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LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS RELATED THE $q-$GAMMA AND THE $q-$DIGAMMA FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper we present several new classes of logarithmically completely monotonic functions. Our functions have in common that they are defined in terms of the $q-$gamma and $q-$digamma functions. As an application of these results, some inequalities for the $q-$gamma and the $q-$digamma functions are established. Some of the given results generalize theorems due to Alzer and Berg and C.-P. Chen and F. Qi.

1. Introduction

It is well-known that the classical Euler gamma function may be defined by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,$$

for $x>0$. The logarithmic derivative of $\Gamma(x)$, denoted $\psi(x)=\frac{\Gamma^{'}(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for $k\in\mathbb{N}$ are called the polygamma functions. The functions $\Gamma(x)$ and $\psi^{(k)}(x)$ for $k\in\mathbb{N}$ are of fundamental importance in mathematics and have been extensively studied by many authors; see for example ([1, 2, 3, 5, 8]) and the references within.

The $q-$analogue of Γ is defined [[4], pp. 493-496] for $x>0$ by

$$\Gamma_q(x)=(1-q)^{1-x}\prod_{j=0}^{\infty}\frac{1-q^{j+1}}{1-q^{j+x}},\;0<q<1, \tag{1}$$

and

$$\Gamma_q(x)=(q-1)^{1-x}q^{\frac{x(x-1)}{2}}\prod_{j=0}^{\infty}\frac{1-q^{-(j+1)}}{1-q^{-(j+x)}},\;q>1. \tag{2}$$

The $q-$gamma function $\Gamma_q(z)$ has the following basic properties:

$$\lim_{q\longrightarrow 1^-}\Gamma_q(z)=\lim_{q\longrightarrow 1^+}\Gamma_q(z)=\Gamma(z), \tag{3}$$

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and

$$\Gamma_q(z) = q^{\frac{(x-1)(x-2)}{2}} \Gamma_{\frac{1}{q}}(z). \tag{4}$$

The $q-$digamma function ψ_q, the $q-$analogue of the psi or digamma function ψ is defined by

$$\begin{aligned}\psi_q(x) &= \frac{\Gamma_q'(x)}{\Gamma_q(z)} \\ &= -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+x}}{1-q^{k+x}} \\ &= -\ln(1-q) + \ln q \sum_{k=1}^{\infty} \frac{q^{kx}}{1-q^{k}} \\ &= -\ln(1-q) - \int_0^{\infty} \frac{e^{-xt}}{1-e^{-t}} d\gamma_q(t),\end{aligned} \tag{5}$$

for $0 < q < 1$, where $d\gamma_q(t)$ is a discrete measure with positive masses $-\ln q$ at the positive points $-k \ln q$ for $k \in \mathbb{N}$, more accurately, (see [9])

$$\gamma_q(t) = \sum_{k=1}^{\infty} \delta(t + k \ln q),\ 0 < q < 1. \tag{6}$$

For $q > 1$ and $x > 0$, the $q-$digamma function ψ_q is defined by

$$\begin{aligned}\psi_q(x) &= -\ln(q-1) + \ln q \left[x - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-(k+x)}}{1-q^{-(k+x)}} \right] \\ &= -\ln(q-1) + \ln q \left[x - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{q^{-kx}}{1-q^{-k}} \right]\end{aligned}$$

Krattenthaler and Srivastava [10] proved that $\psi_q(x)$ tends to $\psi(x)$ on letting $q \longrightarrow 1$ where $\psi(x)$ is the the ordinary psi (digamma) function. Before we present the main results of this paper we recall some definitions, which will be used in the sequel.

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0, \tag{7}$$

for all $x \in I$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ the set of all positive integers.

If the inequality (7) is strict, then f is said to be strictly completely monotonic function.

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n \Big(\ln f(x) \Big)^{(n)} (x) \geq 0,$$

for all $x \in I$ and $n \in \mathbb{N}$.

A positive function f is said to be logarithmically convex on an interval I, or simply log-convex, if its natural logarithm $\ln f$ is convex, that is, for all $x, y \in I$ and $\lambda \in [0,1]$ we have

$$f\left(\lambda x + (1-\lambda) y\right) \leq \left[f(x)\right]^{\lambda} \left[f(y)\right]^{1-\lambda}.$$

We note that every logarithmically completely monotonic function is log-convex. From the definition of $\psi_q(x)$, direct differentiation, and the induction we get

$$(-1)^n\psi_q^{(n+1)}(x) = (-1)^n(\ln q)^{n+2}\sum_{k=1}^{n}\frac{k^{n+1}q^{kx}}{1-q^k} > 0, \tag{8}$$

which implies that the function $\psi_q^{'}$ is strictly completely monotonic function on $(0,\infty)$, for $q\in(0,1)$. The relation (4) and the definition of the $q-$digamma function (5) give

$$\psi_q(x) = \psi_{\frac{1}{q}}(x) + \frac{2x-3}{2}\ln q, \tag{9}$$

for $q>1$. Thus, implies that the function $\psi_q^{'}(x)$ is strictly completely monotonic in $(0,\infty)$ for $q>1$, and consequently the function $\psi_q(x)$ is strictly increasing on $(0,\infty)$.

It is the aim of this paper to provide several new classes of logarithmically completely monotonic functions. The functions we study have in common that they are defined in terms of $q-$gamma and $q-$digamma functions. In the next section we collect some lemmas. Our monotonicity theorems are stated and proved in sections 3.

2. **Useful lemmas**

We begin this section with the following useful lemmas which are needed to complete the proof of the main theorems.

The following monotonicity theorem is proved in [1].

Lemma 1. *Let n be a natural number and c be a real number. The function $x^c|\psi^n(x)|$ is decreasing on $(0,\infty)$ if and only if $c\leq n$.*

Lemma 2. [12] *Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be two continuous functions which are differentiable on (a,b). Further, let $g'\neq 0$ on (a,b). If $\frac{f'}{g'}$ is increasing (or decreasing) on (a,b), then the functions $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are also increasing (or decreasing) on (a,b).*

The next lemma is given in [2, 15].

Lemma 3. *The function ψ_q, $q>0$ has a uniquely determined positive zero, which we denoted by $x_0=x_0(q)\in(1,2)$.*

A proof for the following lemma can be found in [7].

Lemma 4. *Let f be a positive function. If $f^{'}$ is completely monotonic function, then $\frac{1}{f}$ is logarithmically completely monotonic function.*

3. **The main results**

In 1997, Merkle [13] proved that the function $\frac{(\Gamma(x))^2}{\Gamma(2x)}$ is log-convex on $(0,\infty)$. Recently, Alzer and Berg [3], presented a substantial generalization. They established that the function

$$\frac{(\Gamma(ax))^{\alpha}}{(\Gamma(bx))^{\beta}},\ 0<b<a, \tag{10}$$

is completely monotonic on $(0,\infty)$, if and only if, $\alpha\leq 0$ and $\alpha a=\beta b$. The main objective of the next Theorem extend and generalize this results. An application of their result leads to sharp upper and lower bounds for $\frac{\Gamma_q^2(x)}{\Gamma_q(2x)}$ in terms of the ψ_q-function.

Theorem 1. *Let $0 < q < 1$ and $0 < b < a$. Then the function $\frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta}$ is logarithmically completely monotonic on $(0, \infty)$, if and only if $\alpha a = \beta b$ and $\alpha \leq 0$.*

Proof. Let $q \in (0, 1)$. Assume that the function $\frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta}$ is logarithmically completely monotonic on $(0, \infty)$. By definition, we have for all $x > 0$

$$\begin{aligned} f(x) = \left(\ln \frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta} \right)' &= \alpha a \psi_q(ax) - \beta b \psi_q(bx), \\ &= \alpha a \left(\psi_q(ax) - \ln \left(\frac{1 - q^{ax}}{1 - q} \right) \right) - \beta b \left(\psi_q(bx) - \ln \left(\frac{1 - q^{bx}}{1 - q} \right) \right) \\ &\quad + \alpha a \ln \left(\frac{1 - q^{xa}}{1 - q} \right) - \beta b \ln \left(\frac{1 - q^{xb}}{1 - q} \right). \end{aligned} \tag{11}$$

It is worth mentioning that, Moak [14] proved the following approximation for the $q-$digamma function

$$\psi_q(x) = \ln \left(\frac{1 - q^x}{1 - q} \right) + \frac{1}{2} \frac{\ln q \; q^x}{1 - q^x} + O\left(\frac{\ln^2 q \; q^{2x}}{(1 - q^x)^2} \right) \tag{12}$$

holds for $q > 0$ and $x > 0$. So, if $\alpha a - \beta b > 0$, then $\lim_{x \longrightarrow \infty} f(x) = (\beta b - \alpha a) \ln(1 - q) \geq 0$ for $0 < q < 1$ and if $\alpha a - \beta b < 0$, then $\lim_{x \longrightarrow \infty} f(x) = (\beta b - \alpha a) \ln(1 - q) \leq 0$ for $0 < q < 1$. Thus, $\alpha a = \beta b$, and consequently

$$f(x) = \alpha a (\psi_q(ax) - \psi_q(bx)).$$

Since ψ_q is increasing on $(0, \infty)$ and $f(x) \leq 0$ by definition, we conclude that that $\alpha \leq 0$.

Conversely, We show that the function $\frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta}$ is logarithmically completely monotonic on $(0, \infty)$ for $\alpha \leq 0,\ 0 < b < a$ such that $\alpha a = \beta b$. Let $n = 1$, since the function $\psi_q(x)$ is strictly increasing on $(0, \infty)$ we have

$$(-1) \left(\ln \frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta} \right)' = \alpha a \left(\psi_q(bx) - \psi_q(ax) \right) \geq 0. \tag{13}$$

For $n \geq 1$, we get

$$(-1)^{n+1} \left(\ln \frac{(\Gamma_q(ax))^\alpha}{(\Gamma_q(bx))^\beta} \right)^{(n+1)} = (-1)^n \alpha a \left(b^n \psi_q^{(n)}(bx) - a^n \psi_q^{(n)}(ax) \right). \tag{14}$$

Since the function $\psi_q'(x)$ is strictly completely monotonic on $(0, \infty)$, we have for $n \in \mathbb{N}$ and $q > 0$

$$|\psi_q^{(n)}(x)| = (-1)^{n+1} \psi_q^{(n)}(x). \tag{15}$$

Thus

$$\begin{aligned}(-1)^{n+1}x^n\left(\ln\frac{(\Gamma_q(ax))^{\alpha}}{(\Gamma_q(bx))^{\beta}}\right)^{(n+1)} &= (-1)^n\alpha a x^n\left(b^n\psi_q^{(n)}(bx)-a^n\psi_q^{(n)}(ax)\right)\\ &= (-1)^n\alpha a\left((xb)^n(-1)^n|\psi_q^{(n)}(bx)|-(ax)^n(-1)^n|\psi_q^{(n)}(ax)|\right)\\ &= \alpha a\Bigg((ax)^n|\psi_q^{(n)}(ax)|-(bx)^n|\psi_q^{(n)}(bx)|\Bigg),\end{aligned}$$

and the last expression is nonnegative by Lemma 1. Hence, for $q\in(0,1)$ and $n\in\mathbb{N}$

$$(-1)^n\left(\ln\frac{(\Gamma_q(ax))^{\alpha}}{(\Gamma_q(bx))^{\beta}}\right)^{(n)}\geq 0.$$

The proof of Theorem 1 is complete. □

Remark 1. *We note that if interchanging the roles of a and b leads to changing the sign on α.*

Corollary 1. *Let $q>1$ and $0<a<b$. If, $\alpha\geq 0$ and $\alpha a=\beta b$., Then the function $\frac{(\Gamma_q(ax))^{\alpha}}{(\Gamma_q(bx))^{\beta}}$ is logarithmically completely monotonic on $(0,\infty)$.*

Proof. Follows immediately by Theorem 1 and equality (9). □

Corollary 2. *Let $q>0$ and $0<a<b$, the following inequalities*

$$\exp\left[\alpha a(x-x_1)\left(\psi_q(ax_1)-\psi_q(bx_1)\right)\right]\leq\frac{(\Gamma_q(bx_1))^{\beta}}{(\Gamma_q(ax_1))^{\alpha}}\frac{(\Gamma_q(ax))^{\alpha}}{(\Gamma_q(bx))^{\beta}}\leq 1, \tag{16}$$

holds for all $\alpha,\ \beta\geq 0$ such that $\alpha a=\beta b$. and $x>x_1>0$. In particular, the following inequalities holds true for every integer $n\geq 1$:

$$\exp\left[2q(n-1)\frac{\ln q}{1-q}\right]\leq\frac{\Gamma_q^2(n)}{\Gamma_q(2n)}\leq 1. \tag{17}$$

Proof. Let $q>0$ and $0<a<b$. We suppose that $\alpha,\ \beta\geq 0$ such that $\alpha a=\beta b$ and define the function $h_{\alpha,\beta}(q;x)$ by

$$h_{\alpha,\beta}(q;x)=\frac{(\Gamma_q(bx_1))^{\beta}}{(\Gamma_q(ax_1))^{\alpha}}\frac{(\Gamma_q(ax))^{\alpha}}{(\Gamma_q(bx))^{\beta}}$$

where $0<x_1<x$, and $H_{\alpha,\beta,q}(x)=\ln h_{\alpha,\beta}(q;x)$. Since the function $h_{\alpha,\beta}(q;x)$ is logarithmically completely monoyonic on $(0,\infty)$ for $\alpha\geq 0$ and $\alpha a=\beta b$, we conclude that the logarithmic derivative $\frac{(h_{\alpha,\beta}(q;x))'}{h_{\alpha,\beta}(q;x)}$ is increasing on $(0,\infty)$. By Lemma 2 we deduce that the function $\frac{H_{\alpha,\beta}(q;x)}{x-x_1}$ is increasing for all $0<x_1<x$. By l'Hospital's rule and (11) it is easy to deduce that

$$\lim_{x\longrightarrow x_1}\frac{H_{\alpha,\beta}(q;x)}{x-x_1}=\alpha a(\psi_q(ax_1)-\psi_q(bx_1)),$$

from which follows the right side inequality of (16).

As $h_{\alpha,\beta}(q;x)$ is logarithmically completely monotonic on $(0,\infty)$, we deduce that $h_{\alpha,\beta}(q;x)$ is decreasing on $(0,\infty)$. The following inequality hold true for all $0<x_1<x$:

$$h_{\alpha,\beta}(q;x)\leq h_{\alpha,\beta}(q;x_1)=1,$$

we conclude the left side inequality of (16).

Taking $\alpha = b = 2,\ \beta = a = 1$ and $x_1 = 1$ in (16) and using the recurrence formula of ψ_q [[8], p. 1245, Theorem 4.4]

$$\psi_q^{(n-1)}(x+1) - \psi_q^{(n-1)}(x) = -\frac{d^{n-1}}{dx^{n-1}}\left(\frac{q^x}{1-q^x}\right)\ln q \tag{18}$$

we obtain the inequalities (17). □

The main purpose of the next Theorem is to present monotonicity properties of the function

$$g_\beta(q;x) = \frac{1}{1+q}\left[\frac{\Gamma_{q^2}\left(x+\frac{1}{2}\right)}{\Gamma_{q^2}(x+1)}\right]^2 \exp\left[\frac{\beta(1-q^2)q^{2x}}{2(1-q^{2x})} + \psi_q(2x)\right], \tag{19}$$

where $q \in (0,1)$ and $x > 0$.

It is worth mentioning that Ai-Jun Li and Chao-Ping Chen [11] considered the function

$$g_\beta(x) = \frac{1}{2}\left[\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)}\right]^2 \exp\left[\frac{\beta}{2x} + \psi(2x)\right] \tag{20}$$

which is a special case of the function $g_\beta(q;x)$ on letting $q \longrightarrow 1$ and proved that $g_\beta(x)$ is logarithmically completely monotonic on $(0,\infty)$ if $\beta \geq \frac{13}{12}$. The objective of this Theorem is to generalize this result.

Theorem 2. *Let $q \in (0,1)$. The function $g_\beta(q;x)$ defined by (19) is logarithmically completely monotonic on $(0,\infty)$ if $\beta \geq \frac{-13\ln q}{6(1-q^2)}$.*

Proof. It is clear that

$$\ln g_\beta(q;x) = 2\ln\Gamma_{q^2}(x+\frac{1}{2}) - 2\ln\Gamma_{q^2}(x+1) + \psi_q(2x) + \frac{\beta(1-q^2)q^{2x}}{2(1-q^{2x})} - \ln(q+1).$$

Using the $q-$analogue of Legendre's duplication formula [5]

$$\Gamma_q(2x)\Gamma_{q^2}(\frac{1}{2}) = (1+q)^{2x+1}\Gamma_{q^2}(x)\Gamma_{q^2}(x+\frac{1}{2}) \tag{21}$$

we get

$$\ln g_\beta(q;x) = 2\ln\Gamma_{q^2}(x+\frac{1}{2}) - 2\ln\Gamma_{q^2}(x+1) + \frac{1}{2}\psi_{q^2}(x) + \frac{1}{2}\psi_{q^2}(x+\frac{1}{2}) + \frac{\beta(1-q^2)q^{2x}}{2(1-q^{2x})}.$$

In view of (18) and (5) we obtain that

$$\begin{aligned}(-1)^n\left(\ln g_\beta(q;x)\right)^{(n)} &= (-1)^n\Bigg[2\psi_{q^2}^{(n-1)}(x+\frac{1}{2}) - 2\psi_{q^2}^{(n-1)}(x+1) + \frac{1}{2}\psi_{q^2}^{(n)}(x) + \frac{1}{2}\psi_{q^2}^{(n)}(x+\frac{1}{2})\\ &\quad - \frac{\beta(1-q^2)}{4\ln q}\left(\psi_{q^2}^{(n)}(x+1) - \psi_{q^2}^{(n)}(x)\right)\Bigg]\\ &= \frac{1}{2}\int_0^\infty \frac{e^{-xt}}{e^t-1}\Phi_{\beta,q}(t)d\gamma_{q^2}(t)\end{aligned}$$

where

$$\begin{aligned}
\Phi_{\beta,q}(t) &= \left(\frac{-\beta(1-q^2)}{2\ln q} - 1\right) te^t + (4-t)e^{\frac{t}{2}} + \frac{\beta(1-q^2)}{2\ln q}t - 4\\
&= \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!}\left[\frac{-\beta(1-q^2)}{2\ln q} - 1 - \frac{1}{2^n}\right] + 4\sum_{n=1}^{\infty}\frac{t^n}{2^n n!} + \frac{\beta(1-q^2)}{2\ln q}t\\
&= \sum_{n=1}^{\infty} \frac{t^{n+1}}{n!}\left[\frac{-\beta(1-q^2)}{2\ln q} - 1 - \frac{1}{2^n}\right] + 4\sum_{n=0}^{\infty}\frac{t^{n+1}}{2^{n+1}(n+1)!} - 2t\\
&= \sum_{n=1}^{\infty} \frac{t^{n+1}}{n!}\left[\frac{-\beta(1-q^2)}{2\ln q} - 1 - \frac{1}{2^n} + \frac{1}{(n+1)2^{n-1}}\right]
\end{aligned}$$

Since the max $\left(\frac{1}{2^n} + \frac{1}{(n+1)2^{n-1}}\right) = \frac{1}{12}$, we conclude that $\beta \geq \frac{-13\ln q}{6(1-q^2)}$. The proof is completed. □

Theorem 3. *Let $q > 0$, the function $\frac{1}{\psi_q(x)}$ is Logarithmically completely monotonic on (x_0, ∞).*

Proof. Since the function $\psi_q^{'}$ is completely monotonic function on $(0,\infty)$, and the function ψ_q is increasing on $(0,\infty)$ and a uniquely determined zero on $(0,\infty)$ by Lemma 3. We conclude that the function $\psi_q(x) > 0$ for all $x > x_0$, and consequently the function $\frac{1}{\psi_q(x)}$ is Logarithmically completely monotonic on (x_0,∞), by Lemma 4. □

Corollary 3. *Let $q > 0$ and $a > 1$. The following inequality*

$$[\psi_q(x)]^{\frac{1}{a}}\,[\psi_q(y)]^{1-\frac{1}{a}} \leq \psi_q\left[\frac{x}{a} + \left(1 - \frac{1}{a}\right)y\right] \tag{22}$$

holds for all $x > x_0$ and $y > x_0$. In particular, the following inequality holds

$$[\psi_q(2)]^{a-1} \leq \frac{[\psi_q(u+1)]^a}{\psi_q\left(a(x-1)+2\right)} \tag{23}$$

for all $a > 1$ and $u > \frac{-2}{a} + 1$.

Proof. Let $q \in (0,1), a > 1, x > x_0$ and $y > x_0$. By theorem 3 we obtain that the function $\frac{1}{\psi_q(x)}$ is log-convex on (x_0,∞). Thus,

$$[\psi_q(x)]^{\frac{1}{p}}\,[\psi_q(y)]^{\frac{1}{q}} \leq \psi_q\left[\frac{x}{p} + \frac{y}{q}\right],$$

where $p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$. If $p = a$ and $q = \frac{a}{a-1}$. Then we get the inequality (22). Now, let $y = 2$ and $x = a(u-1)+2$ we obtain the inequality (23). □

Remark 2. *Replacing u by $n \in \mathbb{N}$ and a by 2 in inequality (23) and using the identity*

$$\psi_q(n+1) = \frac{\ln q}{1-q}\gamma_q - \ln q H_{n,q}$$

where $\gamma_q = \frac{1-q}{\ln q}\psi_q(1)$ *is the* $q-$*analogue of the Euler-Mascheroni constant* [16] *and* $H_{n,q}$ *is the q-analogue of Harmonic number is defined by* [17] *as*

$$H_{n,q} = \sum_{k=1}^{n} \frac{q^k}{1-q^k},\ n \in \mathbb{N},$$

we obtain

$$\psi_q^2(2)\psi_q(2n) \leq \left[\frac{\ln q}{1-q}\gamma_q - \ln q H_{n,q}\right]^2,\ n \in \mathbb{N}. \tag{24}$$

Theorem 4. *Let* $q > 0$ *The function* $\Gamma_q(x)$ *is logarithmically completely monotonic on* (x_0, ∞). *So, the following inequality*

$$\Gamma_q\left(\frac{x+y}{2}\right) \leq \Gamma_q(x)\Gamma_q(y). \tag{25}$$

holds for all $x, y > x_0$.

Proof. Proving by induction that

$$(-1)^n(\ln\Gamma_q(x))^{(n)} \geq 0, \text{ for all } n \in \mathbb{N}.$$

For $n = 1$, we get

$$(-1)(\ln\Gamma_q(x))' = \psi_q(x) \geq 0, \text{ for all } x > x_0.$$

Suppose that

$$(-1)^k(\ln\Gamma_q(x))^{(k)} \geq 0, \text{ for all } 1 \leq k \leq n \text{ and } x \in (x_0, \infty).$$

Since the function ψ_q' is completely monotonic on $(0, \infty)$, for $q > 0$ we get

$$(-1)^{n+1}(\ln\Gamma_q(x))^{(n+1)} = (-1)^n\psi_q^{(n+1)}(x) \geq 0.$$

We note that every logarithmically completely monotonic function is log-convex on $(0, \infty)$, that is, for all $x, y > x_0$ and $t \in [0, 1]$, we have

$$\Gamma_q(tx + (1-t)y) \leq \Gamma_q(x)^t\Gamma_q(y)^{1-t}.$$

Choosing t=1/2 in the above inequality we obtain the desired result. The proof of Thoerem 4 is completes. □

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