalize the last two results in the Orlicz-sobolev spaces and with the condition $\phi(x, t, s) \leq c(x, t)\overline{M}^{-1} M(\frac{\alpha}{\lambda}|s|)$ and not assuming any other condition (no coercivity condition and no Δ_2 condition on the N-function M). However the uniqueness of solution remains yet open.
To illustrate the type of problems in Orlicz–Sobolev spaces, we cite the model bellow:

$$\begin{cases} \dfrac{\partial|u|^{q(x)-2}}{\partial t} - div\big(\dfrac{\alpha|\nabla u|^{p-2}\nabla u}{1+|u|^\gamma}.log(e+u)\big) - div(c(x,t)|u|^{p-1}) = f & \text{in} \quad Q_T, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \end{cases}$$

where $b(x, u) = |u|^{q(x)-2}u$, where $q : \Omega \to]1, +\infty[$, with $q(x) \leq -|x|^2 + 2$.
$Au = -\Delta_M u = -div\big(\frac{\alpha|\nabla u|^{p-2}\nabla u}{1+|u|^\gamma}.log(e+u)\big)$, here the N-functions M associated to the operator are $M(t) = t^p log^q(e + t)$, and $P(t) = \frac{t^p}{p}$, with $P \ll M$.
$\phi(x, t, u) = c(x, t)|u|^{p-1}$ the term in divergentiel form which is not continuous with respect to x.

This article is organized as follows: In Section 2, we give some technical lemmas. In Section 3 we give the basic assumptions and give the definition of a renormalized solution of (1.1)-(1.3) and in Section 4, we establish (Theorem 4) the existence of such a solutions.

2. Preliminaries and some technical lemmas

Let $M : R^+ \to R^+$ be an N-function, that is, M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$, and $M(t)/t \to +\infty$ as $t \to +\infty$. Equivalently,M admits the representation $M(t) = \int_0^t a(s)ds$, where $a : R^+ \to R^+$ is nondecreasing, right continuous, with $a(0) = 0, a(t) > 0$ for $t > 0$, and $a(t) \to +\infty$ as $t \to +\infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{a}(s)ds$, where $\overline{a} : R^+ \to R^+$, is given by $\overline{a}(t) = sup\{s : a(s) \leq t\}$.

We will extend these N-functions into even functions on all R. Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q, that is, for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to +\infty$. This is the case if and only if $\lim_{t\to+\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$.

The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$ is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))dx < +\infty \quad \text{(resp.} \quad \int_\Omega M(\frac{u(x)}{\lambda})dx < +\infty \quad \text{for some} \quad \lambda > 0).$$

The set $L_M(\Omega)$ is Banach space under the norm

$$\|u\|_{M,\Omega} = inf\{\lambda > 0 : \int_\Omega M(\frac{u(x)}{\lambda})dx \leq 1\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The dual $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uvdx$,and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|u\|_{\overline{M},\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$) [resp. $W^1E_M(\Omega)$] is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_{M,\Omega}$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.
Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$] denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ [resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm.(for more details see [1]).
We recall the following Lemma:

Lemma 1 (see [11] and [10]) For all $u \in W_0^1L_M(Q_T)$ with $meas(Q_T) < +\infty$ one has

$$\int_{Q_T} M(\frac{|u|}{\lambda})dxdt \leq \int_{Q_T} M(|\nabla u|)dxdt? \tag{5}$$

where $\lambda = diamQ_T$, is the diameter of Q_T.

3. Assumptions and statement of main results

Throughout this paper, we assume that the following assumptions hold true:
Let P and M are two N-functions, such that $P << M$, and for all $i = 1, 2$:

$$b_i : \Omega \times R \to R \text{ is a Carathéodory function such that for every } x \in \Omega, \tag{6}$$

$b_i(x, .)$ is a strictly increasing $\mathcal{C}^1(R)$-function and $b_i \in L^\infty(\Omega \times R)$ with $b_i(x, 0) = 0$. Next for any $k > 0$, there exists a constant $\lambda_k^i > 0$ and functions $A_k^i \in L^\infty(\Omega)$ and $B_k^i \in L_M(\Omega)$ such that:

$$\lambda_k^i \leq \frac{\partial b_i(x,s)}{\partial s} \leq A_k^i(x) \quad \text{and} \quad \Big|\nabla_x\Big(\frac{\partial b_i(x,s)}{\partial s}\Big)\Big| \leq B_k^i(x) \quad \text{a.e. } x \in \Omega \text{ and } \forall\, |s| \leq k. \tag{7}$$

For almost every $(x,t)\in Q_T$, for every $s\in R$ and every $\xi,\ \eta\in R^N$

$$|a(x,t,s,\xi)|\leq d_k(x,t)+\beta_{k,1}\overline{M}^{-1}P(\beta_{k,2}|\xi|), \tag{8}$$

$$a(x,t,s,\xi)\xi\geq\alpha M(|\xi|)\quad\text{with }\alpha>0, \tag{9}$$

$$\big(a(x,t,s,\xi)-a(x,t,s,\eta)\big)(\xi-\eta)>0\quad\text{with }\xi\neq\eta, \tag{10}$$

where $d_k(x,t)\in E_{\overline{M}}(Q_T)$, and $\beta_{k,1},\beta_{k,2}>0$ are the given real numbers.
Let $\phi(x,t,s)$ be a Carathéodory function such that for a.e $(x,t)\in Q_T$ for all $s\in R$

$$|\phi_i(x,t,s)|\leq c_i(x,t)\overline{M}^{-1}M(\frac{\alpha_0^i}{\lambda}|s|),\quad c_i(.,.)\in L^\infty(Q_T),\text{where}\quad \|c_i(.,.)\|_\infty\leq\alpha, \tag{11}$$

$f_i:\Omega\times R\times R\to R$ is a Carathéodory function with

$$f_1(x,0,s)=f_2(x,s,0)=0\quad\text{a.e. } x\in\Omega,\forall s\in R, \tag{12}$$

and for almost every $x\in\Omega$, for every $s_1,s_2\in R$,

$$sign(s_i)f_i(x,s_1,s_2)\geq 0. \tag{13}$$

The growth assumptions on f_i are as follows: For each $K>0$, there exists $\sigma_K>0$ and a function F_K in $L^1(\Omega)$ such that

$$|f_1(x,s_1,s_2)|\leq F_K(x)+\sigma_K\ |b_2(x,s_2)|, \tag{14}$$

a.e. in Ω, for all s_1 such that $|s_1|\leq K$, for all $s_2\in R$. For each $K>0$, there exists $\lambda_K>0$ and a function G_K in $L^1(\Omega)$ such that

$$|f_2(x,s_1,s_2)|\leq G_K(x)+\lambda_K\ |b_1(x,s_1)|, \tag{15}$$

for almost every $x\in\Omega$, for every s_2 such that $|s_2|\leq K$, and for every $s_1\in R$.
Finally, we assume the following condition on the initial data $u_{i,0}$:

$$u_{i,0}\text{ is a measurable function such that } b_i(.,u_{i,0})\in L^1(\Omega),\ \text{for } i=1,2. \tag{16}$$

In this paper, for $K>0$, we denote by $T_K: r\mapsto \min(K,max(r,-K))$ the truncation function at height K. For any measurable subset E of Q_T, we denote by $meas(E)$ the Lebesgue measure of E. For any measurable function v defined on Q and for any real number $s,\chi_{\{v<s\}}$ (respectively, $\chi_{\{v=s\}},\chi_{\{v>s\}}$) denote the characteristic function of the set $\{(x,t)\in Q_T\ ;\ v(x,t)<s\}$ (respectively, $\{(x,t)\in Q_T;v(x,t)=s\},\{(x,t)\in Q_T;v(x,t)>s\}$).

Definition 2 A couple of functions (u_1,u_2) defined on Q_T is called a renormalized solution of (6)-(16)if for $i=1,2$ the function u_i satisfies

$$T_K(u_i)\in W_0^{1,x}L_M(Q_T)\quad\text{and}\quad b_i(x,u_i)\in L^\infty(0,T;L^1(\Omega)), \tag{17}$$

$$\int_{\{\ m\leq|u_i|\leq m+1\}} a(x,t,u_i,\nabla u_i)\nabla u_i\,dx\,dt\to 0\quad\text{as } m\to+\infty, \tag{18}$$

For every function S in $W^{2,\infty}(R)$ which is piecewise C^1 and such that S' has a compact support,we have

$$\frac{\partial B_{i,S}(x,u_i)}{\partial t}-div(S'(u_i)a(x,t,u_i,\nabla u_i))+S''(u_i)a(x,t,u_i,\nabla u_i)\nabla u_i$$
$$+div(S'(u_i)\phi_i(x,t,u_i))-S''(u_i)\phi_i(x,t,u_i)\nabla u_i+f_i(x,u_1,u_2)S'(u_i)=0 \tag{19}$$

$$B_{i,S}(x,u_i)(t=0)=B_{i,S}(x,u_{i,0})\quad\text{in }\Omega, \tag{20}$$

where $B_{i,S}(r)=\int_0^r b_i'(x,s)S'(s)\,ds$.

Remark 3

Due to (17), each term in (19) has a meaning in $W^{-1,x}L_{\overline{M}}(Q_T)+L^1(Q_T)$. Indeed, if K such that $suppS\subset[-K,K]$, the following identifications are made in (19)

- $B_{i,S}(x,u_i)\in L^\infty(Q_T)$, since $|B_{i,S}(x,u_i)|\le K\|A^i_K\|_{L^\infty(\Omega)}\|S'\|_{L^\infty(R)}$
- $S'(u_i)a(x,t,u_i,\nabla u_i)$ can be identified with $S'(u_i)a(x,t,T_K(u_i),\nabla T_K(u_i))$ a.e. in Q_T. Since indeed $|T_K(u_i)|\le K$ a.e. in Q_T . As a consequence of (8) , (17) and $S'(u_i)\in L^\infty(Q_T)$, it follows that
$$S'(u_i)a(x,T_K(u_i),\nabla T_K(u_i))\in(L_{\overline{M}}(Q_T))^N.$$
- $S'(u_i)a(x,t,u_i,\nabla u_i)\nabla u_i$ can be identified with $S'(u_i)a(x,t,T_K(u_i),\nabla T_K(u_i))\nabla T_K(u_i)$ a.e. in Q_T with (7) and (17) it has
$$S'(u_i)a(x,t,T_K(u_i),\nabla T_K(u_i))\nabla T_K(u_i)\in L^1(Q_T)$$
- $S'(u_i)\Phi_i(u_i)$ and $S''(u_i)\Phi_i(u_i)\nabla u_i$ respectively identify with $S'(u_i)\Phi_i(T_K(u_i))$ and $S''(u_i)\Phi(T_K(u_i))\nabla T_K(u_i)$. In view of the properties of S and (11), the functions S',S'' and $\Phi\circ T_K$ are bounded on R so that (17) implies that $S'(u_i)\Phi_i(T_K(u_i))\in(L^\infty(Q_T))^N$ and $S''(u_i)\Phi_i(T_K(u_i))\nabla T_K(u_i)\in(L_{\overline{M}}(Q_T))^N$.
- $S'(u_i)f_i(x,u_1,u_2)$ identifies with $S'(u_i)f_1(x,T_K(u_1),u_2)$ a.e. in Q_T (or $S'(u_i)f_2(x,u_1,T_K(u_2))$ a.e. in Q_T). Indeed, since $|T_K(u_i)|\le K$ a.e. in Q_T, assumptions (14) and (15) and using (17) and of $S'(u_i)\in L^\infty(Q)$, one has

$$S'(u_1)f_1(x,T_K(u_1),u_2)\in L^1(Q_T)\quad\text{and}\quad S'(u_2)f_2(x,u_1,T_K(u_2))\in L^1(Q_T).$$

As consequence, (19) takes place in $D'(Q_T)$ and that
$$\frac{\partial B_{i,S}(x,u_i)}{\partial t}\in W^{-1,x}L_{\overline{M}}(Q_T)+L^1(Q_T).\tag{21}$$

Due to the properties of S and (7)
$$B_{i,S}(x,u_i)\in W_0^{1,x}L_M(Q_T).\tag{22}$$

Moreover (21) and (22) implies that $B_{i,S}(x,u_i)\in C^0([0,T],L^1(\Omega))$ so that the initial condition (20) makes sense.

4. Existence result

We shall prove the following existence theorem

Theorem 4 Assume that (6)-(16) hold true. There is at least a renormalized solution (u_1,u_2) of Problem (1).

Proof. We give the prof in 5 steps.

Step 1: Approximate problem.

Let us introduce the following regularization of the data: for $n>0$ and $i=1,2$

$$b_{i,n}(x,s)=b_i(x,T_n(s))+\frac{1}{n}\ s\quad\forall s\in R,\tag{23}$$

$$a_n(x,t,s,\xi)=a(x,t,T_n(s),\xi)\text{ a.e. in }\Omega,\forall s\in R,\forall\xi\in R^N,\tag{24}$$

$$\Phi_{i,n}(x,t,s)=\Phi_{i,n}(x,t,T_n(s))\quad\text{a.e.}\quad(x,t)\in Q_T,\quad\forall s\in IR.\tag{25}$$

$$f_{1,n}(x,s_1,s_2)=f_1(x,T_n(s_1),s_2) \quad \text{a.e. in } \Omega, \forall s_1,s_2\in R, \tag{26}$$
$$f_{2,n}(x,s_1,s_2)=f_2(x,s_1,T_n(s_2)) \quad \text{a.e. in } \Omega, \forall s_1,s_2\in R, \tag{27}$$
$$u_{i,0n}\in C_0^\infty(\Omega), b_{i,n}(x,u_{i,0n})\to b_i(x,u_{i,0}) \quad \text{in } L^1(\Omega) \text{ as } n\to+\infty. \tag{28}$$

Let us now consider the regularized problem

$$\frac{\partial b_{i,n}(x,u_{i,n})}{\partial t}-\mathrm{div}(a_n(x,u_{i,n},\nabla u_{i,n}))-div(\Phi_{i,n}(x,t,u_{i,n}))+f_{i,n}(x,u_{1,n},u_{2,n})=0 \quad \text{in } Q_T, \tag{29}$$
$$u_{i,n}=0 \quad \text{on } (0,T)\times\partial\Omega, \tag{30}$$
$$b_{i,n}(x,u_{i,n})(t=0)=b_{i,n}(x,u_{i,0n}) \quad \text{in } \Omega. \tag{31}$$

In view of (23), for $i=1,2$, we have

$$\frac{\partial b_{i,n}(x,s)}{\partial s}\geq\frac{1}{n}, \quad |b_{i,n}(x,s)|\leq\max_{|s|\leq n}|b_i(x,s)|+1 \quad \forall s\in R,$$

In view of (14)-(15), $f_{1,n}$ and $f_{2,n}$ satisfy: There exists $F_n\in L^1(\Omega), G_n\in L^1(\Omega)$ and $\sigma_n>0,\lambda_n>0$, such that

$$|f_{1,n}(x,s_1,s_2)|\leq F_n(x)+\sigma_n \max_{|s|\leq n}|b_i(x,s)| \quad \text{a.e. in } x\in\Omega,\forall s_1,s_2\in R,$$

$$|f_{2,n}(x,s_1,s_2)|\leq G_n(x)+\lambda_n \max_{|s|\leq n}|b_i(x,s)| \quad \text{a.e. in } x\in\Omega,\forall s_1,s_2\in R.$$

As a consequence, proving the existence of a weak solution $u_{i,n}\in W_0^{1,x}L_M(Q_T)$ of (29)-(31) is an easy task (see e.g. [13]).

Step2: A priori estimates.

Let $t\in(0,T)$ and using $T_k(u_{i,n})\chi_{(0,t)}$ as a test function in problem (29), we get:

$$\begin{aligned}\int_\Omega B^n_{i,k}(x,u_{i,n}(t))dx+\int_{Q_t}a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_k(u_{i,n})dx\,dt+\int_{Q_t}\phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})dx\,dt\\ +\int_{Q_t}f_{i,n}T_k(u_{i,n})\,dx\,dt\leq\int_\Omega B^n_{i,k}(x,u_{i,0n})dx,\end{aligned} \tag{32}$$

where $B^n_{i,k}(x,r)=\displaystyle\int_0^r\frac{\partial b_{i,n}(x,s)}{\partial s}T_k(s)ds.$

Due to definition of $B^n_{i,k}$ we have:

$$\int_\Omega B^n_{i,k}(x,u_{i,n}(t))dx\geq\frac{\lambda_n}{2}\int_\Omega|T_k(u_{i,n})|^2\,dx, \quad \forall k>0, \tag{33}$$

and

$$0\leq\int_\Omega B^n_{i,k}(x,u_{i,0n})dx\leq k\int_\Omega|b_{i,n}(x,u_{i,0n})|dx\leq k||b_i(x,u_{i,0})||_{L^1(\Omega)}, \quad \forall k>0. \tag{34}$$

In view of (13), we have $\int_{Q_t}f_{i,n}T_k(u_{i,n})\,dx\,dt\geq0$

Using Young inequality 11 and lemma 5, we obtain

$$\int_{Q_t}\phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})dx\,dt\leq\|c_i\|_{L^\infty}(\alpha_0^i+1)\int_\Omega M(\nabla T_k(u_{i,n}))dxdt.$$

We conclude that

$$\frac{\lambda_k}{2}\int_\Omega |T_k(u_{i,n})|^2\,dx+\alpha\int_{Q_t} M(\nabla T_k(u_{i,n})\,dx\,dt \leq$$

$$\|c_i\|_{L^\infty}(\alpha_0^i+1)\int_\Omega M(\nabla T_k(u_{i,n}))dxdt+k\big(\|f\|_{L^1(Q_T)}+||b(x,u_{i,0n})||_{L^1(\Omega)}\big).$$

Then

$$\frac{\lambda_k}{2}\int_\Omega |T_k(u_{i,n})|^2\,dx+[\alpha-\|c\|_{L^\infty}(\alpha_0^i+1)]\int_{Q_t} M(\nabla T_k(u_{i,n}))\,dt\,dx \leq C_i.k.$$

If we choose $\|c_i\|_{L^\infty}<\alpha$ and $\alpha_0^i<\dfrac{\alpha-\|c_i\|_{L^\infty}}{\|c_i\|_{L^\infty}}$
we get

$$\int_{Q_t} M(\nabla T_k(u_{i,n}))\,dx\,dt \leq C_i.k, \tag{35}$$

then, we conclude that $T_k(u_{i,n})$ is bounded in $W^{1,x}L_M(Q_T)$ independently of n and for any $k\geq 0$, so there exists a subsequence still denoted by u_n such that

$$T_k(u_{i,n})\to\psi_{i,k} \tag{36}$$

weakly in $W_0^{1,x}L_M(Q_T)$ for $\sigma(\Pi L_M,\Pi E_{\overline{M}})$ strongly in $E_M(Q_T)$ and a.e in Q_T.
Since Lemma (5) and (41), we get also,

$$\begin{aligned} M(\tfrac{k}{\lambda})\ meas\Big\{\{|u_{i,n}|>k\}\cap B_R\times[0,T]\Big\} &\leq \int_0^T\int_{\{|u_{i,n}|>k\}\cap B_R} M(\frac{T_k(u_{i,n})}{\lambda})dxdt\\ &\leq \int_{Q_T} M(\frac{T_k(u_{i,n})}{\lambda})dxdt\\ &\leq \int_{Q_T} M(\nabla T_k(u_{i,n}))dxdt. \end{aligned}$$

Then

$$meas\Big\{\{|u_{i,n}|>k\}\cap B_R\times[0,T]\Big\}\leq\frac{C_i.k}{M(\frac{k}{\lambda})},$$

which implies that:
$\lim_{k\to+\infty} meas\Big\{\{|u_{i,n}|>k\}\cap B_R\times[0,T]\Big\}=0.$ uniformly in n.
Now we turn to prove the almost every convergence of $u_{i,n}$, $b_{i,n}(x,u_{i,n})$ and convergence of $a_{i,n}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))$.

Proposition 5 Let $u_{i,n}$ be a solution of the approximate problem, then:

$$u_{i,n}\to u_i \quad \text{a.e in} \quad Q_T, \tag{37}$$

$$b_{i,n}(x,u_{i,n})\to b_i(x,u_i) \quad \text{a.e in} \quad Q_T.\quad b_i(x,u_i)\in L^\infty(0,T,L^1(\Omega)), \tag{38}$$

$$a_n(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\rightharpoonup X_{i,k} \quad \text{in} \quad (L_{\overline{M}}(Q_T))^N \text{for} \quad \sigma(\Pi L_{\overline{M}},\Pi E_M), \tag{39}$$

for some $X_{i,k}\in(L_{\overline{M}}(Q_T))^N$

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{m\leq|u_{i,n}|\leq m+1} a_i(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}dxdt=0. \tag{40}$$

Proof

Proof of (37) **and** (38)**:**

Now, consider a non decreasing function $g_k\in C^2(R)$ such that $g_k(s)=s$ for $|s|\leq\frac{k}{2}$

and $g_k(s) = k$ for $|s| \geq k$. Multiplying the approximate equation by $g'_k(u_{i,n})$, we get

$$\frac{\partial B^{i,n}_{k,g}(x,u_{i,n})}{\partial t} - div\Big(a_n(x,t,u_{i,n},\nabla u_{i,n})g'_k(u_{i,n})\Big) + a_n(x,t,u_{i,n},\nabla u_{i,n})g''_k(u_{i,n})\nabla u_{i,n}$$
$$+ div\Big(\phi_{i,n}(x,t,u_{i,n})g'_k(u_{i,n})\Big) - g''_k(u_{i,n})\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} + f_{i,n}g'_k(u_n) = 0 \qquad \text{in } D'(Q_T), \tag{41}$$

where $B^{i,n}_{k,g}(x,z) = \int_0^z \frac{\partial b_{i,n}(x,s)}{\partial s} g'_k(s)ds.$

Using (41),we can deduce that $g_k(u_{i,n})$ is bounded in $W^{1,x}_0 L_M(Q_T)$ and $\frac{\partial B^{i,n}_{k,g}(x,u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_{\overline{M}}(Q_T)$ independently of n.
thanks to (11) and properties of g_k, it follows that

$$|\int_{Q_T} \phi_{i,n}(x,t,u_n)g'_k(u_{i,n})dxdt| \leq \|g'_k\|_\infty (\int_{Q_T} \alpha^i_0 M(\nabla T_k(u_{i,n}))dxdt + \int_{Q_T} \overline{M}(\|c_i(x,t)\|_\infty)dxdt$$
$$\leq C^1_{i,k},$$

and

$$|\int_{Q_T} g''_k(u_{i,n})\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}dxdt| \leq \|g''_k\|_\infty \big(\|c_i\|_{L^\infty}(\alpha^i_0+1)\int_\Omega M(\nabla T_k(u_{i,n}))dxdt\big) \leq C^2_{i,k},$$

where $C^1_{i,k}$ and $C^2_{i,k}$ constants independently of n.
We conclude that $\frac{\partial g_k(u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T)+W^{-1,x}L_{\overline{M}}(Q_T)$ for $k < n$. which implies that $g_k(u_{i,n})$ is compact in $L^1(Q_T)$. Due to the choice of g_k, we conclude that for each k, the sequence $T_k(u_{i,n})$ converges almost everywhere in Q_T, which implies that the sequence $u_{i,n}$ converge almost everywhere to some measurable function u_i in Q_T.
Then by the same argument in [5], we have

$$u_{i,n} \to u_i \text{ a.e. } Q_T, \tag{42}$$

where u_i is a measurable function defined on Q_T. and

$$b_{i,n}(x,u_{i,n}) \to b_i(x,u_i) \quad \text{a.e. in} \quad Q_T,$$

by (36) and (42) we have

$$T_k(u_{i,n}) \to T_k(u_i) \tag{43}$$

weakly in $W^{1,x}_0 L_M(Q_T)$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ strongly in $E_M(Q_T)$ and a.e in Q_T.
We now show that $b_i(x,u_i) \in L^\infty(0,T;L^1(\Omega))$. Indeed using $\frac{1}{\varepsilon}T_\varepsilon(u_{i,n})$ as a test function in (29),

$$\frac{1}{\varepsilon}\int_\Omega b^\varepsilon_{i,n}(x,u_{i,n})(t)\,dx + \frac{1}{\varepsilon}\int_{Q_T} a_n(x,u_{i,n},\nabla u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt$$
$$-\frac{1}{\varepsilon}\int_{Q_T}\Phi_{i,n}(x,t,u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt + \frac{1}{\varepsilon}\int_{Q_T} f_{i,n}(x,u_{1,n},u_{2,n})T_\varepsilon(u_{i,n})\,dx\,dt$$
$$= \frac{1}{\varepsilon}\int_\Omega b^\varepsilon_{i,n}(x,u_{i,0n})\,dx, \tag{44}$$

for almost any t in $(0,T)$, where, $b^\varepsilon_{i,n}(r) = \int_0^r b'_{i,n,\varepsilon}(s)T_\varepsilon(s)\,ds$.

Since a_n satisfies (9) and $f_{i,n}$ satisfies (13), we get

$$\int_\Omega b^{\varepsilon}_{i,n}(x,u_{i,n})(t)\,dx \le \int_{Q_T} \Phi_{i,n}(x,t,u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt + \int_\Omega b^{\varepsilon}_{i,n}(x,u_{i,0n})\,dx, \tag{45}$$

By Young inequality and (11), we get

$$\int_{Q_T} \Phi_{i,n}(x,t,u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt \le \int_{|u_{i,n}|\le\varepsilon} \overline{M}(\Phi_{i,n}(x,t,u_{i,n}))\,dx\,dt + \int_{|u_{i,n}|\le\varepsilon} M(\nabla T_\varepsilon(u_{i,n}))\,dx\,dt$$

$$\le \|c_i\|_{L^\infty}(\alpha^i_0+1)\int_{|u_{i,n}|\le\varepsilon} M(\nabla T_\epsilon(u_{i,n}))dxdt. \tag{46}$$

Using the Lebesgue's Theorem and $M(\nabla T_\varepsilon(u_{i,n})) \in W^{1,x}_0 L_M(Q_T)$ in second term of the left hand side of (46) and letting $\varepsilon \to 0$ in (45)we obtain

$$\int_\Omega |b_{i,n}(x,u_{i,n})(t)|\,dx \le \|b_{i,n}(x,u_{i,0n})\|_{L^1(\Omega)} \tag{47}$$

for almost $t \in (0,T)$. thanks to (28) , (37), and passing to the limit-inf in (47), we obtain $b_i(x,u_i) \in L^\infty(0,T;L^1(\Omega))$.

Proof of (39) **:**

Following the same way in([4]), we deduce that $a_n(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))$ is a bounded sequence in $(L_{\overline{M}}(Q_T))^N$, and we obtain (39).

Proof of (40) **:**

Multiplying the approximating equation (29) by the test function $\theta_m(u_{i,n}) = T_{m+1}(u_{i,n}) - T_m(u_{i,n})$

$$\int_\Omega B_{i,m}(x,u_{i,n}(T))dx + \int_{Q_T} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla\theta_m(u_{i,n})dx\,dt + \int_{Q_T}\phi_{i,n}(x,t,u_{i,n})\nabla\theta_m(u_{i,n})dx\,dt \tag{48}$$

$$+\int_{Q_T} f_{i,n}\theta_m(u_{i,n})\,dx\,dt \le \int_\Omega B_{i,m}(x,u_{i,0n})dx,$$

where $B_{i,m}(x,r) = \displaystyle\int_0^r \theta_m(s)\frac{\partial b_{i,n}(x,s)}{\partial s}ds$.

By (11),we have

$$\int_{Q_T}\phi_{i,n}(x,t,u_{i,n})\nabla\theta_m(u_{i,n})dx\,dt \le \|c_i\|_{L^\infty}(\alpha^i_0+1)\int_\Omega M(\nabla\theta_m(u_{i,n}))dxdt$$

Also $\displaystyle\int_{Q_T} f_{i,n}\theta_m(u_{i,n})dxdt \ge 0$ in view of (13). Then, the same argument in step 2 , we obtain,

$$\int_{Q_T} M(\nabla\theta_m(u_{i,n}))dxdt \le C_i\int_\Omega B_{i,m}(x,u_{i,0n})dx$$

passing to limit as $n \to +\infty$, since the pointwise convergence of $u_{i,n}$ and strongly convergence in $L^1(Q_T)$ of $B_{i,m}(x,u_{i,0n})$ we get

$$\lim_{n\to+\infty}\int_{Q_T} M(\nabla\theta_m(u_{i,n}))dxdt \le C_i\int_\Omega B_{i,m}(x,u_{i,0})dx$$

By using Lebesgue's theorem and passing to limit as $m \to +\infty$, in the all term of the right-hand side, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{m\le|u_i|\le m+1} M(\nabla\theta_m(u_{i,n})dxdt = 0, \tag{49}$$

and on the other hand, we have

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} \phi_{i,n}(x,t,u_{i,n})\nabla\theta_m(u_{i,n})dxdt \le \lim_{m\to+\infty}\lim_{n\to+\infty}\int_{m\le|u_i|\le m+1} M(\nabla\theta_m(u_{i,n}))dxdt$$
$$+\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{m\le|u_{i,n}|\le m+1}\overline{M}(\phi_{i,n}(x,t,u_{i,n}))dxdt$$

Using the pointwise convergence of $u_{i,n}$ and by Lebesgue's theorem, in the second term of the right side, we get

$$\lim_{n\to+\infty}\int_{m\le|u_{i,n}|\le m+1}\overline{M}(\phi_{i,n}(x,t,u_{i,n}))dxdt = \int_{m\le|u_i|\le m+1}\overline{M}(\phi_i(x,t,u_i))dxdt$$

and also, by Lebesgue's theorem

$$\lim_{m\to+\infty}\int_{m\le|u_i|\le m+1}\overline{M}(\phi_i(x,t,u_i))dxdt = 0 \tag{50}$$

we obtain with (49) and (50),

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} \phi_{i,n}(x,t,u_{i,n})\nabla\theta_m(u_{i,n})dxdt = 0$$

then passing to the limit in (48), we get (40).

Step 3:

Let $\upsilon_{i,j} \in \mathcal{D}(Q_T)$ be a sequence such that $\upsilon_{i,j} \to u_i$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence.

This specific time regularization of $T_k(\upsilon_{i,j})$ (for fixed $k \ge 0$) is defined as follows. Let $(\alpha_{i,0}^{\mu})_{\mu}$ be a sequence of functions defined on Ω such that

$$\alpha_{i,0}^{\mu} \in L^{\infty}(\Omega)\cap W_0^1L_M(\Omega) \quad \text{for all} \quad \mu > 0 \tag{51}$$
$$\|\alpha_{i,0}^{\mu}\|_{L^\infty(\Omega)} \le k \text{ for all } \quad \mu > 0.$$

and

$$\alpha_{i,0}^{\mu} \text{ converges to } \quad T_k(u_{i,0}) \text{ a.e. in } \quad \Omega \quad \text{ and } \quad \frac{1}{\mu}\|\alpha_{i,0}^{\mu}\|_{M,\Omega} \text{ converges to } \quad 0 \quad \mu\to+\infty.$$

For $k \ge 0$ and $\mu > 0$, let us consider the unique solution $(T_k(\upsilon_{i,j}))_{\mu} \in L^{\infty}(Q) \cap W_0^{1,x}L_M(Q_T)$ of the monotone problem:

$$\frac{\partial (T_k(\upsilon_{i,j}))_{\mu}}{\partial t} + \mu((T_k(\upsilon_{i,j}))_{\mu} - T_k(\upsilon_{i,j})) = 0 \text{ in } D'(\Omega), \tag{52}$$

$$(T_k(\upsilon_{i,j}))_{\mu}(t=0) = \alpha_{i,0}^{\mu} \text{ in } \Omega. \tag{53}$$

Remark that due to

$$\frac{\partial (T_k(\upsilon_{i,j}))_{\mu}}{\partial t} \in W_0^{1,x}L_M(Q_T) \tag{54}$$

We just recall that,

$$(T_k(\upsilon_{i,j}))_{\mu} \to T_k(u_i) \quad \text{a.e. in} \quad Q_T, \quad \text{weakly} * \quad \text{in } L^{\infty}(Q_T) \quad \text{and} \tag{55}$$

$$(T_k(\upsilon_{i,j}))_\mu \to (T_k(u_i))_\mu \quad \text{in} \quad W_0^{1,x}L_M(Q_T) \tag{56}$$

for the modular convergence as $j \to +\infty$.

$$(T_k(u_i))_\mu \to T_k(u_i) \quad \text{in} \quad W_0^{1,x}L_M(Q_T) \tag{57}$$

for the modular convergence as $\mu \to +\infty$.

$$||(T_k(\upsilon_{i,j}))_\mu||_{L^\infty(Q_T)} \leq max(||(T_k(u_i))||_{L^\infty(Q_T)}, ||\alpha_0^\mu||_{L^\infty(\Omega)}) \leq k, \;\; \forall \;\; \mu > 0 \;\; ,\forall \;\; k > 0. \tag{58}$$

Now, we introduce a sequence of increasing $C^\infty(R)$-functions S_m such that, for any $m \geq 1$.

$$S_m(r) = r \text{ for } |r| \leq m, \quad \text{supp}(S'_m) \subset [-(m+1), (m+1)], \quad \|S''_m\|_{L^\infty(R)} \leq 1. \tag{59}$$

Through setting, for fixed $K \geq 0$,

$$W^n_{i,j,\mu} = T_K(u_{i,n}) - T_K(\upsilon_{i,j})_\mu \quad \text{and} \quad W^n_{i,\mu} = T_K(u_{i,n}) - T_K(u_i)_\mu \tag{60}$$

we obtain upon integration,

$$\begin{aligned}
&\int_{Q_T} \langle \frac{\partial b_{i,S_m}(u_{i,n})}{\partial t}, W^n_{i,j,\mu} \rangle \, dx \, dt \\
&+ \int_{Q_T} S'_m(u_{i,n}) a_n(x, u_i^n, \nabla u_{i,n}) \nabla W^n_{i,j,\mu} \, dx \, dt + \int_{Q_T} S''_m(u_{i,n}) W^n_{i,j,\mu} a_n(x, u_{i,n}, \nabla u_{i,n}) \nabla u_{i,n} \, dx \, dt \\
&+ \int_{Q_T} \Phi_{i,n}(x, t, u_{i,n}) S'_m(u_{i,n}) \nabla W^n_{i,j,\mu} \, dx \, dt + \int_{Q_T} S''_m(u_{i,n}) W^n_{i,j,\mu} \Phi_{i,n}(x, t, u_{i,n}) \nabla u_{i,n} \, dx \, dt \\
&+ \int_{Q_T} f_{i,n}(x, u_{1,n}, u_{2,n}) S'_m(u_{i,n}) W^n_{i,j,\mu} \, dx \, dt = 0.
\end{aligned} \tag{61}$$

We pass to limit, as $n \to +\infty$, $j \to +\infty$, $\mu \to +\infty$ and then m tends to $+\infty$, the real number $K \geq 0$ being kept fixed. In order to perform this task we prove below the following results for fixed $K \geq 0$:

$$\liminf_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \int_{Q_T} \langle \frac{\partial b_{i,S_m}(u_{i,n})}{\partial t} , S'_m(u_{i,n}) W^n_{i,j,\mu} \rangle \, dx\, dt \geq 0 \quad \text{for any } m \geq K, \tag{62}$$

$$\lim_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \int_{Q_T} S'_n(u_{i,n}) \Phi_{i,n}(x,t,u_{i,n}) \nabla W^n_{i,j,\mu} \, dx\, dt = 0 \quad \text{for any } m \geq 1, \tag{63}$$

$$\lim_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \int_{Q_T} S''_m(u_{i,n}) W^n_{i,\mu} \Phi_{i,n}(x,t,u_{i,n}) \nabla u_{i,n} \, dx\, dt = 0 \quad \text{for any } m, \tag{64}$$

$$\lim_{m\to+\infty} \overline{\lim_{\mu\to+\infty}} \lim_{j\to+\infty} \overline{\lim_{n\to+\infty}} \Big| \int_{Q_T} S''_m(u_{i,n}) W^n_{i,j,\mu} a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} \, dx\, dt \Big| = 0, \tag{65}$$

$$\lim_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \int_{Q_T} f_{i,n}(x,u_{1,n},u_{2,n}) S'_m(u_{i,n}) W^n_{i,j,\mu} \, dx\, dt = 0 \quad \text{for any } m \geq 1. \tag{66}$$

$$\limsup_{n\to+\infty} \int_{Q_T} a(x,t,u_{i,n},\nabla T_K(u_{i,n})) \nabla T_K(u_{i,n}) \, dx\, dt \leq \int_{Q_T} X_{i,K} \nabla T_K(u_i) \, dx\, dt. \tag{67}$$

$$\int_{Q_T} [a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n})) - a(x,t,T_k(u_{i,n}),\nabla T_k(u_i))][\nabla T_k(u_{i,n}) - \nabla T_k(u_i)] dx\, dt \to 0. \tag{68}$$

Proof of (62):
Lemma 6

$$\int_{Q_T} \langle \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t}, S'_m(u_{i,n}) W^n_{i,j,\mu} \rangle dx\, dt \geq \epsilon(n,j,\mu,m) \tag{69}$$

Proof This follows from the proof in [13].
Proof of (63):
If we take $n > m+1$, we get

$$\phi_{i,n}(x,t,u_{i,n}) S'_m(u_{i,n}) = \phi_i(x,t,T_{m+1}(u_{i,n})) S'_m(u_{i,n})$$

Using (11), we have:

$$\begin{aligned} \overline{M}(\phi_{i,n}(x,t,T_{m+1}(u_{i,n}) S'_m(u_{i,n})) &\leq (m+1)\overline{M}(\phi_i(x,t,T_{m+1}(u_{i,n}))) \\ &\leq (m+1)\overline{M}(\|c_i(x,t)\|_{L^\infty(Q_T)} \overline{M}^{-1} M(\frac{\alpha_0}{\lambda}(m+1))) \end{aligned}$$

Then $\phi_{i,n}(x,t,u_n) S_m(u_{i,n})$ is bounded in $L_{\overline{M}}(Q_T)$, thus, by using the pointwise convergence of $u_{i,n}$ and Lebesgue's theorem we obtain $\phi_{i,n}(x,t,u_{i,n}) S_m(u_{i,n}) \to \phi_i(x,t,u_i) S_m(u_i)$ with the modular convergence as $n \to +\infty$, then $\phi_{i,n}(x,t,u_{i,n}) S_m(u_{i,n}) \to \phi(x,t,u_i) S_m(u_i)$ for $\sigma(\prod L_{\overline{M}}, \prod L_M)$.
On the other hand $\nabla W^n_{i,j,\mu} = \nabla T_k(u_{i,n}) - \nabla (T_k(v_{i,j}))_\mu$ for converge to $\nabla T_k(u_i) - \nabla (T_k(v_{i,j}))_\mu$ weakly in $(L_M(Q_T))^N$, then

$$\int_{Q_T} \phi_{i,n}(x,t,u_{i,n}) S_m(u_{i,n}) \nabla W^n_{i,j,\mu} \, dx\, dt \to \int_{Q_T} \phi_i(x,t,u_i) S_m(u_i) \nabla W_{i,j,\mu} \, dx\, dt$$

as $n \to +\infty$.
By using the modular convergence of $W_{i,j,\mu}$ as $j \to +\infty$ and letting μ tends to infinity, we get (63).
Proof of (64):
For $n > m+1 > k$, we have $\nabla u_{i,n} S''_m(u_{i,n}) = \nabla T_{m+1}(u_{i,n})$ a.e. in Q_T. By the almost every where convergence of $u_{i,n}$ we have $W^n_{i,j,\mu} \to W_{i,j,\mu}$ in $L^\infty(Q_T)$ weak-* and since the sequence $(\phi_{i,n}(x,t,T_{m+1}(u_{i,n})))_n$ converges strongly in $E_{\overline{M}}(Q_T)$, then

$$\phi_{i,n}(x,t,T_{m+1}(u_{i,n}))\ W^n_{i,j,\mu} \to \phi_i(x,t,T_{m+1}(u_i))\ W_{i,j,\mu}$$

converge strongly in $E_{\overline{M}}(Q_T)$ as $n \to +\infty$. By virtue of $\nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u_i)$ weakly in $(L_M(Q_T))^N$ as $n \to +\infty$ we have

$$\int_{m\le|u_{i,n}|\le m+1} \phi_{i,n}(x,t,T_{m+1}(u_{i,n}))\nabla u_{i,n} S''_m(u_{i,n}) W^n_{i,j,\mu}\,dx\,dt \to \int_{m\le|u_i|\le m+1} \phi(x,t,u_i))\nabla u_i W_{i,j,\mu}\,dx\,dt$$

as $n \to +\infty$.
With the modular convergence of $W_{i,j,\mu}$ as $j \to +\infty$ and letting $\mu \to +\infty$ we get (64).

Proof of (65):
For any $m \ge 1$ fixed, we have

$$\begin{aligned}&\Big|\int_{Q_T} S''_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} W^n_{i,j,\mu}\,dx\,dt\Big|\\ &\le \|S''_m\|_{L^\infty(R)}\|W^n_{i,j,\mu}\|_{L^\infty(Q_T)}\int_{\{m\le|u_{i,n}|\le m+1\}} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt,\end{aligned}$$

for any $m \ge 1$, and any $\mu > 0$. In view (58) and (59), we can obtain

$$\begin{aligned}&\limsup_{n\to+\infty}\Big|\int_{Q_T} S''_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} W^n_{i,j,\mu}\,dx\,dt\Big|\\ &\le 2K \limsup_{n\to+\infty}\int_{\{m\le|u_{i,n}|\le m+1\}} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt,\end{aligned} \tag{70}$$

for any $m \ge 1$. Using (40) we pass to the limit as $m \to +\infty$ in (70) and we obtain (65).
Proof of (66):

For fixed $n \ge 1$ and $n > m+1$, we have

$$f_{1,n}(x,u_{1,n},u_{2,n})S'_m(u_{1,n}) = f_1(x,T_{m+1}(u_{1,n}),T_n(u_{2,n}))S'_m(u_{1,n}),$$

$$f_{2,n}(x,u_{1,n},u_{2,n})S'_m(u_{2,n}) = f_2(x,T_n(u_{1,n}),T_{m+1}(u_{2,n}))S'_m(u_{2,n}),$$

In view (14),(15),(43) and Lebegue's the theorem allow us to get, for

$$\lim_{n\to+\infty}\int_{Q_T} f_{i,n}(x,u_{1,n},u_{2,n})S'_m(u_{i,n})W^n_{i,j,\mu}\,dx\,dt = \int_{Q_T} f_i(x,u_1,u_2)S'_m(u_i)W_{i,j,\mu}\,dx\,dt$$

Using (56), we follow a similar way we get as $j \to +\infty$

$$\lim_{j\to+\infty}\int_{Q_T} f_i(x,u_1,u_2)S'_m(u_i)W_{i,j,\mu}\,dx\,dt = \int_{Q_T} f_i(x,u_1,u_2)S'_m(u_i)(T_K(u_i)-T_K(u_i)_\mu)\,dx\,dt$$

we fixed $m > 1$, and using (57), we have

$$\lim_{\mu\to+\infty}\int_{Q_T} f_i(x,u_1,u_2)S'_m(u_i)(T_K(u_i) - T_K(u_i)_\mu)\,dx\,dt = 0$$

Then we conclude the proof of (66).

Proof of (67):

If we pass to the lim-sup when n ,j and μ tends to $+\infty$ and then to the limit as m tends to $+\infty$ in (61). We obtain using (62)-(66), for any $K \geq 0$,

$$\lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{j\to+\infty}\limsup_{n\to+\infty}\int_{Q_T} S'_m(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\big(\nabla T_K(u_{i,n}) - \nabla T_K(\upsilon_{i,j})_\mu\big)\,dx\,dt \leq 0.$$

Since $S'_m(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_K(u_{i,n}) = a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_K(u_{i,n})$ for $n > K$ and $K \leq m$.

Then, for $K \leq m$,

$$\begin{aligned}&\limsup_{n\to+\infty}\int_{Q_T} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_K(u_{i,n})\,dx\,dt\\ &\leq \lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{j\to+\infty}\limsup_{n\to+\infty}\int_{Q_T} S'_m(u_{i,n})a_n(x,u_{i,n},\nabla u_{i,n})\nabla T_K(\upsilon_{i,j})_\mu\,dx\,dt\end{aligned} \tag{71}$$

Thanks to (59), we have in the right hand side of (71) for $n > m+1$ that,

$$S'_m(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n}) = S'_m(u_{i,n})a\Big(x,t,T_{m+1}(u_{i,n}),\nabla T_{m+1}(u_{i,n})\Big) \text{ a.e. in } Q_T.$$

Using (39), and fixing $m \geq 1$, we get

$$S'_m(u_{i,n})a_n(u_{i,n},\nabla u_{i,n}) \rightharpoonup S'_m(u_i)X_{i,m+1} \text{ weakly in } (L_{\overline{M}}(Q_T))^N.$$

when $n \to +\infty$.

We can pass to limit as $j \to +\infty$ and $\mu \to +\infty$, and using (56)-(57)

$$\begin{aligned}&\limsup_{\mu\to+\infty}\limsup_{j\to+\infty}\limsup_{n\to+\infty}\int_{Q_T} S'_m(u_{i,n}))a_n(x,t,u_{i,n}),\nabla u_{i,n}))\nabla T_K(\upsilon_{i,j})_\mu\,dx\,dt\\ &= \int_{Q_T} S'_m(u_i)X_{i,m+1}\nabla T_K(u_i)\,dx\,dt\\ &= \int_{Q_T} X_{i,m+1}\nabla T_K(u_i)\,dx\,dt\end{aligned} \tag{72}$$

where $K \leq m$, since $S'_m(r) = 1$ for $|r| \leq m$.

On the other hand, for $K \leq m$, we have

$$a\big(x,t,T_{m+1}(u_{i,n}),\nabla T_{m+1}(u_{i,n})\big)\chi_{\{|u_{i,n}|<K\}} = a\big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\big)\chi_{\{|u_{i,n}|<K\}},$$

a.e. in Q_T. Passing to the limit as $n \to +\infty$, we obtain

$$X_{i,m+1}\chi_{\{|u_i|<K\}} = X_{i,K}\chi_{\{|u_i|<K\}} \quad \text{a.e. in } Q_T - \{|u_i| = K\} \text{ for } K \leq n. \tag{73}$$

Then

$$X_{m+1}\nabla T_K(u_i) = X_K\nabla T_K(u_i) \quad \text{a.e. in } Q_T. \tag{74}$$

Then we obtain (67).

Proof of (68):
Let $K \geq 0$ be fixed. Using (10) we have

$$\int_{Q_T} \big[a(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})) - a(x,t,T_K(u_{i,n}),\nabla T_K(u_i))\big]\big[\nabla T_K(u_{i,n}) - \nabla T_K(u_i)\big]\,dx\,dt \geq 0, \tag{75}$$

In view (4) and (43), we get

$$a(x,t,T_K(u_{i,n}),\nabla T_K(u_i)) \to a(x,t,T_K(u_i),\nabla T_K(u_i)) \quad \text{a.e. in } Q_T,$$

as $n \to +\infty$, and by (8) and Lebesgue's theorem, we obtain

$$a\big(x,t,T_K(u_{i,n}),\nabla T_K(u_i)\big) \to a\big(x,t,T_K(u_i),\nabla T_K(u_i)\big) \quad \text{strongly in } (L_{\overline{M}}(Q_T))^N. \tag{76}$$

Using (67), (43), (39) and (76), we can pass to the lim-sup as $n \to +\infty$ in (75) to obtain (68).
To finish this step, we prove this lemma:
Lemma 7 For $i = 1,2$ and fixed $K \geq 0$, we have

$$X_{i,K} = a\big(xt,,T_K(u_i),\nabla T_K(u_i)\big) \quad \text{a.e. in } Q. \tag{77}$$

Also, as $n \to +\infty$,

$$a\big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\big)\nabla T_K(u_{i,n}) \rightharpoonup a\big(x,t,T_K(u_i),DT_K(u_i)\big)\nabla T_K(u_i), \tag{78}$$

weakly in $L^1(Q_T)$.

Proof

Proof of (77):
It's easy to see that

$$a_n(x,t,T_K(u_{i,n}),\xi) = a(x,t,T_K(u_{i,n}),\xi) = a_K(x,t,T_K(u_{i,n}),\xi) \quad \text{a.e. in } Q_T$$

for any $K \geq 0$, any $n > K$ and any $\xi \in R^N$.
In view of (39), (68) and (76) we obtain

$$\begin{aligned}&\lim_{n\to+\infty}\int_{Q_T} a_K\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n})\,dx\,dt\\ &= \int_{Q_T} X_{i,K}\nabla T_K(u_i)\,dx\,dt.\end{aligned} \tag{79}$$

Since (4), (8) and (43), imply that the function $a_K(x,s,\xi)$ is continuous and bounded with respect to s. Then we conclude that (77).
Proof of (78):
Using (10) and (68), for any $K \geq 0$ and any $T' < T$, we have

$$\big[a(x,t,T_K(u_{i,n},\nabla T_K(u_{i,n}))-a(x,t,T_K(u_{i,n}),\nabla T_K(u))\big]\big[\nabla T_K(u_{i,n})-\nabla T_K(u_i)\big] \to 0 \tag{80}$$

strongly in $L^1(Q_{T'})$ as $n \to +\infty$.
On the other hand with (43), (39), (76) and (77),we get

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_i) \rightharpoonup a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$

weakly in $L^1(Q_T)$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_i)\Big)\nabla T_K(u_{i,n}) \rightharpoonup a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$

weakly in $L^1(Q_T)$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_i)\Big)\nabla T_K(u_i)\to a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i),$$

strongly in $L^1(Q)$, as $n\to+\infty$.
It's results from (80), for any $K\geq 0$ and any $T'<T$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n})\rightharpoonup a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i) \tag{81}$$

weakly in $L^1(Q_{T'})$ as $n\to+\infty$.then for $T'=T$, we have (78).

Finally we should prove that u_i satisfies (18).

Step 4: Pass to the limit.
we first show that u satisfies (18)

$$\begin{aligned}&\int_{m\leq|u_{i,n}|\leq m+1\}} a(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt\\ &=\int_{Q_T} a_n(x,t,u_{i,n},\nabla u_{i,n})\Big[\nabla T_{m+1}(u_{i,n})-\nabla T_m(u_{i,n})\Big]\,dx\,dt\\ &=\int_{Q_T} a_n\Big(x,t,T_{m+1}(u_{i,n}),\nabla T_{m+1}(u_{i,n})\Big)\nabla T_{m+1}(u_{i,n})\,dx\,dt\\ &\quad-\int_{Q_T} a_n\Big(x,t,T_m(u_{i,n}),\nabla T_m(u_{i,n})\Big)\nabla T_m(u_{i,n})\,dx\,dt\end{aligned}$$

for $n>m+1$. According to (78), one can pass to the limit as $n\to+\infty$; for fixed $m\geq 0$ to obtain

$$\begin{aligned}&\lim_{n\to+\infty}\int_{m\leq|u_{i,n}|\leq m+1\}} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt\\ &=\int_Q a\Big(x,t,T_{m+1}(u_i),\nabla T_{m+1}(u_i)\Big)\nabla T_{m+1}(u_i)\,dx\,dt\\ &\quad-\int_Q a\Big(x,t,T_m(u_i),\nabla T_m(u_i)\Big)\nabla T_m(u_i)\,dx\,dt\\ &=\int_{m\leq|u_i|\leq m+1\}} a(x,t,u_i,\nabla u_i)\nabla u_i\,dx\,dt\end{aligned} \tag{82}$$

Pass to limit as m tends to $+\infty$ in (82) and using (40) show that u_i satisfies (18).

Now we shown that u_i to satisfy (19) and (20).
Let S be a function in $W^{2,\infty}(R)$ such that S' has a compact support. Let K be a positive real number such that $\operatorname{supp} S'\subset[-K,K]$. the Pointwise multiplication of the approximate equation (1) by $S'(u_{i,n})$ leads to

$$\begin{aligned}&\frac{\partial B^n_{i,S}(u_{i,n})}{\partial t}-\operatorname{div}\Big(S'(u_{i,n})a_n(x,u_{i,n},\nabla u_{i,n})\Big)+S''(u_{i,n})a_n(x,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\\ &-\operatorname{div}\Big(S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\Big)+S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}=f_{i,n}(x,u_{1,n},u_{1,n})S'(u_{i,n})\end{aligned} \tag{83}$$

in $D'(Q_T)$, for $i=1,2$.
Now we pass to the limit in each term of (83).

Limit of $\frac{\partial B^n_{i,S}(u_{i,n})}{\partial t}$: Since $B^n_{i,S}(u_{i,n})$ converges to $B_{i,S}(u_i)$ a.e. in Q_T and in $L^\infty(Q_T)$ weak $\star$ and S is bounded and continuous. Then $\frac{\partial B^n_{i,S}(u_{i,n})}{\partial t}$ converges to $\frac{\partial b_{i,S}(u_i)}{\partial t}$ in $D'(Q_T)$ as n tends to $+\infty$.

Limit of $div\Big(S'(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\Big)$**:** Since $\operatorname{supp} S' \subset [-K,K]$, for $n > K$, we have

$$S'(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n}) = S'(u_{i,n})a_n\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big) \quad \text{a.e. in } Q_T.$$

Using the pointwise convergence of $u_{i,n}$, (59),(39) and (77),imply that

$$S'(u_{i,n})a_n\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big) \rightharpoonup S'(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)$$

weakly in $(L_{\overline{M}}(Q_T))^N$, for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ as $n \to +\infty$, since $S'(u_i) = 0$ for $|u_i| \geq K$ a.e. in Q_T. And

$$S'(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big) = S'(u_i)a(x,t,u_i,\nabla u_i) \quad \text{a.e. in } Q_T.$$

Limit of $S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}$**.** Since $\operatorname{supp} S'' \subset [-K,K]$, for $n > K$, we have

$$S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} = S''(u_{i,n})a_n\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n}) \quad \text{a.e. in } Q_T.$$

The pointwise convergence of $S''(u_{i,n})$ to $S''(u_i)$ as $n \to +\infty$, (59) and (78) we have

$$S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} \rightharpoonup S''(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$

weakly in $L^1(Q_T)$, as $n \to +\infty$. And

$$S''(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i) = S''(u_i)a(x,t,u_i,\nabla u_i)\nabla u_i \quad \text{a.e. in } Q_T.$$

Limit of $S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})$**:** We have $S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n}) = S'(u_{i,n})\Phi_{i,n}(x,t,T_K(u_{i,n}))$ a.e. in Q_T,Since $\operatorname{supp} S' \subset [-K,K]$.Using (11), (45) and (37), it's easy to see that $S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n}) \rightharpoonup S'(u_i)\Phi_i(x,t,T_K(u_i))$ weakly for $\sigma(\Pi L_{\overline{M}}, \Pi L_M)$ as $n \to +\infty$. And $S'(u_i)\Phi_i(x,t,T_K(u_i)) = S'(u_i)\Phi_i(x,t,u_i)$ a.e. in Q_T.
Limit of $S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}$**:** Since $S' \in W^{1,\infty}(R)$ with $\operatorname{supp} S' \subset [-K,K]$, we have $S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} = \Phi_{i,n}(x,t,T_K(u_{i,n}))\nabla S'(T_K(u_{i,n}))$ a.e. in Q_T. The weakly convergence of truncation allows us to prove that

$$S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} \rightharpoonup \Phi_i(x,t,u_i)\nabla S'(u_i) \text{ strongly in } L^1(Q_T).$$

Limit of $f_{i,n}(x,u_{1,n},u_{2,n})S'(u_{i,n})$**:** Using (14), (15), (26) and (27), we have $f_{i,n}(x,u_{1,n},u_{2,n})S'(u_{i,n}) \to f_i(x,u_1,u_2)S'(u_i)$ strongly in $L^1(Q_T)$, as $n \to +\infty$.
It remains to show that $B_S(x,u_i)$ satisfies the initial condition (20) for i=1,2.
To this end, firstly remark that, in view of the definition of S'_M, we have $B_M(x,u_{i,n})$ is bounded in $L^\infty(Q_T)$.
Secondly, by (62) we show that $\dfrac{\partial B_M(x,u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T)+W^{-1,x}L_{\overline{M}}(Q_T))$.
As a consequence, an Aubin's type Lemma (see e.g., [14], Corollary 4) implies that $B_M(x,u_{i,n})$ lies in a compact set of $C^0([0,T];L^1(\Omega))$.
It follows that, on one hand,$B_M(x,u_i,n)(t=0)$ converges to $B_M(x,u_i)(t=0)$ strongly in $L^1(\Omega)$. On the order hand, the smoothness of B_M imply that $B_M(x,u_{i,n})(t=0)$ converges to $B_M(x,u_i)(t=0)$ strongly in $L^1(\Omega)$, we conclude that $B_M(x,u_{i,n})(t=$

$0) = B_M(x, u_{i,0n})$ converges to $B_M(x, u_i)(t = 0)$ strongly in $L^1(\Omega)$, we obtain $B_M(x, u_i)(t = 0) = B_M(x, u_{i,0})$ a.e. in Ω and for all $M > 0$, now letting M to $+\infty$, we conclude that $b(x, u_i)(t = 0) = b(x, u_{i,0})$ a.e. in Ω.

As a conclusion, the proof of Theorem (4) is complete.

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