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ON K-ATOMIC DECOMPOSITIONS IN BANACH SPACES

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Abstract. L.Gavruta [9] first introduced frames for an operator K called K-frames in Hilbert spaces. In this paper, we define K-atomic decompositions for Banach spaces and obtain various results related to the existence of K-atomic decompositions. Also, we discuss several methods for constructing K-atomic decompositions together with perturbation results for K-atomic decompositions.

1. Introduction

Danis Gabör [8] introduced a fundamental approach to signal decomposition in terms of elementary signals. Duffin and Schaeffer [6] while addressing some deep problems in non-harmonic Fourier series, abstracted Gabor's method to define frames for Hilbert space. Feichtinger and Gröcheing [7] extended the notion of atomic decomposition to Banach space. Gröcheing [10] introduced a more general concept for Banach spaces called Banach frame. Banach frames and atomic decompositions were further studied in [4].
Christensen [3] proved perturbation results for Banach frames and atomic decompositions. Casazza et al. [2] studied X_d-frames and X_d-Bessel sequences in Banach spaces. Stoeva [5] gave some perturbation results for X_d-frames and atomic decompositions. Gavruta [9] introduced the notion of atomic system for an operator K and the notion of K-frame in a Hilbert space. X.Xiao et al. [16] discussed relationship between K-frames and ordinary frames in Hilbert spaces. Terekhin [15] introduced and studied frames in Banach spaces.

In the present paper, we define K-atomic decomposition for a Banach space and prove some results on the existence of K-atomic decompositions. Also, we discuss several methods to construct K-atomic decomposition for Banach Spaces and finally obtain some perturbation results for K-atomic decompositions.

2. Preliminaries

Throughout this paper, E will denote a Banach space over the scalar field K($\mathbb{R}$ or $\mathbb{C}$), E^* the dual space of E, E_d a BK-space and $L(E)$ will denote the set of all

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bounded linear operators from E into E. For $T \in L(E)$, T^* denotes the adjoint of T, $\pi : E \longrightarrow E^{**}$ is the natural canonical projection from E onto E^{**}.

Definition 2.1. [10] Let E be a Banach space and E_d be a BK-space. A sequence $(x_n, f_n)(\{x_n\} \subset E, \{f_n\} \subset E^*)$ is called an *atomic decomposition* for E with respect to E_d if the following statements hold:

(a) $\{f_n(x)\} \in E_d$, for all $x \in E$.
(b) There exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \;\leq\; \|\{f_n(x)\}\|_{E_d} \;\leq\; B\|x\|_E, \text{ for all } x \in E \tag{1}$$

(c) $x = \sum\limits_{n=1}^{\infty} f_n(x)x_n$, for all $x \in E$.

Definition 2.2. [2] A sequence $\{f_n\} \subseteq E^*$ is called an *E_d-frame* for E if

(a) $\{f_n(x)\} \in E_d$, for all $x \in E$.
(b) There exist constants A and B with $0 < A \leq B < \infty$ such that

$$A \parallel x \parallel_E \leq \parallel \{f_n(x)\} \parallel_{E_d} \leq B \parallel x \parallel_E, \text{ for all } x \in E. \tag{2}$$

The constants A and B are called *E_d-frame* bounds. If atleast (a) and the upper bound condition in (2.2) are satisfied, then $\{f_n\}$ is called an *E_d-Bessel sequence* for E.

If $\{f_n\}$ is an *E_d-frame* for E and if there exists a bounded linear operator $T : E_d \longrightarrow E$ such that $T(\{f_n(x)\}) = x$, for all $x \in E$, then $(\{f_n\}, T)$ is called a *Banach frame* for E with respect to E_d.

Definition 2.3. [12] Let $T \in L(E)$. We say that an operator $S \in L(E)$ is a pseudo inverse of T if $TST = T$. Also, $S \in L(E)$ is called the generalized inverse of T if $TST = T$ and $STS = S$.

Next, we state some results in the form of lemmas which will be used in the subsequent results.

Lemma 2.4. [14, 17] *Let X, Y be Banach spaces and $T : X \longrightarrow Y$ be a bounded linear operator. Then, the following conditions are equivalent:*

(a) There exist two continuous projection operators $P : X \to X$ and $Q : Y \to Y$ such that

$$P(X) = kerT \text{ and } Q(Y) = T(X). \tag{3}$$

(b) T has a pseudo inverse operator T^+.

If two continuous projection operators $P : X \to X$ and $Q : Y \to Y$ satisfies (2.3), then there exists a pseudo inverse operator T^+ of T such that $T^+T = I_X - P$ and $TT^+ = Q$, where I_X is the identity operator on X.

Lemma 2.5. [1, 13] *Let E be a Banach space. If $T \in L(E)$ has a generalized inverse $S \in L(E)$, then TS, ST are projections and $TS(E) = T(E)$ and $ST(E) = S(E)$.*

Lemma 2.6. [11] *Let E be a Banach space and $\{f_n\} \subset E^*$ be a sequence such that $\{x \in E : f_n(x) = 0, for\ all\ n \in \mathbb{N}\} = \{0\}$. Then E is linearly isometric to the Banach space $X_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_{X_d} = \|x\|_E$, $x \in E$.*

3. K-Atomic Decompositions

Definition 3.1. Let E be a Banach Space, $\{x_n\} \subset E, \{f_n\} \subset E^*$ and $K \in L(E)$. A pair (x_n, f_n) is called a *K-atomic decomposition* for E with respect to E_d if

(a) $\{f_n(x)\} \in E_d$, for all $x \in E$.

(b) There exist constants A and B with $0 < A \leq B < \infty$ such that

$$A \parallel K(x) \parallel_E \leq \parallel \{f_n(x)\} \|_{E_d} \leq B \parallel x \parallel_E, \text{ for all } x \in E.$$

(c) $\sum\limits_{n=1}^{\infty} f_n(x) x_n$ converges for all $x \in E$ and $K(x) = \sum\limits_{n=1}^{\infty} f_n(x) x_n$.

The constants A and B are called lower and upper bounds of the K-atomic decomposition (x_n, f_n).

Remark 3.2. Let (x_n, f_n) be a K-atomic decomposition for E with respect to E_d and with bounds A and B.

(I). If $K = I_E$, then (x_n, f_n) is an atomic decomposition for E with respect to E_d with bounds A and B.

(II). If K is invertible, then $(K^{-1}(x_n), f_n)$ is an atomic decomposition for E with respect to E_d.

(III). If K is invertible, then there exists a bounded linear operator $T : E_d \longrightarrow E$ such that $(\{f_n\}, T)$ is a Banach frame with respect to some BK-space E_d.

In the following example, we show the existence of K-atomic decomposition for a Banach space E with respect to an associated BK space E_d .

Example 3.3. Let E be a Banach Space. Let$\{x_n\} \subseteq E$, $\{f_n\} \subseteq E^*$ such that $\sum\limits_{n=1}^{\infty} f_n(x) x_n$ converges for all $x \in E$ and $x_n \neq 0$, for all $n \in \mathbb{N}$. Also, let $E_d = \{\{\alpha_n\} | \sum\limits_{n=1}^{\infty} \alpha_n x_n \ converges\}$. Then E_d is a BK-space with norm $\|\{\alpha_n\}\|_{E_d} = \sup\limits_{1 \leq n < \infty} \parallel \sum\limits_{k=1}^{n} \alpha_k x_k \parallel$. Define $T : E_d \longrightarrow E$ as $T\{\alpha_n\} = \sum\limits_{n=1}^{\infty} \alpha_n x_n$ and $S : E \longrightarrow E_d$ as $S(x) = \{f_n(x)\}$, $x \in E$. Take $K = TS$. Then $K : E \longrightarrow E$ is such that $K(x) = TS(x) = \sum\limits_{n=1}^{\infty} f_n(x) x_n$, for all $x \in E$. Clearly, $\{f_n(x)\} \in E_d$ and

$$\begin{aligned} \|K(x)\|_E &= \left\| \sum_{n=1}^{\infty} f_n(x) x_n \right\| \leq \sup_{1 \leq n < \infty} \left\| \sum_{k=1}^{n} f_k(x) x_k \right\| \\ &= \|\{f_n(x)\}\|_{E_d} \leq \sigma \parallel x \parallel_E, \ for \ all \ x \in E, \end{aligned}$$

where $S_n(x) = \sum\limits_{k=1}^{n} f_k(x) x_k$ and $\sigma = \sup\limits_{1 \leq n < \infty} \parallel S_n \parallel$.

Hence, (x_n, f_n) is a K-atomic decomposition for E with respect to E_d.

Next, we give an example of a K-atomic decomposition for E which is not an atomic decomposition for E.

Example 3.4. Let $E = c_0$ *and* $E_d = l_\infty$. Let $\{x_n\} \subset E$ be the sequence of standard unit vectors in E and $\{f_n\} \subseteq E^*$ be such that for $x = \{\alpha_n\} \in E, f_1(x) = 0, f_2(x) = \alpha_2, ..., f_n(x) = \alpha_n, ...$ It is clear that $\sum\limits_{n=1}^{\infty} f_n(x) x_n$ converges for $x \in E$.

Define $K : E \longrightarrow E$ by $K(x) = \sum\limits_{n=1}^{\infty} f_n(x)x_n$, $x \in E$. Then $\{f_n(x)\} \in E_d$ is such that (x_n, f_n) is a K-atomic decomposition for E with respect to E_d. But (x_n, f_n) is not an atomic decomposition for E.

Next, we give several methods to construct K-atomic decompositions for E.

Theorem 3.5. *Let (x_n, f_n) be an atomic decomposition for E with respect to E_d with bounds A and B. Let $K \in L(E)$ with $K \neq 0$. Then*
(a) (Kx_n, f_n) is a K-atomic decomposition for E with respect to E_d.
(b) $(x_n, K^*(f_n))$ is a K-atomic decomposition for E with respect to E_d.

Proof. (a) For each $x \in E$, $K(x) = \sum\limits_{n=1}^{\infty} f_n(x)K(x_n)$. Also, we have $\| K(x) \|_E \leq \| K \| \| x \|_E$, for all $x \in E$. This gives

$$\frac{A}{\| K \|} \| K(x) \|_E \leq \| \{f_n(x)\}\|_{E_d} \leq B \| x \|_E, \text{ for all } x \in E.$$

(b) For each $x \in E$ and $n \in \mathbb{N}$, we have

$$K(x) = \sum_{n=1}^{\infty} f_n(K(x))x_n = \sum_{n=1}^{\infty} g_n(x)x_n,$$

where $g_n = K^*f_n,\ n \in \mathbb{N}$. Also

$$\{g_n(x)\} = \{(K^*f_n)(x)\} = \{f_n(K(x))\} \in E_d, \text{for } all\ x \in E.$$

Note that

$$A\|K(x)\|_E \leq \| \{f_n(K(x))\|_{E_d} = \|\{K^*f_n(x)\}\|_{E_d}, \text{for all } x \in E.$$

and

$$\|\{(K^*f_n)(x)\}\|_{E_d} = \|\{f_n(K(x))\}\|_{E_d} \leq B\|K(x)\|_E, \text{ for all } \ x \in E.$$

Hence

$$A\|K(x)\|_E \leq \| \{g_n(x)\}\|_{E_d} \leq B' \| x \|_E, \text{for all } x \in E,$$

where $B' = B\|K\|$. □

Theorem 3.6. *Let (x_n, f_n) be a K-atomic decomposition for E with respect to E_d and $T \in L(E)$. Then*
(a) (Tx_n, f_n) is a TK-atomic decomposition for E with respect to E_d.
(b) (x_n, T^*f_n) is a KT-atomic decomposition for E with respect to E_d.

Proof. (a)Straight forward.

(b)Since (x_n, f_n) is an K-atomic decomposition for E,

$$KT(x) = \sum_{n=1}^{\infty}(T^*f_n)(x)x_n = \sum_{n=1}^{\infty} g_n(x)x_n,$$

where $g_n = T^*f_n$ and $x \in E$. Also, we have

$$\{g_n(x)\} = \{f_n(T(x))\} \in E_d, for\ all\ x \in E.$$

Further, for $x \in E$, we have

$$\|\{g_n(x)\}\|_{E_d} = \|\{f_n(T(x))\}\|_{E_d} \leq B\|T\|\|x\|_E.$$

and

$$A\|KT(x)\|_E \leq \|\{f_n(T(x))\}_{E_d} = \|\{(T^*f_n)(x)\}\|_{E_d} = \|\{g_n(x)\}\|_{E_d}.$$

Hence

$$A\|KT(x)\|_E \leq \|\{g_n(x)\}\|_{E_d} \leq B\|T\|\|x\|_E,\ x \in E.$$

□

Theorem 3.7. *If (x_n, f_n) is a K-atomic decomposition for E with respect to E_d and K has pseudo inverse K^+, then there exists $\{g_n\} \subseteq E^*$ such that (x_n, g_n) is a K-atomic decomposition for E with respect to E_d.*

Proof. Let A and B be positive constants such that

$$A\|K(x)\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E,\ x \in E.$$

Also, for each $x \in E$, we have

$$K(x) = \sum_{n=1}^{\infty} f_n(K^+K(x))x_n = \sum_{n=1}^{\infty}((K^+K)^*(f_n))(x)x_n.$$

For each $n \in \mathbb{N}$, define $g_n = (K^+K)^*(f_n)$. Then

$$\|K(x)\|_E \leq \frac{1}{A} \parallel \{f_n(K^+K(x))\}\|_{E_d} = \frac{1}{A}\|\{g_n(x)\}\|_{E_d},\ x \in E$$

and

$$\|\{g_n(x)\}\|_{E_d} = \|\{f_n(K^+K(x))\}\|_{E_d} \leq B\|K^+\|\|K\|\|x\|_E,\ x \in E.$$

Hence, we conclude that (x_n, g_n) is a K-atomic decomposition for E with respect to E_d. □

In the next two results, we give necessary conditions under which an E_d frame gives rise to a bounded operator K with respect to which there is a K-atomic decomposition for E.

Theorem 3.8. *Let $\{f_n\} \subseteq E^*$ be an E_d-frame for E with bounds A and B. Let $\{x_n\} \subseteq E$ with $\sup_{1 \leq n < \infty} \|x_n\| < \infty$ and let $\sum_{n=1}^{\infty} |f_n(x)| < \infty$, for all $x \in E$. Then there exists an operator $K \in L(E)$ such that (x_n, f_n) is a K-atomic decomposition for E with respect to E_d.*

Proof. Let $n, m \in \mathbb{N}$ with $n \leq m$.Then

$$\|\sum_{k=n}^{m} f_k(x)x_k\|_E \leq \sup_{1 \leq j < \infty} \|x_j\|_E \sum_{k=n}^{m} |f_k(x)|,\ \text{for all } x \in E.$$

Hence $\sum_{n=1}^{\infty} f_n(x)x_n$ converges for all $x \in E$.

Define $K : E \longrightarrow E$ by $K(x) = \sum_{n=1}^{\infty} f_n(x)x_n$, $x \in E$. Then K is a bounded linear operator such that

$$\|K(x)\|_E \leq \sup_{1 \leq n < \infty} \|\sum_{k=1}^{n} f_k(x)x_k\|_E \leq \sigma\|x\|_E,$$

where $\sigma = \sup_{1\leq n<\infty} \sum_{k=1}^{n} f_k(x)x_k$. Thus

$$\frac{A}{\sigma}\|K(x)\|_E \leq \| \{f_n(x)\}\|_{E_d} \leq B \parallel x \parallel_E, \text{ for all } x \in E.$$

Hence, (x_n, f_n) is a K-atomic decomposition for E with respect to E_d with bounds $\frac{A}{\sigma}$ and B. □

Theorem 3.9. *Let $\{f_n\} \subseteq E^*$ be an E_d-frame with bounds A, B and let $\{x_n\} \subseteq E$. Let $T : E_d \longrightarrow E$ given by $T(\{\alpha_n\}) = \sum_{n=1}^{\infty} \alpha_n x_n$ be a well defined operator. Then, there exists a linear operator $K \in L(E)$ such that (x_n, f_n) is a K-atomic decomposition for E with respect to E_d.*

Proof. Define $U : E \longrightarrow E_d$ by $U(x) = \{f_n(x)\}$, $x \in E$. Then U is well defined and $\|U\| \leq B$. Take $K = TU$. Then $K(x) = \sum_{n=1}^{\infty} f_n(x)x_n$, $x \in E$. Therefore, by uniform boundedness principle, we have

$$\|K(x)\|_E \leq \sup_{1\leq n<\infty} \| \sum_{k=1}^{n} f_k(x)x_k\|_E \leq \sigma\|x\|_E, \ x \in E,$$

where $\sigma = \sup_{1\leq n<\infty} \| \sum_{k=1}^{n} f_k(x)x_k\|_E$. Thus, we have

$$\frac{A}{\sigma}\|K(x)\| \ \leq \| \{f_n(x)\}\| \leq B \parallel x \parallel, \text{ for all } x \in E.$$

Hence (x_n, f_n) is a K-atomic decomposition for E with respect to E_d with bounds $\frac{A}{\sigma}$ and B. □

Next, we give the existence of a K-atomic decomposition from an E_d Bessel sequence.

Theorem 3.10. *Let E be a reflexive Banach space and E_d be a BK-space which has a sequence of canonical unit vectors $\{e_n\}$ as a basis. Let $\{f_n\} \subseteq E^*$ be an E_d-Bessel sequence with bound B and let $\{x_n\} \subseteq E$. If $\{f(x_n)\} \in (E_d)^*$ for all $f \in E^*$, then there exists a bounded linear operator $K \in L(E)$ such that (x_n, f_n) is a K-atomic decomposition for E with respect to E_d.*

Proof. Clearly $U : E \longrightarrow E_d$ given by $U(x) = \{f_n(x)\}$, $x \in E$ is well defined. Define a map $R : E^* \longrightarrow (E_d)^*$ by $R(f) = \{f(x_n)\}$, $x \in E$. Then, its adjoint $R^* : (E_d)^{**} \longrightarrow E^{**}$ is given by $R^*(e_j)(f) = e_j(R(f)) = f(x_j)$. Let $T = (R^*)|_{E_d}$ and $\{\alpha_n\} \in E_d$. Then

$$T(\{\alpha_n\}) = \sum_{n=1}^{\infty} \alpha_n T(e_n) = \sum_{n=1}^{\infty} \alpha_n x_n.$$

But $\{f_n(x)\} \in E_d$. So $T(\{f_n(x)\}) = \sum_{n=1}^{\infty} f_n(x)x_n$. Take $K = TU$. Then $K \in L(E)$ and $K(x) = \sum_{n=1}^{\infty} f_n(x)x_n$. Moreover, T is a bounded linear operator such that

$\|K(x)\| \leq \|T\| \|\{f_n(x)\}\|$. Hence

$$\frac{1}{\|T\|}\|K(x)\| \leq \|\{f_n(x)\}\| \leq B\|x\|, \ x \in E$$

□

Next, we construct a K^*-atomic decomposition for E^* from a given K-atomic decomposition for E.

Theorem 3.11. *Let E_d be a BK-space with dual $(E_d)^*$ and let E_d and $(E_d)^*$ have sequences of canonical unit vectors $\{e_n\}$ and $\{v_n\}$ respectively as basis. Let (x_n, f_n) be a K-atomic decomposition for E with respect to E_d. Let $S : E_d \longrightarrow E$ given by $S(\{d_n\}) = \sum\limits_{n=1}^{\infty} d_n x_n$ be a well defined mapping. Then, $(f_n, \pi(x_n))$ is a K^*-atomic decomposition for E^* with respect to $(E_d)^*$.*

Proof. For each $x \in E$, $K(x) = \sum\limits_{n=1}^{\infty} f_n(x)x_n$. Thus $f(K(x)) = \sum\limits_{n=1}^{\infty} f_n(x)f(x_n)$. Take $n,\ m \in \mathbb{N}$ with $m \leq n$. Then for $f \in E^*$

$$\|\sum_{k=m}^{n} f(x_k)f_k\| = \sup_{x\in E, \|x\|=1} |\sum_{k=m}^{n} f(x_k)f_k(x)|.$$

Therefore, $\sum\limits_{n=1}^{\infty} f(x_n)f_n$ converges for all $f \in E^*$. Also, for $x \in E$, we have

$$(K^*(f))(x) = f(\sum_{n=1}^{\infty} f_n(x)x_n) = (\sum_{n=1}^{\infty} f(x_n)f_n)(x).$$

This gives $K^*(f) = \sum\limits_{n=1}^{\infty} f(x_n)f_n$, for $f \in E^*$. Note that
$S^*(f)(e_j) = f(S(e_j)) = f(x_j), f \in E^*$. So, $S^*(f) = \{f(x_n)\}$ and $\{f(x_n)\} = \{f(S(e_n))\} \in (E_d)^*, f \in E^*$. Also

$$\|\{f(x_n)\}\|_{(E_d)^*} = \|S^*(f)\| \leq \|S\|\|f\|_{E^*},\ f \in E^*.$$

Define $R : E \longrightarrow E_d$ by $R(x) = \{f_n(x)\}, x \in E$. Then, $R^*(v_j)(x) = v_j(R(x)) = f_j(x)$, $x \in E$. So, $R^*(v_j) = f_j$, for all $j \in \mathbb{N}$ and for $\{\alpha_n\} \in (E_d)^*$ we have

$$R^*(\{\alpha_n\}) = R^*(\sum_{n=1}^{\infty} \alpha_n v_n) = \sum_{n=1}^{\infty} \alpha_n R^*(v_n) = \sum_{n=1}^{\infty} \alpha_n f_n.$$

Therefore, we have

$$R^*S^*(f) = R^*(\{f(x_n)\}) = \sum_{n=1}^{\infty} f(x_n)f_n,\ f \in E^*.$$

Moreover, $K^* = R^*S^*$ and so

$$\|K^*(f)\|_{E^*} = \|R^*S^*(f)\|_{E^*} \leq \|R^*\|\|\{f(x_n)\}\|_{(E_d)^*},\ f \in E^*.$$

This gives

$$\frac{1}{\|R^*\|}\|K^*(f)\|_{E^*} \leq \|\{f(x_n)\}\|_{(E_d)^*} \leq \|S\|\|f\|_{E^*},\ f \in E^*. \tag{4}$$

Hence, $(f_n, \pi(x_n))$ is a K^*-atomic decomposition for E^* with respect to $(E_d)^*$. □

Next, we give the following result characterizing the class of K-atomic decompositions.

Theorem 3.12. *Let (x_n, f_n) be a K-atomic decomposition for E with respect to E_d with bounds A and B. Let $T : E_d \longrightarrow E$ given by $T(\{\alpha_n\}) = \sum_{n=1}^{\infty} \alpha_n x_n$ is well defined for $\{\alpha_n\} \in E_d$ and let $U : E \longrightarrow E_d$ be the mapping given by $U(x) = \{f_n(x)\}$. If K is invertible, then the following statements are equivalent.*

(a) T is the pseudo inverse of U.
(b) (x_n, f_n) is an atomic decomposition for E with respect to E_d.
(c) T is a linear extension of $U^{-1} : U(E) \longrightarrow E$.
(d) $U(E)$ is a complemented subspace of E_d.
(e) $KerT$ is a complemented subspace of E_d and T is surjective.

Proof. $(a) \Rightarrow (b)$ By hypothesis, $\{x \in E : f_n(x) = 0,\ for\ all\ n \in \mathbb{N}\} = \{0\}$. So, $KerU = \{0\}$. Since T is the pseudo inverse of U, by Lemma 2.4 there exists a continuous projection operator $\theta : E \longrightarrow E$ such that $TU = I_E - \theta$ and $kerU = \theta(E)$. Thus, for each $x \in E$, we have

$$TU(x) = (I_E - \theta)(x) = x,\ x \in E.$$

Hence, for every $x \in E$, $\sum_{n=1}^{\infty} f_n(x)x_n = x$.
$(b) \Rightarrow (a)$ For $x \in E$, we have

$$UTU(x) = UT(\{f_n(x)\}) = U(\sum_{n=1}^{\infty} f_n(x)x_n) = U(x).$$

Hence, $UTU = U$.
$(c) \Rightarrow (b)$ If T is a linear extension of $U^{-1} : U(E) \longrightarrow E$, then $TU : E \longrightarrow E$ is the identity map on E. So, $TU(x) = x$ and $\sum_{n=1}^{\infty} f_n(x)x_n = x$.
$(c) \Rightarrow (a)$ Obvious, since $UTU = UI_E = U$.
(d)⇒(b) Suppose $E_d = U(E) \oplus G$, where G is a closed subspace of E_d. Let P be a projection of E_d onto $U(E)$ along G.
Then, $P(\{\alpha_n\}) = \{f_n(\sum_{k=1}^{\infty} \alpha_k x_k)\}$, for all $\{\alpha_n\} \in E_d$. Therefore

$$\begin{aligned} U^{-1} \circ P(\{\alpha_n\}) &= U^{-1}\{f_n(\sum_{k=1}^{\infty} \alpha_k x_k)\} = \sum_{k=1}^{\infty} \alpha_n x_n \\ &= T(\{\alpha_n\}),\ for\ all\ \{\alpha_n\} \in E_d. \end{aligned}$$

This gives, T=$U^{-1} \circ P$ and

$$T(\{f_n(x)\}) = U^{-1} \circ P(\{f_n(x)\}) = U^{-1}(\{f_n(x)\}.$$

Hence, $x = \sum_{n=1}^{\infty} f_n(x)x_n$, for all $x \in E$.
(b)⇒(d) Obvious.
(e)⇒(b) Let $E_d = kerT \oplus M$, where M is a closed subspace of E_d. Take $\Upsilon = kerT \oplus U(E)$. Let $Q : E_d \longrightarrow M$ be a projection from E_d onto M along $kerT$.

Define $L : E_d \longrightarrow \Upsilon$ by $L(\alpha) = (\alpha - Q(\alpha), UT(\alpha))$, for $\alpha = \{\alpha_n\} \in E_d$. Let $L(\alpha) = 0$. This gives $Q(\alpha) = \alpha$. So $\alpha \in M$. Let $UT(\alpha) = 0$. Then

$$U(\sum_{n=1}^{\infty} \alpha_n x_n) = \{f_n(\sum_{k=1}^{\infty} \alpha_k x_k)\} = 0, \ for \ n \in \mathbb{N}.$$

This gives $\sum\limits_{n=1}^{\infty} \alpha_n x_n = 0$ and so, $\alpha \in kerT$. Thus, $\alpha \in kerT \cap M = \{0\}$. Hence, L is one-one.
Let $(\alpha_0, U(x)) \in kerT \oplus U(E)$, for $\alpha_0 \in kerU$ and $U(x) \in U(E)$.
Since, T is onto, for each $x \in E$, there exists $\beta \in E_d$ such that $T(\beta) = x$ and this gives $UT(\beta) = U(x)$. Take $\alpha = \alpha_0 + Q(\beta)$. Then $Q(\alpha) = Q(\alpha_0) + Q^2(\beta) = Q(\beta)$ and $\alpha_0 = \alpha - Q(\alpha)$. Also, we have

$$UT(\alpha) = UT(\alpha - \alpha_0) = UT(Q(\beta)) = UT(\beta) = U(x). \tag{5}$$

Thus $L(\alpha) = (\alpha_0, UT(x))$ and L is an isomorphism from E_d onto Υ. So, there is a projection $P = UT : E_d \longrightarrow U(E)$ onto $U(E)$ along $kerT$. This gives

$$U^{-1} \circ P = T \ \ and \ \ U^{-1} \circ P(\{f_n(x)\}) = T(\{f_n(x)\}).$$

Finally, we have

$$U^{-1}(\{f_n(x)\}) = \sum_{n=1}^{\infty} f_n(x) x_n \ \ and \ \ x = \sum_{n=1}^{\infty} f_n(x) x_n.$$

Therefore, (x_n, f_n) is an atomic decomposition for E with respect to E_d.
(b)⇒(e) Obvious. □

Next, we prove a duality type result for a K-atomic decomposition for E.

Theorem 3.13. *Let E_d be a reflexive BK-space with its dual $(E_d)^*$ and let sequences of canonical unit vectors $\{e_n\}$ and $\{v_n\}$ be bases for E_d and $(E_d)^*$, respectively. Let $(f_n, \pi(x_n))$ be a K-atomic Decomposition for E^* with respect to $(E_d)^*$. If $S : (E_d)^* \longrightarrow E^*$ given by $S(\{d_n\}) = \sum\limits_{n=1}^{\infty} d_n f_n$ is well defined for $\{d_n\} \in E_d^*$, then there exists a linear operator $L \in L(E)$ such that (x_n, f_n) is L-atomic decomposition for E with respect to E_d.*

Proof. For $f \in E^*$, we have $K(f) = \sum\limits_{n=1}^{\infty} f(x_n) f_n$. Let $m, \ n \in \mathbb{N}$ with $m \le n$ and $x \in E$. Then

$$\| \sum_{k=m}^{n} f_k(x) x_k \|_E = \sup_{f \in E^*, \|f\|=1} | \sum_{k=m}^{n} f_k(x) f(x_k)|$$

Thus, $\sum\limits_{n=1}^{\infty} f_n(x) x_n$ converges, for all $x \in E$. Define$L : E \longrightarrow E$ by $L(x) = \sum\limits_{n=1}^{\infty} f_n(x) x_n, \ x \in E$. Note that $S(v_n) = f_n, \ n \in \mathbb{N}$ and for $x \in E$, the linear bounded operator $S^* : E^{**} \longrightarrow (E_d)^{**}$ satisfies

$$S^*(\pi(x))(v_n) = \pi(x) S(v_n) = f_n(x).$$

So, $\{f_n(x)\}$ is identified with $S^*(\pi(x)) \in (E_d)^{**} = E_d$. Further, we have

$$\|\{f_n(x)\}\|_{E_d} = \|S^*(\pi(x))\|_{E_d} \le \| S \| \| x \|_E, \ x \in E. \tag{6}$$

Letting $U = S^* |_E$, we have $U(x) = \{f_n(x)\}$ and $\|U\| \leq \|S\|$.
Define $R : E^* \longrightarrow (E_d)^*$ by $R(f) = \{f(x_n)\}, f \in E^*$. Then

$$R^*(e_j)(f) = e_j(R(f)) = f(x_j), \quad f \in E^*.$$

So, $R^*(e_j) = x_j$, for all $j \in \mathbb{N}$. Take $T = (R^*)|_{E_d}$. Then, for $\{\alpha_n\} \in E_d$ we have

$$T(\{\alpha_n\}) = T(\sum_{n=1}^{\infty} \alpha_n e_n) = \sum_{n=1}^{\infty} \alpha_n T(e_n) = \sum_{n=1}^{\infty} \alpha_n x_n.$$

Thus, $TU(x) = \sum\limits_{n=1}^{\infty} f_n(x)x_n$, for all $x \in E$ and this gives $TU = L$ on E. Therefore, $\frac{1}{\|T\|}\|L(x)\|_E \leq \|\{f_n(x)\}\|_{E_d}$. Then

$$\frac{1}{\|T\|}\|L(x)\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq \|S\|\|x\|_E.$$

Hence, (x_n, f_n) is L-atomic decomposition for E with respect to E_d. □

Next, we give the results related to perturbation of K-atomic decomposition for E.

Theorem 3.14. *Let (x_n, f_n) be an atomic decomposition for E with respect to E_d with bounds A and B. Let (y_n, f_n) be a K-atomic decomposition for E with respect to E_d with bounds C and D. If there exists $\lambda > 0$ with $\frac{\lambda D}{C} < 1$, then there exist a sequence $\{g_n\} \subseteq E^*$ such that $(x_n + \lambda y_n, g_n)$ is an atomic decomposition for E with respect to E_d with bounds $\frac{AC}{C + \lambda A}$ and $\frac{DC}{C - \lambda D}$.*

Proof. Take $L = I_E + \lambda K$. Then, $L : E \longrightarrow E$ is given by
$L(x) = \sum\limits_{n=1}^{\infty} f_n(x)(x_n + \lambda y_n)$. Also, we have

$$\begin{aligned} \|L(x)\|_E = \|(I_E + \lambda K)(x)\|_E &\leq \|x\|_E + \lambda\|K(x)\|_E \\ &\leq \frac{C + \lambda A}{AC}\|\{f_n(x)\}\|_{E_d} \end{aligned}$$

and $\|L\| \leq \frac{D(C + \lambda A)}{AC}$. This yields

$$\frac{AC}{C + \lambda A}\|L(x)\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq D\|x\|_E.$$

So, $(x_n + \lambda y_n, f_n)$ is an L-atomic decomposition with respect to E_d with bounds $\frac{AC}{C + \lambda A}$ and D. Also, since (y_n, f_n) is a K-atomic decomposition, we have:

$$\|(I_E - L)(x)\|_E = \lambda\|\sum_{n=1}^{\infty} f_n(x)y_n\|_E = \lambda\|K(x)\|_E \leq \frac{\lambda D}{C}\|x\|_E.$$

This gives $\|I_E - L\| \leq 1$. Thus L is invertible.

$$\text{Also, } \|x\|_E - \|L(x)\|_E \leq \frac{\lambda D}{C}\|x\|_E \tag{7}$$

So, $\|L^{-1}\| \leq \dfrac{C}{C-\lambda D}$. For $n \in \mathbb{N}$, take $g_n = (L^{-1})^* f_n$. Then, for $x \in E$, we have

$$\begin{aligned} x &= LL^{-1}(x) = L(L^{-1}(x)) = \sum_{n=1}^{\infty} f_n(L^{-1}(x))(x_n + \lambda y_n) \\ &= \sum_{n=1}^{\infty} ((L^{-1})^*(f_n))(x)(x_n + \lambda y_n) = \sum_{n=1}^{\infty} g_n(x)(x_n + \lambda y_n). \end{aligned}$$

For $x \in E$, $\{g_n(x)\} = \{f_n(L^{-1}(x))\} \in E_d$.
Also, if $x \in E$, then

$$\frac{AC}{C+\lambda A}\|x\|_E = \frac{AC}{C+\lambda A}\|L(L^{-1}(x))\| \leq \|\{f_n(L^{-1}(x))\}\|_{E_d}.$$

and

$$\begin{aligned} \|\{g_n(x)\}\|_{E_d} &= \|\{f_n(L^{-1}(x))\}\|_{E_d} \leq D\|L^{-1}(x)\|_E \leq D\|L^{-1}\|\|x\|_E \\ &\leq \frac{DC}{C-\lambda D}\|x\|. \end{aligned}$$

Thus, for $x \in E$, we have

$$\frac{AC}{C+\lambda A}\|x\|_E \ \leq \|\{g_n(x)\}\|_{E_d} \leq \frac{DC}{C-\lambda D}\|x\|_E.$$

Hence, $(x_n + \lambda y_n, g_n)$ is an atomic decomposition for E with respect to E_d with bounds $\dfrac{AC}{C+\lambda A}$ and $\dfrac{DC}{C-\lambda D}$. □

Theorem 3.15. *Let E_d be a BK-space with a sequence of canonical vectors as basis. Let (x_n, f_n) be a K-atomic decomposition for E with respect to E_d with bounds A, B and let K has a generalized inverse K^+. Let $\alpha, \beta, \gamma \in [0, \infty)$ with $max\{\beta, (\alpha + \gamma B\|K^+\|\|K\|)\} < 1$ and $\{y_n\} \subseteq E$. If*
$\|\sum_{k=1}^{n} d_k(x_k - y_k)\|_E \leq \alpha\|\sum_{k=1}^{n} d_k x_k\|_E + \beta\|\sum_{k=1}^{n} d_k y_k\|_E + \gamma\|\{d_k\}_{k=1}^n\|_{E_d}$ *for any finite scalars $d_1, d_2, d_3, ..., d_n, n \in \mathbb{N}$, then there exists $\{g_n\} \subseteq E^*$ and a linear operator $T \in L(E)$ such that (y_n, g_n) is a T-atomic decomposition for E with respect to E_d with bounds $\dfrac{A(1-\beta)}{1+\alpha+\gamma B\|K^+\|\|K\|}$ and $\dfrac{B(1+\beta)\|T\|\|K^+\|\|K\|}{[1-(\alpha+\gamma B\|K^+\|\|K\|)]}$.*

Proof. For $x \in E$, $K(x) = \sum_{n=1}^{\infty} f_n(x)x_n$. Also, $\sum_{n=1}^{\infty} f_n(x)y_n$ converges for all $x \in E$. Let $L : E \longrightarrow E$ be defined by $L(x) = \sum_{n=1}^{\infty} f_n(x)y_n, x \in E$. For $x \in E$, we have

$$\begin{aligned} \|K(x) - L(x)\|_E &= \|\sum_{n=1}^{\infty} f_n(x)(x_n - y_n)\|_E \\ &\leq \alpha\|K(x)\|_E + \beta\|L(x)\|_E \\ &+ \gamma\|\{f_n(x)\}\|_{E_d} \end{aligned} \tag{8}$$

Also, for $x \in K(E)$, we have

$$\|x\|_E = \|KK^+(x)\|_E = \|KK^+K(x)\|_E \leq \|K\|\|K^+\|\|K(x)\|_E$$

and

$$\|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E \leq B\|K\|\|K^+\|\|K(x)\|_E. \tag{9}$$

From 8 and 9, we have

$$\|K(x)-L(x)\|_E \leq (\alpha+\gamma B\|K\|\|K^+\|)\|K(x)\|_E + \beta\|L(x)\|_E.$$

Thus, for any $x \in K(E)$, we have

$$\begin{aligned}\frac{1-(\alpha+\gamma B\|K\|\|K^+\|)}{1+\beta}\|K(x)\|_E &\leq \|L(x)\|_E \\ &\leq \frac{1+\alpha+\gamma B\|K\|\|K^+\|}{1-\beta}\|K(x)\|_E\end{aligned}$$

and

$$\frac{[1-(\alpha+\gamma B\|K\|\|K^+\|)]}{(1+\beta)\|K\|\|K^+\|}\|x\| \leq \|L(x)\| \leq \frac{[1+\alpha+\gamma B\|K\|\|K^+\|]}{(1-\beta)AB^{-1}}\|x\| \tag{10}$$

Take $V = L|_{K(E)}$. We shall show that $V(K(E))$ is closed. Let $\{s_n\} \subseteq V(K(E))$ such that $s_n \to s \in E$. For each s_n, there exists $t_n \in K(E)$ such that $s_n = V(t_n)$, for all $n \in \mathbb{N}$. Now, we have

$$\|t_{n+m}-t_n\| \leq C^{-1}\|V(t_{n+m}-t_n)\| \leq C^{-1}\|s_{n+m}-s_n\|,$$

where $C = \dfrac{[1-(\alpha+\gamma B\|K\|\|K^+\|)]\|K\|^{-1}\|K^+\|^{-1}}{1+\beta}$. Since $\{s_n\}$ is a Cauchy sequence, it follows that $\{t_n\}$ is also a Cauchy sequence. But $K(E)$ is closed. So, there exists $t \in K(E)$ such that $t_n \to t$ and

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} V(t_n) = V(t) \in V(K(E)).$$

From (10), we conclude that V is injective on $K(E)$. Therefore, $V : K(E) \longrightarrow V(K(E))$ is invertible. Let $T : E \longrightarrow V(K(E))$ be an orthogonal projection from E to $V(K(E))$. Define $g_n = (V^{-1}T)^* f_n$, $n \in \mathbb{N}$. Then for $x \in E$, we have

$$\begin{aligned}T(x) &= VV^{-1}(T(x)) = V(V^{-1}T(x)) = \sum_{n=1}^{\infty} f_n((V^{-1}T)(x))y_n \\ &= \sum_{n=1}^{\infty}((V^{-1}T)^* f_n)(x)y_n = \sum_{n=1}^{\infty} g_n(x)y_n.\end{aligned}$$

Also, for $x \in E$ we have $\{g_n(x)\} = \{(f_n(L^{-1}T))(x)\} \in E_d$ and

$$\begin{aligned}\|T(x)\|_E &= \|V(V^{-1}T(x))\|_E \\ &\leq \frac{1+\alpha+\gamma B\|K\|\|K^+\|}{1-\beta}\|K(V^{-1}T(x)\|_E. \\ &\leq \frac{1+\alpha+\gamma B\|K\|\|K^+\|}{A(1-\beta)}\|\{f_n(V^{-1}T(x))\}\|_{E_d}\end{aligned}$$

For $x \in E$ we have

$$\|\{g_n(x)\}\|_{E_d} = \|\{f_n(V^{-1}T(x))\}\|_{E_d} \leq B\|V^{-1}T(x)\|_E \tag{11}$$

Also, for $y \in V(K(E))$, we have

$$\|V^{-1}(y)\|_E \leq \frac{(1+\beta)\|K\|\|K^+\|}{1-(\alpha+\gamma B\|K\|\|K^+\|)}\|y\|_E. \tag{12}$$

From (11) and (12), we conclude that

$$\begin{aligned}\|\{g_n(x)\}\|_{E_d} &\leq \frac{B(1+\beta)\|K\|\|K^+\|}{1-(\alpha+\gamma B\|K\|\|K^+\|)}\|T(x)\|_E \\ &\leq \frac{B(1+\beta)\|K\|\|K^+\|}{1-(\alpha+\gamma B\|K\|\|K^+\|)}\|T\|\|x\|_E,\ x\in E\end{aligned}$$

Hence

$$\begin{aligned}\frac{A(1-\beta)}{1+\alpha+\gamma B\|K\|\|K^+\|}\|T(x)\|_E &\leq \|\{g_n(x)\}\|_{E_d} \\ &\leq \frac{B(1+\beta)\|T\|\|K\|\|K^+\|}{1-(\alpha+\gamma B\|K\|\|K^+\|)}\|x\|_E.\end{aligned}$$

□

Finally, we prove the following result related to the perturbation of an atomic decomposition for E.

Theorem 3.16. *Let (x_n, f_n) be an atomic decomposition for E with respect to E_d with bounds A and B. Let (x_n, g_n) be a K-atomic decomposition for E with respect to E_d with bounds C and D. Let $T : E_d \longrightarrow E$ given by $T(\{\alpha_n\}) = \sum_{n=1}^{\infty} \alpha_n x_n$ be a well defined map for $\{\alpha_n\} \in E_d$. If there exists $\lambda > 0$ such that $\frac{\lambda D}{C} < 1$, then there exists $\{y_n\} \subseteq E$ such that $(y_n, f_n + \lambda g_n)$ is an atomic decomposition for E with respect to E_d with bounds $\frac{C-\lambda D}{C\|T\|}$ and $B + \lambda D$.*

Proof. Define an operator $L = I_E + \lambda K : E \longrightarrow E$ by
$L(x) = \sum_{n=1}^{\infty}(f_n + \lambda g_n)(x)x_n,$ for all $x \in E$. Then

$$\{(f_n + \lambda g_n)(x)\} = \{f_n(x)\} + \lambda\{g_n(x)\} \in E_d$$

and

$$\begin{aligned}\|\{(f_n+\lambda g_n)(x)\}\|_{E_d} &\leq \|\{f_n(x)\}\|_{E_d} + \lambda\|\{g_n(x)\}\|_{E_d} \\ &\leq (B+\lambda D)\|x\|_E.\end{aligned}$$

Now define $U : E \longrightarrow E_d$ by $U(x) = \{(f_n + \lambda g_n)(x)\}$. Then, U is well defined and $\|U\| \leq B + \lambda D$. Since

$$TU(x) = T(\{(f_n+\lambda g_n)(x)\}) = \sum_{n=1}^{\infty}(f_n+\lambda g_n)(x)x_n,\ x\in E,$$

we conclude that $L = TU$. Moreover, we have

$$\|L(x)\|_E = \|TU(x)\|_E \leq \|T\|\|\{(f_n+\lambda g_n)(x)\}\|_{E_d}.$$

Thus

$$\frac{1}{\|T\|}\|L(x)\|_E \leq \|\{(f_n+\lambda g_n)(x)\}\|_{E_d} \leq (B+\lambda D)\|x\|_E,\ x\in E.$$

Therefore, $(x_n, f_n + \lambda g_n)$ is L-atomic decomposition for E with respect to E_d. Since

$$\|(I_E - L)(x)\|_E = \lambda\|K(x)\|_E \leq \frac{\lambda D}{C}\|x\|_E,\ x\in E,$$

L is invertible. Thus, we have

$$\|x\|_E - \|L(x)\|_E \leq \frac{\lambda D}{C}\|x\|_E,\ x \in E.$$

This gives, $\|L^{-1}\| \leq \dfrac{C}{C-\lambda D}$. Define $y_n = L^{-1}(x_n)$, for $n \in \mathbb{N}$. Then, for $x \in E$, we have

$$\begin{aligned} x &= L^{-1}L(x) = L^{-1}(\sum_{n=1}^{\infty}(f_n+\lambda g_n)(x)x_n) \\ &= \sum_{n=1}^{\infty}(f_n+\lambda g_n)(x)L^{-1}(x_n) = \sum_{n=1}^{\infty}(f_n+\lambda g_n)(x)y_n. \end{aligned}$$

So

$$\begin{aligned} \|x\|_E &= \|L^{-1}L(x)\|_E \leq \|L^{-1}\|\|T\|\|\{(f_n+\lambda g_n)(x)\}\|_{E_d},\ x \in E \\ &\leq \frac{C}{C-\lambda D}\|T\|\|\{(f_n+\lambda g_n)(x)\}\|_{E_d} \end{aligned}$$

Therefore

$$\frac{C-\lambda D}{C\|T\|}\|x\|_E \leq \|\{(f_n+\lambda g_n\}\|_{E_d} \leq (B+\lambda D)\|x\|_E,\ x \in E.$$

Hence, $(y_n, f_n+\lambda g_n)$ is an atomic decomposition for E with respect to E_d with bounds $\dfrac{C-\lambda D}{C\|T\|}$ and $B+\lambda D$.

□

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