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A NOTE ON SEMIMULTIPLIERS IN PRIME RINGS

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Abstract. Let R be a ring and g be a surjective map of R. An additive mapping $F : R \to R$ is called a semimultiplier if (1) $F(xy) = F(x)g(y) = g(x)F(y)$ (2) $F(g(x)) = g(F(x))$ for all $x, y \in R$. In this paper, we introduce the notion of semimultiplier of a ring R, and investigate the commutativity of prime rings admitting semimultipliers satisfying (1) $F([x, y]) - [x, y] = 0$ (2) $F([x, y]) + [x, y] = 0$ (3) $F(x \circ y) - x \circ y = 0$ (4) $F(x \circ y) + x \circ y = 0$ (5) $F(xy) = xy$ (6) $F(xy) = yx$ for all x, y in some appropriate subset of R.

1. Introduction

Many considerable works have been done on left (right) multipliers in prime and semiprime rings during the last couple of decades([9-11]). An additive mapping $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y + yd(x)$ holds for all $x, y \in R$. Following [5], an additive mapping $F : R \to R$ is called a *generalized derivation* on R if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for every $x, y \in R$. Obviously, a generalized derivation with $d = 0$ covers the concept of left multiplicars. Over the last few decares, several authors have investigated the relationship between the commutativity of the ring R and certain specific types of derivations of R. The first result in this direction is due to E. C. Posner [8] who proved that if a ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. This result was subsequently, refined and extended by a number of authors. In [7], Bresar and Vuckman showed that a prime ring must be commutative if it admits a nonzero left derivation. Recently, many authors have obtained commutativity theorems for prime and semiprime rings admitting derivation, generalized derivation. In this paper, we introduce the notion of a semimultiplier of R, and investigate the commutativity of prime rings satisfying certain identities involving semimultiplier.

2. Preliminaries

Throughout R will represent an associative ring with center $Z(R)$. For all $x, y \in R$, as a usual commutator, we shall write $[x, y] = xy-yx$, and $x \circ y = xy+yx$. Also,

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we make use of the following two basic identities without any specific mention:

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$

$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$$

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z.$$

Recall that R is *prime* if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. A nonempty subset I of R is called a *right semigroup ideal* if $IR \subseteq I$. Similarly, A nonempty subset I of R is called a *left semigroup ideal* if $RI \subseteq I$. If I is both a left and a right semigroup ideal of R, then I is called a *semigroup ideal* of R. An additive mapping $F : R \to R$ is called a *left multiplier* if $F(xy) = F(x)y$ holds for every $x, y \in R$. Similarly, an additive mapping $F : R \to R$ is called a *right multiplier* if $F(xy) = xF(y)$ holds for every $x, y \in R$. If F is both a left and a right multiplier of R, then it is called a *multiplier* of R.

3. Semimultipliers in prime and semiprime rings

Definition 3.1. Let R be a ring. An additive mapping $F : R \to R$ is called a *semimultiplier* associated with a surjective function $g : R \to R$ if
(a) $F(xy) = F(x)g(y) = g(x)F(y)$,
(b) $F(g(x)) = g(F(x))$, for every $x, y \in R$.

Lemma 3.2. *Let R be a prime ring and I be a nonzero right* (resp. *left*) *semigroup ideal of R and F be a semimultiplier of R associated with g. If $F(x) = 0$ for every $x \in I$, then $F = 0$.*

Proof. By hypothesis, we have $F(x) = 0$ for any $x \in I$. Replacing x by xr with $r \in R$ in the last relation, we get

$$g(x)F(r) = 0, \ \forall \ x \in I, r \in R. \tag{1}$$

Since g is onto, we get $xF(r) = 0$ for all $x \in I$ and $r \in R$. Now, replacing x by xs in (1), we have $xsF(r) = 0$ for every $x \in I$ and $r, s \in R$. Thus, we obtain $xRF(r) = \{0\}$ for every $x \in I$ and $r \in R$. Since R is prime and I is a nonzero right semigroup ideal of R, it implies that $F = 0$.

□

Lemma 3.3. *Let R be a prime ring and I be a nonzero semigroup ideal of R. Suppose that F is a semimultiplier of R associated with g and $a \in R$. If $aF(x) = 0$ for every $x \in R$, then $a = 0$ or $F = 0$.*

Proof. By hypothesis, we have $aF(x) = 0$ for any $x \in I$ and $a \in R$. Replacing x by xr in the last relation, we get

$$ag(x)F(r) = 0, \ \forall \ x \in I, r \in R. \tag{2}$$

Since g is onto, we have $axF(r) = 0$ for all $x \in I$ and $r \in R$. Now, replacing x by xs in (2), we have $axsF(r) = 0$ for every $x \in I$ and $r, s \in R$. Thus, we obtain $axRF(r) = \{0\}$ for every $x \in I$ and $r \in R$. Since R is prime and I is a nonzero right semigroup ideal of R, it implies that $ax = 0$ for all $x \in I$ or $F(r) = 0$ for every $r \in R$. Hence

$$aI = 0 \text{ or } F = 0.$$

Assume that $F \neq 0$. Then we get $ax = 0$ for every $x \in I$. Replacing x by rx with $r \in R$ in the last equation, we have $arx = 0$ for every $x \in I, r \in R$. Thus

$$aRx = \{0\}, \ \forall \ \ x \in I.$$

Since R is prime and I is a nonzero right ideal of R, we obtain $a = 0$.

□

Lemma 3.4. *Let R be a prime ring and I be a nonzero semigroup ideal of R and $a, b \in R$. If $aIb = 0$, then $a = 0$ or $b = 0$.*

Proof. By hypothesis, we have $axb = 0$ for any $x \in I$. Replacing x by xr with $r \in R$ in the last relation, we get $axrb = 0$ for all $x \in I$ and $r \in R$. Thus

$$axRb = \{0\}, \ \forall \ x \in I. \tag{3}$$

Since R is prime, we have $ax = 0$ or $b = 0$. Suppose that $b \neq 0$. Then it means that $ax = 0$ for all $x \in I$. Taking rx with $r \in R$ instead of x in the last relation, it holds that $arx = 0$ for $x \in I, r \in R$. Hence we have

$$aRx = \{0\}, \ \forall \ x \in I.$$

Since R is prime and I is a nonzero semigroup ideal of R, we have $a = 0$.

□

Theorem 3.5. *Let R be a prime ring and let I be a nonzero semigroup ideal of R. Suppose that R admits a nonzero semimultiplier F associated with g such that $[F(x), y] = 0$ for every $x, y \in I$. Then R is commutative.*

Proof. By hypothesis, we have $[F(x), y] = 0$ for any $x, y \in I$. Replacing x by xz with $z \in I$, in this relation, we have

$$[F(x)g(z), y] = F(x)[g(z), y] + [F(x), y]g(z) = 0$$

for every $x, y, z \in I$. Using the given hypothesis and the fact that g is onto, we obtain $F(x)[z, y] = 0$ for every $x, y, z \in I$. Now, replacing y by ys with $s \in R$, in the last relation, we obtain

$$F(x)y[z, s] = 0$$

for every $x, y, z \in I$ and $s \in R$. This implies that $F(x)I[z, s] = \{0\}$ for every $x, z \in I$ and $s \in R$. Thus, by Lemma 3.4, we get $F(x) = 0$ or $[z, s] = 0$ for every $x, z \in I$ and $s \in R$. Since $F \neq 0$, we have $[z, s] = 0$ for every $z \in I$ and $s \in R$. Again, replacing z by zr with $r \in R$, in the last relation, we have $[zr, s] = z[r, s] + [z, s]r = z[r, s] = 0$. This implies that $xz[r, s] = 0$ for $0 \neq x \in I$, and hence $xI[r, s] = 0$. By Lemma 3.4, we have $[r, s] = 0$ for every $r, s \in R$, which implies that R is commutative.

□

Theorem 3.6. *Let R be a prime ring and let I be a nonzero semigroup ideal of R. Suppose that R admits a nonzero semimultiplier F associated with g such that $F(I) \subseteq Z(R)$. Then R is commutative.*

Proof. By hypothesis, we have $F(xy) \in Z(R)$ for any $x, y \in I$, and so $F(x)g(y) \in Z(R)$ for every $x, y \in I$. This implies that $[F(x)g(y), r] = 0$ for all $x, y \in I$ and $r \in R$. This can be rewritten as following relation,

$$F(x)[g(y), r] + [F(x), r]g(y) = 0, \ \forall \ x, y \in I, r \in R. \tag{4}$$

Replacing r by $F(x)$ in (4), we have

$$F(x)[g(y), F(x)] = 0, \ \forall \ x, y \in I. \tag{5}$$

Since g is surjective, we have

$$F(x)[y, F(x)] = 0, \ \forall \ x, y \in I. \tag{6}$$

Again, replacing y by yz with $z \in I$, in (6), we get $F(x)y[z, F(x)] = 0$ for every $x, y, z \in I$. This implies that $F(x)I[z, F(x)] = \{0\}$ for every $x, z \in I$. By Lemma 3.4, we have $F(x) = 0$ or $[z, F(x)] = 0$ for every $x, z \in I$. Since $F \neq 0$, we have $[z, F(x)] = 0$ for all $x, z \in I$, which implies that R is commutative by Theorem 3.5.

□

Theorem 3.7. *Let R be a prime ring and let I be a nonzero semigroup ideal of R. Suppose that R admits a nonzero semimultiplier F associated with g such that $[F(x), F(y)] = 0$, for every $x, y \in I$. Then R is commutative.*

Proof. By hypothesis, we have

$$[F(x), F(y)] = 0, \ \forall \ x, y \in I. \tag{7}$$

Replacing y by yz with $z \in I$, in (7), we have $[F(x), F(y)g(z)] = 0$, which implies that

$$F(y)[F(x), g(z)] = 0.$$

Since g is onto, we have

$$F(y)[F(x), z] = 0, \ \forall \ x, y, z \in I. \tag{8}$$

Now, replacing z by zs with $s \in R$, we have $F(y)z[F(x), s] = 0$ for every $x, y \in I$ and $s \in R$. This implies that $F(y)I[F(x), s] = \{0\}$ for every $y \in I$ and $s \in R$. Thus, by Lemma 3.4, we have $F(y) = 0$ for any $y \in I$ and $[F(x), s] = 0$ for $x \in I$ and $s \in R$. Since $F \neq 0$, we have $[F(x), s] = 0$, which implies that $F(x) \in Z(R)$ for any $x \in I$. That is, $F(I) \subseteq Z(R)$. Hence, by Theorem 3.6, R is commutative.

□

Theorem 3.8. *Let R be a prime ring and let I be a nonzero semigroup ideal of R. Suppose that R admits a nonzero semimultiplier F associated with g. If $F(x) \circ F(y) = 0$ holds for every $x, y \in I$, then R is commutative.*

Proof. By hypothesis, we have

$$F(x) \circ F(y) = 0, \ \forall \ x, y \in I. \tag{9}$$

Replacing y by zy with $z \in I$, in (9), we have $F(x) \circ F(yz) = F(x) \circ F(y)g(z) = 0$ for every $x, y, z \in I$, which implies that

$$(F(x) \circ F(y))g(z) - F(y)[F(x), g(z)] = 0$$

for every $x, y, z \in I$. Using the given relation, we have $F(y)[F(x), g(z)] = 0$ for every $x, y, z \in I$. Since g is onto, we have $F(y)[F(x), z] = 0$ for every $x, y, z \in I$. Replacing z by zy, where $x \in I$, in the last equation, we have $F(y)z[F(x), y] = 0$, which implies that $F(y)I[F(x), y] = \{0\}$ for every $x, y \in I$. By Lemma 3.4, we have $F(y) = 0$ or $[F(x), y] = 0$ for every $x, y \in I$. Since $F \neq 0$, we have $[F(x), y] = 0$ for every $x, y \in I$. Hence, by Theorem 3.5, R is commutative.

□

Theorem 3.9. *Let R be a prime ring and let I be a nonzero semigroup ideal of R. Suppose that R admits a nonzero semimultiplier F associated with g. If $F([x, y]) = 0$ holds for every $x, y \in I$, then R is commutative.*

Proof. By hypothesis, we have

$$F([x, y]) = 0, \ \forall \ x, y \in I. \tag{10}$$

Replacing y by zy with $z \in I$, in (10), we have $F[x, yz]) = 0$ for every $x, y, z \in I$, which implies that

$$F(y[x, z] + [x, y]z) = 0$$

for every $x, y, z \in I$. Hence we have $F(y)g([x, z]) + F([x, y])g(z) = 0$ for all $x, y, z \in I$, and so

$$F(y)g([x, z]) = 0, \ \forall \ x, z \in I. \tag{11}$$

Since g is onto, we have $F(y)[x, z] = 0$ for every $x, y, z \in I$. Replacing z by zr, where $r \in R$, in the last equation, we have $F(y)z[x, r] = 0$, which implies that $F(y)I[x, r] = \{0\}$ for every $x, y \in I$ and $r \in R$. By Lemma 3.4, we have $F(y) = 0$ or $[x, r] = 0$ for every $x, y \in I$ and $r \in R$. Since $F \neq 0$, we have $[x, r] = 0$ for every $x \in I$ and $r \in R$. Again, replacing x by xs in the last relation, we have $x[s, r] = 0$ for all $x \in I$ and $s, r \in R$. Hence $I[s, r] = \{0\}$ for all $r, s \in R$, which implies that $IR[s, r] = \{0\}$ for all $s, r \in R$. Since $I \neq 0$, we have $[s, r] = 0$ for all $s, r \in R$, which implies that R is commutative.

□

Theorem 3.10. *Let R be a prime ring and let I be a nonzero semigroup ideal of R. Suppose that R admits a nonzero semimultiplier F associated with g. If $F(x \circ y) = 0$ holds for every $x, y \in I$, then R is commutative.*

Proof. By hypothesis, we have

$$F(x \circ y) = 0, \ \forall \ x, y \in I. \tag{12}$$

Replacing y by yz with $z \in I$, in (12), we have $F(x \circ yz) = 0$ for every $x, y, z \in I$, which implies that

$$F((x \circ y)z - y[x, z]) = 0$$

for every $x, y, z \in I$. Hence we have $F(x \circ y)g(z) - F(y)g([x, z]) = 0$ for all $x, y, z \in I$, and so

$$F(y)g([x, z]) = 0, \ \forall \ x, z \in I. \tag{13}$$

Since g is onto, we have $F(y)[x, z] = 0$ for every $x, y, z \in I$. Replacing z by zr, where $r \in R$, in the last equation, we have $F(y)z[x, r] = 0$, which implies that $F(y)I[x, r] = \{0\}$ for every $x, y \in I$ and $r \in R$. By Lemma 3.4, we have $F(y) = 0$ or $[x, r] = 0$ for every $x, y \in I$ and $r \in R$. Since $F \neq 0$, we have $[x, r] = 0$ for every $x \in I$ and $r \in R$. Again, replacing x by xs in the last relation, we have $x[s, r] = 0$ for all $x \in I$ and $s, r \in R$. Hence $I[s, r] = \{0\}$ for all $r, s \in R$, which implies that $IR[s, r] = \{0\}$ for all $s, r \in R$. Since $I \neq 0$, we have $[s, r] = 0$ for all $s, r \in R$, which implies that R is commutative.

□

Theorem 3.11. *Let R be a prime ring and let I be a nonzero semigroup ideal of R. Suppose that R admits a semimultiplier F associated with g and $F(x) \neq x$ for all $x \in I$. If $F(xy) = F(x)F(y)$ holds for every $x, y \in I$, then $F = 0$.*

Proof. By hypothesis, we have

$$F(xy) = F(x)g(y) = F(x)F(y), \ \forall \ x \in I. \tag{14}$$

Replacing x by xw in (14), we have $F(xw)g(y) = F(xw)F(y)$, that is, $F(x)g(w)g(y) = F(x)g(w)F(y)$ for all $x, y, w \in I$. This implies that $F(x)g(w)(g(y) - F(y)) = 0$ for all $x, y, w \in I$. Since g is onto, we have $F(x)R(y - F(y)) = \{0\}$ for all $x, y \in R$. Since R is prime, we have $F(x) = 0$ or $y - F(y) = 0$ for all $x, y \in R$. But $F(x) \neq x$ for all $x \in I$, and so $F(x) = 0$ for all $x \in I$, which implies that $F = 0$ by Lemma 3.2.

□

Theorem 3.12. *Let R be a prime ring and let I be a nonzero semigroup ideal of R. Suppose that R admits a nonzero semimultiplier F associated with g and $g(x) \neq x$ for all $x \in I$. If $F(xy) = [x, y]$ holds for every $x, y \in I$, then R is commutative.*

Proof. By hypothesis, we have

$$F(xy) = [x, y], \ \forall \ x \in I. \tag{15}$$

Replacing x by xy in (15), we have $F(xy)g(y) = [xy, y]$, that is, $[x, y]g(y) = [x, y]y$ for all $x, y \in I$. This implies that $[x, y](g(y) - y) = 0$ for all $x, y \in I$. Also, replacing x by sx in the last relation, we have $[s, y]x(g(y) - y) = 0$ for all $x, y \in I$ and $s \in R$. This implies that $[s, y]I(g(y) - y) = \{0\}$ for all $x, y \in I$ and $s \in R$. Since R is prime, we have $[s, y] = 0$ for all $y \in I$ and $s \in R$ or $g(y) - y = 0$ for all $y \in I$. But $g(x) \neq x$ for all $x \in I$, and so $[s, y] = 0$ for all $x, y \in I$ and $s \in R$. Again, replacing y by ry with $r \in R$ in this relation, we have $[s, r]y = 0$, which implies that $[s, r]I = \{0\}$ for all $r, s \in R$. Hence $[s, r]RI = \{0\}$. Since $I \neq 0$, we have $[s, r] = 0$, which means that R is commutative.

□

Theorem 3.13. *Let R be a prime ring and let I be a nonzero semigroup ideal of R. Suppose that R admits a nonzero semimultiplier F associated with g and $g(x) \neq x$ for all $x \in I$. If $F(xy) = x \circ y$ holds for every $x, y \in I$, then R is commutative.*

Proof. By hypothesis, we have

$$F(xy) = x \circ y, \ \forall \ x \in I. \tag{16}$$

Replacing x by xy in (16), we have $F(xy)g(y) = (x \circ y)y$, that is, $(x \circ y)g(y) = (x \circ y)y$ for all $x, y \in I$. This implies that $(x \circ y)(g(y) - y) = 0$ for all $x, y \in I$. Also, replacing x by xy in the last relation, we have $(x \circ y)y(g(y) - y) = 0$ for all $x, y \in I$. This implies that $(x \circ y)I(g(y) - y) = \{0\}$ for all $x, y \in I$. By Lemma 3.4, we have $x \circ y = 0$ for all $x, y \in I$ or $g(y) - y = 0$ for all $y \in I$. But $g(x) \neq x$ for all $x \in I$, and so $x \circ y = 0$ for all $x, y \in I$. Again, replacing y by ys with $s \in R$ in this relation, we have $y[x, s] = 0$ for all $x \in I$ and $s \in R$. Taking xr instead of x with $r \in R$, in the last relation, we have $yx[r, s] = 0$, that is, $yI[r, s] = 0$ for all $r, s \in R$. Since $I \neq 0$, we have $[r, s] = 0$ for all $r, s \in R$. Hence R is commutative.

□

Theorem 3.14. *Let R be a prime ring and let I be a nonzero semigroup ideal of R. Suppose that R admits a nonzero semimultiplier F associated with g and $g(x) \neq x$ for all $x \in I$. If $F([x, y]) = x \circ y$ holds for every $x, y \in I$, then R is commutative.*

Proof. By hypothesis, we have

$$F([x, y]) = x \circ y, \ \forall \ x, y \in I. \tag{17}$$

Replacing y by yz in (17), we have $F(y[x, z] + [x, y]z) = x \circ yz$, that is, $F(y)g([x, z]) + F([x, y])g(z) = (x \circ y)z - y[x, z]$ for all $x, y, z \in I$. Taking z instead of x in this relation, we have $F([z,y])g(z) = (z \circ y)z$ for all $y, z \in I$. By hypothesis, we obtain $(z \circ y)(g(z) - z) = 0$ for all $y, z \in I$.

Using the similar arguments of the last part proof Theorem 3.13, we get the required result.

□

Theorem 3.15. *Let R be a prime ring and let I be a nonzero semigroup ideal of R. Suppose that R admits a nonzero semimultiplier F associated with g and $g(x) \neq x$ for all $x \in I$. If $F(x \circ y) = [x, y]$ holds for every $x, y \in I$, then R is commutative.*

Proof. By hypothesis, we have

$$F(x \circ y) = [x, y], \ \forall \ x, y \in I. \tag{18}$$

Replacing y by yz in (18), we have $F((x \circ y)z - y[x, z]) = [x, yz]$, that is, $F(x \circ y)g(z) - F(y)g([x, z]) = y[x, z] + [x, y]z$ for all $x, y, z \in I$. Taking x instead of z in this relation, we have

$$F(x \circ y)g(x) = [x, y]x$$

for all $x, y \in I$. By the hypothesis, we get $[x, y]g(x) = [x, y]x$, and so $[x, y](g(x) - x) = 0$ for all $x, y \in I$.

Using the similar arguments of the last part proof Theorem 3.12, we get the required result.

□

Theorem 3.16. *Let R be a prime ring and let I be a nonzero semigroup ideal of R. Suppose that R admits a nonzero semimultiplier F associated with g such that F satisfies any one of the following conditions:*

(a) $[F(x), F(y)] = xy$ *for every* $x, y \in I$,

(b) $[F(x), F(y)] = yx$ *for every* $x, y \in I$.

Then R is commutative.

Proof. (a) By hypothesis, we have

$$[F(x), F(y)] = xy, \ \forall \ x, y \in I. \tag{19}$$

Replacing y by yz in (19), we get $[F(x), F(yz)] = [F(x), F(y)g(z)] = xyz$ for every $x, y \in I$. Using (19) and the fact that g is onto, we obtain

$$F(y)[F(x), z] = 0, \ \forall \ x, y \in I. \tag{20}$$

Again, replacing z by zs with $s \in R$, in (20), we have $F(y)z[F(x), s] = 0$, which implies that $F(y)I[F(x), s] = \{0\}$ for any $x, y \in I$ and $s \in R$. Thus, by Lemma 3.4, we have $F(y) = 0$ or $[F(x), s] = 0$ for every $x, y \in I$ and $s \in R$. Since $F \neq 0$, we have $[F(x), s] = 0$, which implies that $F(I) \subseteq Z(R)$. Hence, by Theorem 3.6, R is commutative.

(b) By hypothesis, we have

$$[F(x), F(y)] = yx, \ \forall \ x, y \in I. \tag{21}$$

Replacing x by xz with $z \in I$, in (21), we get $[F(xz), F(y)] = [F(x)g(z), F(y)] = 0$ for every $x, y \in I$. Using (21) and the fact that g is onto, we obtain

$$F(x)[z, F(y)] = 0, \ \forall \ x, y, z \in I. \tag{22}$$

Using the same methods as we used in the last part proof of (a), we get the required result.

□

Theorem 3.17. *Let R be a prime ring and let I be a nonzero semigroup ideal of R. Suppose that R admits a nonzero semimultiplier F associated with g such that F satisfies any one of the following conditions:*
(a) $F(x)F(y) = [x, y]$ *for every* $x, y \in I$,
(b) $F(y)F(x) = [x, y]$ *for every* $x, y \in I$,
(c) $F(x)F(y) = x \circ y$ *for every* $x, y \in I$.
Then R is commutative.

Proof. (a) By hypothesis, we have

$$F(x)F(y) = [x, y], \ \forall \ x, y \in I. \tag{23}$$

Replacing y by yz in (23), we get $F(x)F(yz) = [x, yz]$ for every $x, y \in I$. This implies that $F(x)F(y)g(z) = y[x, z] + [x, y]z$ for every $x, y, z \in I$. Using (23), we obtain

$$[x, y]g(z) = y[x, z] + [x, y]z, \ \forall \ x, y, z \in I. \tag{24}$$

Taking y in place of x, we have $y[y, z] = 0$ for all $y, z \in I$. Replacing z by zs with $z \in I$, in (24), we have $yz[y, s] = 0$, which implies that $yI[x, s] = \{0\}$ for every $x, y \in I$ and $s \in R$. Thus, by Lemma 3.4, we have $y = 0$ for $[y, s] = 0$ for all $x, y \in I$ and $s \in R$. If $y = 0$, then $I = \{0\}$, a contradiction, and so $[x, s] = 0$ for every $x \in I$ and $s \in R$. Now, replacing x by xr in the last relation, we have $x[r, s] = 0$, which means that $I[r, s] = \{0\}$ for all $r, s \in R$. Hence we get $xI[r, s] = \{0\}$ for $0 \neq x \in I$ and $r, s \in R$. By Lemma 3.4, we have $[r, s] = 0$, which implies that R is commutative.

In cases of (b) and (c), using the same methods as we used in the last part proof of (a), we get the required result.

□

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