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THE EQUICONVERGENCE OF THE EIGENFUNCTION EXPANSION FOR A SINGULAR STURM-LIOUVILLE PROBLEM WITH SIGN-VALUED WEIGHT

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Abstract. The purpose of this paper is to prove the equiconvergence formula of the eigenfunction expansion for a singular Sturm-Liouville problem with sign valued weight on a finite interval $[0, \pi]$. Our methodology depends on asymptotic calculation and the method of contour integration.

1. Introduction

The theory of the equiconvergence of the eigenfunction expansion is one of interesting an analytical problem that arising in the field of spectral analysis of differential operator see [1],[2]. From many years ago, the class of spectral problem of Sturm-Liouville with discontinuous weight founded great interest by Gasimov and his disciples see [4-6]. Consider the following Sturm-Liouville problem

$$-y'' + q(x)\, y = \lambda\, \rho(x)\, y \;\; 0 \leq x \leq \pi \tag{1}$$

$$y(0) = 0\;,\; y'(\pi) + H\, y(\pi) = 0, \tag{2}$$

with $q(x)$ being non-negative real function has a second piecewise integrable derivatives on $(0, \pi)$, Let also, H is positive number, λ is a spectral parameter and weighted function or the explosive factor $\rho(x)$ has the following form

$$\rho(x) = \begin{cases} 1\;;\; 0 \leq x \leq a < \pi \\ -1\;;\; a < x \leq \pi. \end{cases} \tag{3}$$

The author in [7] discuss the asymptotic behavior of the eigenvalues which are real and simple, and the eigenfunctions of the problem(1)-(2), also he studied the orthogonality of eigenfunction expansion with respect to $\rho(x)$. In[8] the author calculated the regularized trace formula,consequently he studied the eigenfunction expansion of same problem see [9], we should mention here the more difficulty that we obtained in our problem due to the definition of $\rho(x)$in the form of (3) because it divided our problem into two problems see [10],[11] which the author studied the

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equicovergence theorem with same $\rho(x)$,indeed the authors in [5],[6] obtained the equiconvergence theorem but in different $\rho(x)$ which define by the following form

$$\rho(x) = \begin{cases} \alpha^2 \ ; \ \alpha \neq \ 10 \leq x \leq a \\ 1 \ ; \ x > a. \end{cases} \quad (4)$$

Although all authors following same methodology there's a change on the boundary conditions that contained which led to different in the results that obtained. In this paper we prove the equiconvergence formula for problem (1)-(2)using countour integration over the quadratic contour Γ_n which is defined in [8] as follow

$$\Gamma_n = \left\{ |Re\ s| < \frac{\pi}{a}(n - \frac{1}{4}) + \frac{\pi}{2a} \ , \ |Im\ s| \leq \frac{\pi}{\pi - a}(n - \frac{1}{4}) + \frac{\pi}{2(\pi - a)} \right\}. \quad (5)$$

2. Basic definitions and results

In this section we mention some basic definitions and results which obtained by the author in [7-9]which we need in our work.

- Let the functions $\varphi(x, \lambda)$,$\psi(x, \lambda)$ are solutions of equation (1) under the initial conditions:

$$\varphi(0, \lambda) \ = \ 0, \ \varphi'(0, \lambda) \ = \ 1 \quad (6)$$

$$\psi(\pi, \lambda) \ = \ 1, \ \psi'(\pi, \lambda) \ = \ -H. \quad (7)$$

  Where $\varphi(x, \lambda), \psi(x, \lambda)$ are entire in λ and satisfied boundary conditions (2) at $x = 0$ and $x = \pi$ respectively. The Wronskian of two solutions $\varphi(x, \lambda), \psi(x, \lambda)$ of the equation (1) is define as

$$W(\lambda) = < \varphi(x, \lambda), \psi(x, \lambda) >= \ \varphi(x, \lambda) \ \psi'(x, \lambda) \ - \ \varphi'(x, \lambda) \ \psi(x, \lambda). \quad (8)$$

  Where $W(\lambda) \neq 0$ if and only if the two solutions $\varphi(x, \lambda), \psi(x, \lambda)$are linearly independent and the eigenvalues coincide with the roots of the function $W(\lambda) = 0$ which are simple, indeed $W(\lambda)$ doesn't dependent on x and it's appropriate to put $x \ = \ a$ in (8).
- In [9]the author define the next formula:

$$G(x, t, \lambda) \ = \ \frac{-1}{W(\lambda)} \begin{cases} \varphi(x, \lambda) \ \psi(t, \lambda) \ x \leq \ t, \\ \varphi(t, \lambda) \ \psi(x, \lambda) \ x \geq \ t, \end{cases} \quad (9)$$

  which is called Green's function (the kernel of the resolvent of Sturm-Liouville problem (1)-(2) and this function admit for $\lambda = \lambda_k$ the following formula

$$G(x, t, \lambda) \ = \ \frac{-1}{\lambda - \lambda_k} \frac{\varphi(x, \lambda_k) \ \varphi(t, \lambda_k)}{a_k} \ + \ G_1(x, t, \lambda). \quad (10)$$

  Where $G_1(x, t, \lambda)$ is regular in the neighborhood of $\lambda = \lambda_k$ and$a_k = \int_0^\pi \ \rho(t)\varphi^2(x, \lambda_k)dx \neq 0$. which is called the normalization numbers of (1)-(2).

- Also the author in [9] studied the extended asymptotic formulas of the eigenfunctions $\varphi(x,\lambda)$,and $\psi(x,\lambda)$ for the problem (1)-(2) over the interval $[0,\pi]$ as follow:

$$\varphi(x,\lambda) \ = \ \begin{cases} \frac{\sin sx}{s} \ + \ O\left(\frac{e^{|Ims|x)}}{|s^2|}\right), \ 0 \le x \le a, \\ \\ \frac{1}{s}\left[\sin sa \ \cosh s(a-x) \ - \ \cos sa \ \sinh s(a-x)\right] \\ + \ O\left(\frac{e^{|Ims|a+|Res|(a-x))}}{|s^2|}\right), \ a < x \le \pi, \end{cases} \tag{11}$$

$$\psi(x,\lambda) \ = \ \begin{cases} \frac{u(x)}{u(a)}\left[\cos s(x-a) \ \cosh s(\pi-a) \ - \ \sin s(x-a) \ \sinh s(\pi-a)\right] \\ + \ O\left(\frac{e^{|Ims|(x-a)+|Res|(\pi-a))}}{|s|}\right), \ 0 \le x \le a, \\ \\ \cosh s(\pi-x) \ + \ O\left(\frac{e^{|Res|(\pi-x))}}{|s|}\right), \ a < x \le \pi, \end{cases} \tag{12}$$

where

$$u(x) \ = \ \frac{1}{2}\int_0^x \ q(t) \ dt.$$

3. The simple form for Sturm-Liouville (1)-(2)

Consider the Sturm-Liouville problem in the simple form ($q(x) = 0$), then the problem (1)-(2) can be written as

$$-\,y'' \ = \ \lambda \ \rho(x) \ y \ 0 \le x \le \pi \tag{13}$$

$$y(0) \ = \ 0 \ , \ y'(\pi) \ = \ 0. \tag{14}$$

Let $\varphi_o(x,\lambda), \psi_o(x,\lambda)$ are the solutions of problem (13)-(14)in cases $\rho(x) = 1, \rho(x) = -1$ respectively where

$$\varphi_o(x,\lambda) \ = \ \frac{\sin sx}{s} \ 0 \le x \le a \tag{15}$$

$$\psi_o(x,\lambda) \ = \ \cosh s(\pi-x) \ a < x \le \pi, \tag{16}$$

we need to extended the solutions $\varphi_o(x,\lambda), \psi_o(x,\lambda)$ to all interval $[0,\pi]$ because these formulas in (15),(16) defined on parts of the interval.In the following lemma we will deduce this extension formula.

lemma 1 The asymptotic formula of the solutions $\varphi_o(x,\lambda)$, and $\psi_o(x,\lambda)$ have the next form

$$\varphi_o \ (x,\lambda) \ = \ \begin{cases} \frac{\sin sx}{s}; \ 0 \le x \le a; \\ \frac{\sin sa}{s} \ \cosh s(x-a) \ + \ \frac{cossa}{s} \sinh s(x-a); \ a < x \le \pi; \end{cases} \tag{17}$$

$$\psi_o(x,\lambda) \ = \ \begin{cases} \cos s(x-a) \ \cosh s(\pi-a) \ - \ \sin s(x-a) \ \sinh s(\pi-a); \ 0 \le x \le a; \\ \cosh s(\pi-x); \ a < x \le \pi. \end{cases} \tag{18}$$

Proof. we starting with equation

$$-\,y'' \ = \ s^2 \ y, \ 0 \le x \le a. \tag{19}$$

The fundamental system of solutions of (19) is $y_1(x,s) = \sin sx$, $y_2(x,s) = \cos sx$, moreover the equation

$$y'' = s^2 y, \; a < x \leq \pi. \tag{20}$$

have also fundamental system of solutions of is $z_1(x,s) = \sinh s(\pi - x)$, $z_2(x,s) = \cosh s(\pi - x)$, hence the solutions $\varphi_o(x,\lambda)$, and $\psi_o(x,\lambda)$ over interval $[0,\pi]$ can be represented by

$$\varphi_o\ (x,\lambda) = \begin{cases} \frac{\sin sx}{s};\ 0 \leq x \leq a; \\ c_1\ z_1(x,s) + c_2\ z_2(x,s);\ a < x \leq \pi; \end{cases} \tag{21}$$

$$\psi_o(x,\lambda) = \begin{cases} m_1\ y_1(x,s) + m_2\ y_2(x,s);\ 0 \leq x \leq a; \\ \cosh s(\pi - x);\ a < x \leq \pi. \end{cases} \tag{22}$$

To calculate the constants $c_1, c_2, m_1,\ and m_2$ differentiation both equations (21),(22) with respect to x at $x = a$, and using the continuity property of these derivatives with solutions $\varphi_o(x,\lambda)$, and $\psi_o(x,\lambda)$, we get

$$\begin{aligned} c_1 &= -\tfrac{\sin sa}{s}\ \sinh s(\pi - a) - \tfrac{\cos sa}{s}\ \cosh s(\pi - a), \\ c_2 &= \tfrac{\sin sa}{s}\ \cosh s(\pi - a) + \tfrac{\cos sa}{s}\ \sinh s(\pi - a), \end{aligned} \tag{23}$$

substituting from(23) into (21)we obtain (17), hence by applying same methodology, we get

$$\begin{aligned} m_1 &= \sin sa\ \cosh s(\pi - a) - \cos sa\ \sinh s(\pi - a), \\ m_2 &= \cos sa\ \cosh s(\pi - a) + \sin sa\ \sinh s(\pi - a), \end{aligned} \tag{24}$$

substituting from(24) into (22)we obtain (18), which complete our proof.

4. The Green's function in terms of the simple Green's function

The study of the equiconvergence theorem of problem (1)-(2) required to find the asymptotic formula of Green's function of problem (1)-(2) in terms of the corresponding simple Green's function in case of $q(x) = 0$ for the problem (13)-(14). Consider the Green's function of the problem (13)-(14) as follow:

$$G_o(x,t,\lambda) = \frac{-1}{W_o(\lambda)} \begin{cases} \varphi_o(x,\lambda)\ \psi_o(t,\lambda)\ x \leq\ t, \\ \varphi_o(t,\lambda)\ \psi_o(x,\lambda)\ x \geq\ t, \end{cases}, \tag{25}$$

where,

$$W_o(\lambda) = -\sin sa\ \sinh s(\pi - a) - \cos sa\ \cosh s(\pi - a), \tag{26}$$

which satisfied the next inequality on the contour Γ_n, which is defined by (5)

$$|W_o(\lambda)| \geq C\ e^{|Im\ s|a\ +\ |Re\ s|(\pi - a)}. \tag{27}$$

In the next lemma we calculate the asymptotic formula for the Green's function $G(x,t,\lambda)$ in terms of $G_o(x,t,\lambda)$.

lemma 2 The Green's function $G(x,t,\lambda)$ admits the next formula

$$G(x,t,\lambda) = G_o(x,t,\lambda) + g(x,t,\lambda), \tag{28}$$

under the following conditions

- $q(x) \in L_1[0,\pi]$,
- the asymptotic formula of (11), and (12) where $g(x,t,\lambda), \lambda \in \Gamma_n,\ n \to \infty$ holds the next inequality

$$g(x,t,\lambda) \ = \ \begin{cases} O\left(\frac{e^{-|Im\ s||x-t|}}{|s^2|}\right), \ for\ x,t\ \in\ [0,a] \\ O\left(\frac{e^{-|Re\ s||x-t|}}{|s^2|}\right), \ for\ x,t\ \in\ (a,\pi] \\ O\left(\frac{e^{-|Im\ s|(x-a)-|Re\ s|(a-t))}}{|s^2|}\right), \ for\ 0 \le x \le a < t \le \pi, \\ O\left(\frac{e^{-|Im\ s|(t-a)-|Re\ s|(a-x)}}{|s^2|}\right), \ for\ 0 \le t \le a < x \le \pi, \end{cases} \tag{29}$$

Proof. The author in [7] obtained the following

$$W(\lambda) \ = \ \varphi(a,\lambda)\ \psi'(a,\lambda) - \varphi'(a,\lambda)\ \psi(a,\lambda),$$

keep in mind (11),(12),(26), and(27), we have after some calculations

$$W(\lambda) \ = \ W_o(\lambda) \ + \ O\left(\frac{e^{|Im\ s|a+|Re\ s|(\pi-a))}}{|s|}\right), \tag{30}$$

which equivalent to

$$W(\lambda) \ = \ W_o(\lambda)\left[1+O\left(\frac{1}{|s|}\right)\right]. \tag{31}$$

urging as before in [9], we have six possibilities to study

- (i) first three possibilities for $x \le t$, we have
  (1) $0 \le x \le t \le a$, (2) $a < x \le t \le \pi$, and (3) $0 \le x \le a \le t \le \pi$,
- (ii) second three possibilities for $t \le x$, we have
  (4) $0 \le t \le x \le a$, (5) $a < t \le x \le \pi$, and (6) $0 \le t \le a \le x \le \pi$.

Starting with the calculations of the first three possibilities in case(i) for $x \le t$,as follow

(1) In the case (1) using (9),(11), and(12),we get

$$\begin{aligned} G(x,t,\lambda) &= \frac{-1}{W(\lambda)}\ \varphi(x,\lambda)\ \psi(t,\lambda) \\ &= \frac{-1}{W(\lambda)}\ \left[\varphi_o(x,\lambda)\ \psi_o(t,\lambda) + O\left(\frac{e^{|Im\ s|(x+a-t)+|Re\ s|(\pi-a))}}{|s^2|}\right)\right]. \end{aligned}$$

by the aid of (30),(31),(27), and (25) after substituting, we obtain

$$\begin{aligned} G(x,t,\lambda) &= \frac{-1}{W_o(\lambda)}\ \left[\varphi_o(x,\lambda)\ \psi_o(t,\lambda)\right] \ + \ O\left(\frac{e^{|Im\ s|(x-t)}}{|s^2|}\right) \\ &= G_o(x,t,\lambda) \ + \ O\left(\frac{e^{|Im\ s|(x-t)}}{|s^2|}\right). \end{aligned} \tag{32}$$

(2) In the case (2) using (9),(11), and (12),we get

$$\begin{aligned} G(x,t,\lambda) &= \frac{-1}{W(\lambda)}\ \varphi(x,\lambda)\ \psi(t,\lambda) \\ &= \frac{-1}{W(\lambda)}\ \left[\varphi_o(x,\lambda)\ \psi_o(t,\lambda) + O\left(\frac{e^{|Im\ s|a+|Re\ s|(\pi-a+x-t)}}{|s^2|}\right)\right]. \end{aligned}$$

by the aid of (30),(31),(27), and (25) after substituting, we obtain

$$\begin{aligned} G(x,t,\lambda) &= \frac{-1}{W_o(\lambda)} \; [\varphi_o(x,\lambda)\;\psi_o(t,\lambda)] \;+\; O\left(\frac{e^{|Re\;s|(x-t)}}{|s^2|}\right) \\ &= G_o(x,t,\lambda) \;+\; O\left(\frac{e^{|Re\;s|(x-t)}}{|s^2|}\right). \end{aligned} \tag{33}$$

(3) In the case (3) using same methodology, we have

$$\begin{aligned} G(x,t,\lambda) &= \frac{-1}{W(\lambda)}\;\varphi(x,\lambda)\;\psi(t,\lambda) \\ &= \frac{-1}{W(\lambda)} \left[\varphi_o(x,\lambda)\;\psi_o(t,\lambda) + O\left(\frac{e^{|Im\;s|x+|Re\;s|(\pi-t))}}{|s^2|}\right)\right]. \end{aligned}$$

as before, we obtain

$$\begin{aligned} G(x,t,\lambda) &= \frac{-1}{W_o(\lambda)} \; [\varphi_o(x,\lambda)\;\psi_o(t,\lambda)] \;+\; O\left(\frac{e^{|Im\;s|(x-a)+|Re\;s|(a-t)}}{|s^2|}\right) \\ &= G_o(x,t,\lambda) \;+\; O\left(\frac{e^{|Im\;s|(x-a)+|Re\;s|(a-t)}}{|s^2|}\right). \end{aligned} \tag{34}$$

Second, we will discuss the other three cases for $t \le x$ as follow

(4) In the case (4) using (9),(11), and (12),we have

$$\begin{aligned} G(x,t,\lambda) &= \frac{-1}{W(\lambda)}\;\varphi(t,\lambda)\;\psi(x,\lambda) \\ &= \frac{-1}{W(\lambda)} \left[\varphi_o(t,\lambda)\;\psi_o(x,\lambda) + O\left(\frac{e^{|Im\;s|(a+t-x)+|Re\;s|(\pi-a)}}{|s^2|}\right)\right]. \end{aligned}$$

by the aid of (30),(31),(27), and (25) after substituting, we obtain

$$G(x,t,\lambda) \;=\; G_o(x,t,\lambda) \;+\; O\left(\frac{e^{|Im\;s|(t-x)}}{|s^2|}\right). \tag{35}$$

(5) Moreover in the case (5) urging as before, we obtain

$$G(x,t,\lambda) \;=\; \frac{-1}{W(\lambda)} \left[\varphi_o(t,\lambda)\;\psi_o(x,\lambda) + O\left(\frac{e^{|Im\;s|a+|Re\;s|(\pi-a+t-x)}}{|s^2|}\right)\right].$$

similarly, we have

$$G(x,t,\lambda) \;=\; G_o(x,t,\lambda) \;+\; O\left(\frac{e^{|Re\;s|(t-x)}}{|s^2|}\right). \tag{36}$$

Finally, in the case (6),we get

$$G(x,t,\lambda) \;=\; \frac{-1}{W(\lambda)} \left[\varphi_o(t,\lambda)\;\psi_o(x,\lambda) + O\left(\frac{e^{|Im\;s|t+|Re\;s|(\pi-x)}}{|s^2|}\right)\right].$$

by same manner, we obtain

$$G(x,t,\lambda) \;=\; G_o(x,t,\lambda) \;+\; O\left(\frac{e^{|Im\;s|(t-a)+|Re\;s|(a-x)}}{|s^2|}\right). \tag{37}$$

By the virtue of (32), and(35) together we have

$$G(x,t,\lambda) \;=\; G_o(x,t,\lambda) \;+\; O\left(\frac{e^{-|Im\ s||x-t|}}{|s^2|}\right),\;\; x,t \in\; [0,a]. \tag{38}$$

Similarly from (33), and(36), we get

$$G(x,t,\lambda) \;=\; G_o(x,t,\lambda) \;+\; O\left(\frac{e^{-|Re\ s||x-t|}}{|s^2|}\right),\;\; x,t \in\; [a,\pi]. \tag{39}$$

Final step from (38),(39),(34),and (37) together we get the inequality in (29) which end our proof.

5. Equiconvergence

In this section we prove the equiconvergence of the eigenfunction expansion of Sturm-Liouville problem (1)-(2), now we will claim what we do to prove the equiconvergence as follow

- suppose that $f(x) \in\; L_2[0,\pi]$, we choose

$$A_{n,f} \;=\; \sum_{k=0}^{n} \frac{1}{a_k^+}\,\varphi(x,\lambda_k^+)\int_0^{\pi}\;\varphi(t,\lambda_k^+)\;f(t)\;\rho(t)dt+\sum_{k=0}^{n}\frac{1}{a_k^-}\,\varphi(x,\lambda_k^-)\int_0^{\pi}\;\varphi(t,\lambda_k^-)\;f(t)\;\rho(t)dt, \tag{40}$$

  where $a_k^{\pm}\;\neq 0$ from [7]. Arguing as in [9] the series in (40) convergence uniformly to any function $\;f(x) \in\; (0,\pi,\rho(x))$ as$n\to\infty$. Let $A_{n,f}^{(o)}$ denoted the corresponding function for Sturm-Liouville problem (13)-(14) (where $q(x) = 0$).
- The essentially required to prove the equiconvergence of the eigenfunction expansion of problem (1)-(2) that is the difference $|A_{n,f} \;-\; A_{n,f}^{(o)}|$ uniformly convergence to 0 as $n\to\infty,\;x\in\;[0,\pi]$, and in next theorem we will explain that.

theorem 1 The next equiconveregence formula which state that:

$$\lim_{n\to\infty}\;\sup_{0\le x\le\pi}\;|A_{n,f}(x)\;-\;A_{n,f}^{(o)}(x)|\;=\;0, \tag{41}$$

admits under the conditions of lemma (3), and lemma (4). **Proof.** From lemma (4), we have

$$G(x,t,\lambda)\;=\;G_o(x,t,\lambda)\;+\;g(x,t,\lambda),$$

multiply both sides by $\rho(t)\;f(t)$, and hence integrate from 0 to π , we get

$$\int_0^{\pi}\;G(x,t,\lambda)\;\rho(t)\;f(t)\;dt\;=\;\int_0^{\pi}\;G_o(x,t,\lambda)\;\rho(t)\;f(t)\;dt+\int_0^{\pi}\;g(x,t,\lambda)\;\rho(t)\;f(t)\;dt, \tag{42}$$

now to apply the Caushy residues formula to (42) we must integrate over a closed contour, so that according to definition of the quadratic contour Γ_n in(5) , suppose that Γ_n^+ the upper half of contour Γ_n, $Ims\ge 0$, and L_n is the image of the contour Γ_n^+ under the mapping $\lambda\;=\;s^2$.

Now multiply (42) by $\frac{1}{2\pi i}$ and integrating over the contour L_n in λ domain, we get

$$\frac{1}{2\pi i}\oint_{L_n}\left\{\int_0^\pi G(x,t,\lambda)\ \rho(t)\ f(t)\ dt\right\}d\lambda\ =$$

$$\frac{1}{2\pi i}\oint_{L_n}\left\{\int_0^\pi G_o(x,t,\lambda)\ \rho(t)\ f(t)\ dt\right\}d\lambda+\frac{1}{2\pi i}\oint_{L_n}\left\{\int_0^\pi g(x,t,\lambda)\ \rho(t)\ f(t)\ dt\right\}d\lambda, \tag{43}$$

notice that the poles of the function $G(x,t,\lambda)$ coincide with the roots of the function $W(\lambda)$ following from (10). Now in (43) we want to calculate the three integrals, then to obtain the first integral $\frac{1}{2\pi i}\oint_{L_n}\left\{\int_0^\pi G(x,t,\lambda)\ \rho(t)\ f(t)\ dt\right\}d\lambda$, applying the residues formula, we have

$$\frac{1}{2\pi i}\oint_{L_n}\left\{\int_0^\pi G(x,t,\lambda)\ \rho(t)\ f(t)\ dt\right\}d\lambda\ =\sum_{k=0}^{n} Res_{\lambda=\lambda_k}\left\{\int_0^\pi G(x,t,\lambda_k^\pm)\ \rho(t)\ f(t)\ dt\right\}, \tag{44}$$

using the formula (10), then (44) have the following form

$$\frac{1}{2\pi i}\oint_{L_n}\left\{\int_0^\pi G(x,t,\lambda)\ \rho(t)\ f(t)\ dt\right\}d\lambda\ =$$

$$\sum_{k=0}^{n}\frac{\varphi(x,\lambda_k^+)}{a_k^+}\int_0^\pi \varphi(t,\lambda_k^+)\ f(t)\ \rho(t)dt+\sum_{k=0}^{n}\frac{\varphi(x,\lambda_k^-)}{a_k^-}\int_0^\pi \varphi(t,\lambda_k^-)\ f(t)\ \rho(t)dt=A_{n,f}(x). \tag{45}$$

Similarly, applying same methodology to the second integral $\frac{1}{2\pi i}\oint_{L_n}\left\{\int_0^\pi G_o(x,t,\lambda)\ \rho(t)\ f(t)\ dt\right\}d\lambda$, we have

$$\frac{1}{2\pi i}\oint_{L_n}\left\{\int_0^\pi G_o(x,t,\lambda)\ \rho(t)\ f(t)\ dt\right\}d\lambda\ =\ A_{n,f}^{(o)}(x). \tag{46}$$

Substituting from (45),(46) into(43), we obtain

$$A_{n,f}(x)\ -\ A_{n,f}^{(o)}(x)\ =\ \frac{1}{2\pi i}\oint_{L_n}\left\{\int_0^\pi g(x,t,\lambda)\ \rho(t)\ f(t)\ dt\right\}d\lambda, \tag{47}$$

affected by modules to both sides of (47), we get

$$|A_{n,f}(x)\ -\ A_{n,f}^{(o)}(x)|\ \le\ \frac{1}{2\pi}\oint_{L_n}\left\{\int_0^\pi |g(x,t,\lambda)|\ |f(t)|\ dt\right\}|d\lambda|. \tag{48}$$

To get our purpose of the theorem and prove the equicongeregence we must show that the right hand side of (48) must tends to zero uniformly with respect to $x\in[0,\pi]$, arguing as in [6],[11], apply same methodology , we have

$$\oint_{L_n}\left\{\int_0^\pi |g(x,t,\lambda)|\ |f(t)|\ dt\right\}|d\lambda|\ =$$

$$\oint_{L_n}\left\{\int_0^a |g(x,t,\lambda)|\ |f(t)|\ dt\right\}|d\lambda|\ +\ \oint_{L_n}\left\{\int_a^\pi |g(x,t,\lambda)|\ |f(t)|\ dt\right\}|d\lambda|. \tag{49}$$

From lemma(4), we get

$$\oint_{L_n}\left\{\int_0^{\pi} |g(x,t,\lambda)||f(t)|\ dt\right\}|d\lambda| \ \le$$

$$H_1 \oint_{L_n}\left\{\int_0^{a} \frac{e^{-|Im\ s|x-t|}}{|s|^2}|f(t)|dt\right\}|d\lambda| + H_2 \oint_{L_n}\left\{\int_a^{\pi} \frac{e^{-|Im\ s|(x-a)-|Re\ s|(a-t)}}{|s|^2}|f(t)|\ dt\right\}\ |d\lambda|. \tag{50}$$

since H_1, H_2 are constants, here we have two integrals $\int_0^a$,and $\int_a^\pi$ we must deal with them, therefore starting with calculation of the integral $\int_0^a$ so that,let $\lambda = s^2$, and suppose that $\delta > 0$ be sufficiently small number, then, for $x, t \in [o, a]$, we have

$$\oint_{L_n}\ \left\{\int_0^{a} \frac{e^{-|Im\ s|x-t|}}{|s|^2}\ |f(t)|\ dt\right\}\ |d\lambda|$$

$$= \int_{\Gamma_n^+}\frac{|ds|}{|s|}\left\{\int_{|x-t|\le\delta} e^{-|Im\ s||x-t|}\ |f(t)|\ dt\ +\ \int_{|x-t|>\delta} e^{-|Im\ s||x-t|}\ |f(t)|\ dt\right\}$$

$$\le \int_{\Gamma_n^+}\frac{|ds|}{|s|}\ \int_{|x-t|\le\delta} |f(t)|\ dt\ + \int_0^{\pi} |f(t)|\ dt\ \int_{\Gamma_n^+} e^{-|Im\ s|\delta}\ \frac{|ds|}{|s|}$$

$$\le\ 4\ \int_{|x-t|\le\delta} |f(t)|\ dt\ +\ \int_0^{\pi} |f(t)|\ dt\ \left[\frac{2}{\delta(n-\frac{1}{2})}\ +\ 2\ e^{-\delta(n-\frac{1}{4})}\right],$$

which led to the next relation

$$\begin{aligned} &H_1 \oint_{L_n}\ \left\{\int_0^{a} \frac{e^{-|Im\ s|x-t|}}{|s|^2}\ |f(t)|\ dt\right\}\ |d\lambda| \\ &\le\ M_1 \int_{|x-t|\le\delta} |f(t)|\ dt\ +\ \frac{M_2}{\delta n}\ +\ M_3\ e^{-\delta\ n}. \end{aligned} \tag{51}$$

Where M_1, M_2, and M_3 are independent of x,n, and δ. By the same manner we evaluated the next integral of $\int_a^\pi$ in (50), we obtain

$$\begin{aligned} &H_2 \oint_{L_n}\ \left\{\int_a^{\pi} \frac{e^{-|Im\ s|(x-a)-|Re\ s|(a-t)}}{|s|^2}\ |f(t)|\ dt\right\}\ |d\lambda| \\ &\le\ M_4 \int_{|x-t|\le\delta} |f(t)|\ dt\ +\ \frac{M_5}{\delta n}\ +\ M_6\ e^{-\delta\ n}. \end{aligned} \tag{52}$$

Where M_4, M_5, and M_6 are independent of x,n, and δ, hence substituting (51),(52) into (50), we obtain that

$$\oint_{L_n}\ \left\{\int_0^{\pi} |g(x,t,\lambda)|\ |f(t)|\ dt\right\}\ |d\lambda|\ \le\ B\ \int_{|x-t|\le\delta} |f(t)|\ dt + \frac{C}{\delta n} + D\ e^{-\delta n}. \tag{53}$$

Where A,B,and C are constants also independent of x,n,and δ, now from (53) into(48), we get

$$|A_{n,f}(x)\ -\ A_{n,f}^{(o)}(x)|\ \le\ B\ \int_{|x-t|\le\delta} |f(t)|\ dt + \frac{C}{\delta n} +\ D\ e^{-\delta n}. \tag{54}$$

As a final step of our proof, for $f(x) \in L_1[0, \pi]$ applying the property of absolute continuity of Lesbuge integral to f(x), $\forall\ \epsilon\ > 0, \exists\ \delta\ > 0$ is sufficiently small such that $\int_{|x-t| \le \delta} |f(t)|\, dt \le \epsilon$, where $\epsilon$is independent of x which means that (the set $\{|x - t \le \delta\}$ is measurable), also fixed δ in (54), there exists N such that $\forall\ n > N$, $\frac{1}{\delta n} < \epsilon$ and $e^{-\delta n}\ \epsilon$, then the formula (54) becomes

$$|A_{n,f}(x) - A^{(o)}_{n,f}(x)| \le (B + C + D)\ \epsilon,\ n > N. \tag{55}$$

Choose ϵ is sufficiently small in 955), then we get $|A_{n,f}(x) - A^{(o)}_{n,f}(x)| \to 0,\ as\ n \to \infty$, uniformly with respect to $x \in [0, \pi]$. Which finish our vision of the proof of the equiconvergence theorem for sinqular Sturm-Liouville problem(1)-(2).

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