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ON A MULTIVARIABLE EXTENSION OF LAGUERRE MATRIX PLOYNOMIALS

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Abstract. This paper deals with the series expansion formula, summation formula and recurrence relations for Laguerre matrix polynomials of several variables. New generating functions and several new relation between Laguerre matrix polynomials of different number of variables are also given.

1. Introduction and preliminaries

Special matrix functions is an emerge field of study, with important result in statistics, theoretical physics, group representation theory and number theory [1, 2, 8, 9, 25]. Concept of orthogonal matrix polynomials have been considered in [13], in the survey on orthogonal matrix polynomials by Rodman [22] also, in papers [3, 4, 5, 6, 7] and the references therein. The classical orthogonal polynomials have been extended to the orthogonal matrix polynomials in [16, 19]. Hermite, Laguerre, Gegenbauer and Chebyshev matrix polynomials was introduced and studied in [14, 16, 20, 21, 24], Hermite matrix polynomials have been introduced and studied in [15]. The hypergeometric matrix function $F(A, B; C; z)$, hypergeometric matrix differential equation in [17] and the explicit closed form general solution of it has been given in [18].

Throughout this study, we consider the complex space $\mathbb{C}^{N\times N}$ of all square complex matrices of common order N, we say that a matrix A in $\mathbb{C}^{N\times N}$ is a positive stable, if $Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$, where $\sigma(A)$ is the set of all eigenvalues of A. If $A_0, A_1, \cdots, A_n$ are elements of $\mathbb{C}^{N\times N}$ and $A_n \neq 0$, then $P(x) = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$ a matrix polynomial of degree n in x. If $A + nI$ is invertible for every integer $n \geq 0$ then, from [17] the matrix version of the pochhammer symbol is

$$(A)_n = \frac{\Gamma(A + nI)}{\Gamma(A)},$$

$$(A)_n = \begin{cases} I, & \text{if } n = 0 \\ A(A+I)(A+2I)\cdots(A+(n-1)I), & \text{if } n = 1, 2, \cdots. \end{cases} \tag{1}$$

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From (1), it is easy to find that

$$(A)_{n-k} = \frac{(-1)^k (A)_n}{(I - A - nI)_k};\ 0 \leq k \leq n. \tag{2}$$

From [11, pp. 58], one obtains

$$\frac{(-1)^k}{(n-k)!} I = \frac{(-n)_k}{n!} I = \frac{(-nI)_k}{n!};\ 0 \leq k \leq n. \tag{3}$$

If $A + nI$, $B + nI$, and $A + B + nI$ all are invertible

$$\beta(A, B) = \frac{\Gamma(A)\Gamma(B)}{\Gamma(A+B)};\ 0 \leq k \leq n, \tag{4}$$

where $\beta(A, B)$ denote beta matrix function for the pair A, B. The hypergeometric matrix function $F(A, B; C; z)$ has been given in the form [17] for matrix A, B, and C in $\mathbb{C}^{N\times N}$ such that $C + nI$ is invertible for all integer $n \geq 0$

$$F(A, B; C; z) = \sum_{n\geq 0} \frac{(A)_n (B)_n}{(C)_n n!} z^n, \tag{5}$$

it converges for $|z| < 1$, for any matrix A in $\mathbb{C}^{N\times N}$ we will exploit the following relation due to [17].

$$(1 - x)^{-A} = \sum_{n\geq 0} \frac{(A)_n x^n}{n!};\ |x| < 1. \tag{6}$$

From [10], the matrices $A(k, n)$ and $B(k, n)$ in $\mathbb{C}^{N\times N}$ where $n \geq 0$, $k \geq 0$, the following relation are satisfied.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(k, n - 2k) \tag{7}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k, n - k), \tag{8}$$

similarly, we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n + 2k), \tag{9}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n + k). \tag{10}$$

If $\Re(C - A - B) > 0$ and if C is neither zero matrix nor a negative stable matrix,

$${}_2F_1 \left[\begin{array}{cc} A, B; & 1 \\ C; & \end{array} \right] = \frac{\Gamma(C)\Gamma(C - A - B)}{\Gamma(C - A)(C - B)}. \tag{11}$$

Also, if $|z| < 1$ and $|z/(1 - z)| < 1$

$${}_2F_1 \left[\begin{array}{cc} A, B; & z \\ C; & \end{array} \right] = (1 - z)^{-A} {}_2F_1 \left[\begin{array}{cc} A, C - B; & \frac{z}{z-1} \\ C; & \end{array} \right], \tag{12}$$

$$ {}_2F_1\left[\begin{array}{cc} A, B; & z \\ C; & \end{array}\right] = (1-z)^{C-A-B}\,{}_2F_1\left[\begin{array}{cc} C-A, C-B; & z \\ C; & \end{array}\right]. \tag{13} $$

If λ is a complex number with $\Re(\lambda) > 0$ and A is a matrix in $\mathbb{C}^{N\times N}$ with $A+nI$ invertible for every integer $n \geq 1$ the n^{th} Laguerre matrix polynomial $L_n^{(A,\lambda)}(x)$ is defined by [20]

$$ L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k}{k!(n-k)!}(A+I)_n[(A+I)_k]^{-1}x^k. \tag{14} $$

The generating function of Laguerre matrix polynomials is given by [20]

$$ (1-t)^{-(A+I)} exp\big(\frac{-\lambda x t}{(1-t)}\big) = \sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x)t^n, x, t \in \mathbb{C}; |t| < 1, \tag{15} $$

and the generating function of Laguerre matrix polynomials of two variables is given by [28]

$$ (1-yt)^{-(A+I)} exp\big(\frac{-\lambda x t}{(1-yt)}\big) = \sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x,y)t^n, x, y, t \in \mathbb{C}; |yt| < 1. \tag{16} $$

The Laguerre polynomials of two variables given by S.F.Ragub [26]

$$ L_n^{(\alpha,\beta)}(x,y) = \frac{(1+\alpha)_n(1+\beta)_n}{(n!)^2} \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-n)_{r+s}x^s y^r}{(1+\alpha)_s(1+\beta)_r s!r!}. \tag{17} $$

In view of Eqs. (14), (16) and (17), the Laguerre Matrix polynomials of two variables can be cast in the form

$$ L_n^{(A,B)}(x,y) = \frac{(I+A)_n(I+B)_n}{(n!)^2} \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-nI)_{r+s}x^s y^r}{(I+A)_s(I+B)_r s!r!}. \tag{18} $$

The Laguerre Matrix polynomials of three variables can be cast in the form

$$ L_n^{(A,B,C)}(x,y,z) = \frac{(I+A)_n(I+B)_n(I+C)_n}{(n!)^3} \sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{k=0}^{n-r-s} \frac{(-nI)_{r+s+k}x^s y^r z^k}{(I+A)_s(I+B)_r(I+C)_k s!r!k!}. \tag{19} $$

2. Generating Functions of $L_n^{(A,B)}(x,y)$

One can easily derive the following generating function for the Laguerre matrix polynomials of two variables, which is given by (18)

$$ \sum_{n=0}^{\infty} \frac{n!(\lambda I)_n L_n^{(A-nI,B-nI)}(x,y)}{(\mu I)_n} t^n = (1-xt)^B(1-yt)^A $$
$$ \times F^{(3)}\left[\begin{array}{cc} - :: -A; -B; \mu I - \lambda I : \lambda I; -; - : & \frac{t}{(xt-1)(yt-1)}, \frac{yt}{(yt-1)}, \frac{xt}{(xt-1)} \\ \mu I :: -; -; - : -; -; - : & \end{array}\right], \tag{20} $$

where $\mu I, \lambda I, A, B$ are matrices in $\mathbb{C}^{N\times N}$ and $F^{(3)}[x,y,z]$ denotes a general triple hypergeometric series defined by [12]. The special case of (20) for $\mu I = \lambda I$, the equation (20) reduces to the following form

$$\sum_{n=0}^{\infty} n! L_n^{(A-nI,B-nI)}(x,y)t^n = (1-xt)^B(1-yt)^A {}_2F_0 \left[\begin{array}{cc} -A;-B: & \frac{t}{(1-xt)(1-yt)} \\ -: & \end{array} \right]. \tag{21}$$

Further, the following generating relations for Laguerre matrix polynomials of two variables hold

$$\sum_{n=0}^{\infty} \frac{n! L_n^{(A-nI,B)}(x,y)t^n}{(I+B)_n} = exp(-xt)(1+t)^A {}_1F_1 \left[\begin{array}{cc} -A: & \frac{yt}{(1+t)} \\ I+B: & \end{array} \right], \tag{22}$$

$$\sum_{n=0}^{\infty} \frac{n! L_n^{(A,B-nI)}(x,y)t^n}{(I+A)_n} = exp(-yt)(1+t)^B {}_1F_1 \left[\begin{array}{cc} -B: & \frac{xt}{(1+t)} \\ I+A: & \end{array} \right], \tag{23}$$

also (22) and (23) can be written as follows, by using definition of Laguerre polynomials [11]

$$\sum_{n=0}^{\infty} \frac{n! L_n^{(A-nI,B)}(x,y)t^n}{(I+B)_n} = \frac{\Gamma(I+A)\Gamma(I+B)}{\Gamma(I+A+B)} exp(-xt)(1+t)^A L_A^{(B)}\left(\frac{ty}{1+t}\right), \tag{24}$$

$$\sum_{n=0}^{\infty} \frac{n! L_n^{(A,B-nI)}(x,y)t^n}{(I+A)_n} = \frac{\Gamma(I+A)\Gamma(I+B)}{\Gamma(I+A+B)} exp(-yt)(1+t)^B L_B^{(A)}\left(\frac{tx}{1+t}\right), \tag{25}$$

where $A, B \in \{I, 2I, 3I, \cdots\}$. Now, we introduce two interesting expression of Laguerre Matrix polynomials of two variables form the generating functions (22) (23) as follows

$$L_n^{(A-nI,B)}(x+z,y) = \frac{(I+B)_n}{n!} \sum_{k=0}^{n} \frac{k!(-x)^{n-k}}{(n-k)!(I+B)_k} L_n^{(A-kI,B)}(z,y), \tag{26}$$

$$L_n^{(A-nI,B)}(x,y+z) = \frac{(I+A)_n}{n!} \sum_{k=0}^{n} \frac{k!(-y)^{n-k}}{(n-k)!(I+A)_k} L_n^{(A,B-kI)}(x,z), \tag{27}$$

the above expression can also be written as

$$L_n^{(A-nI,B)}(x+z,y) = \frac{(I+B)_n}{n!} \sum_{k=0}^{n} \frac{k!(-z)^{n-k}}{(n-k)!(I+B)_k} L_n^{(A-kI,B)}(x,y), \tag{28}$$

$$L_n^{(A,B-nI)}(x,y+z) = \frac{(I+A)_n}{n!} \sum_{k=0}^{n} \frac{k!(-z)^{n-k}}{(n-k)!(I+A)_k} L_n^{(A,B-kI)}(x,y). \tag{29}$$

Furthermore, we obtain the following generating functions of Laguerre Matrix polynomials of two variables

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!(-x)^k t^n}{k!(I+A)_n(I+A)_{k-n}(I+B)_{-k}} L_{n-k}^{(A+kI,B-nI)}(x,y)$$
$$= (1-xt)^B(1-yt)^A {}_2F_0 \left[\begin{array}{cc} -A,-B & \frac{t}{(1-xt)(1-yt)} \\ -: & \end{array} \right], \tag{30}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!(-y)^k t^n}{k!(I+B)_n(I+B)_{k-n}(I+A)_{-k}}L_{n-k}^{(A-nI,BkI)}(x,y)$$
$$=(1-xt)^B(1-yt)^A{}_2F_0\left[\begin{array}{ll}-A,-B & \frac{t}{(1-xt)(1-yt)}\\ -: & \end{array}\right], \tag{31}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!(-x)^k t^n}{k!(I+A)_n(I+A)_{k-n}(I+B)_{n-k}}L_{n-k}^{(A+kI,B)}(x,y)$$
$$=exp(-xt)(1+t)^A{}_1F_1\left[\begin{array}{ll}-A; & \frac{ty}{(1+t)}\\ I+B; & \end{array}\right], \tag{32}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!(-y)^k t^n}{k!(I+B)_n(I+B)_{k-n}(I+A)_{n-k}}L_{n-k}^{(A,B+kI)}(x,y)$$
$$=exp(-yt)(1+t)^B{}_1F_1\left[\begin{array}{ll}-B; & \frac{tx}{(1+t)}\\ I+A; & \end{array}\right]. \tag{33}$$

Proof of (30).
Taking L.H.S of (30) and substituting the value of $L_n^{A,B}(x,y)$ from (18), we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!(-x)^k t^n}{k!(I+A)_n(I+A)_{k-n}(I+B)_{-k}}\frac{(I+A+kI)_{n-k}(I+B-nI)_{n-k}}{(n-k)!^2}$$
$$\times\sum_{r=0}^{n-k}\sum_{s=0}^{n-k-r}\frac{(-nI+kI)_{r+s}x^s y^r}{(I+A+kI)_s(I+B-nI)_r s!r!},$$

putting $s=p-k$, where p is new parameter of summation and using result (11), we find

$$\sum_{n=0}^{\infty}\frac{t^n}{n!(I+A)_n(I+B)_{-n}}\sum_{r=0}^{n}\sum_{p=0}^{n-r}\frac{(-nI)_{p+r}x^p y^r}{(I+A-nI)_p(I+B-nI)_r p!r!},$$
$$=\sum_{n=0}^{\infty}n!L_n^{(A-nI,B-nI)}(x,y)t^n.$$

Making use of the equation (20), we get R.H.S of (30). Similarly, we can prove (31), (32), (33).
Again, we can obtain the following generating functions by series rearrangement techniques or series manipulation and use of linear transformations (Euler's transformations) (12), (13) of the Gaussian hypergeometric function

$$\sum_{n=0}^{\infty}\frac{n!(\lambda I)_n}{(\mu I)_n(I+B)_n}L_n^{(A,B)}(x,yt^n$$
$$=\frac{1}{(1-t)^{I+A}}F^{(3)}\left[\begin{array}{ll}I+A::-;-;\lambda I:\mu I-\lambda I;-;-: & \frac{t}{(t-1)},\frac{yt}{(t-1)},\frac{xt}{(t-1)}\\ \mu I::-;-;-:-;I+B;I+A: & \end{array}\right], \tag{34}$$

$$\sum_{n=0}^{\infty}\frac{n!(\lambda I)_n}{(\mu I)_n(I+B)_n}L_n^{(A,B)}(x,y)t^n=(1-t)^{\mu I-\lambda I-A-I}$$
$$\times F^{(3)}\left[\begin{array}{c}-::-;-;\lambda I,I+A:\mu I-\lambda I,\mu I-A-I;-;-:\\ \mu I::-;-;-:-;I+B;I+A:\end{array}\ t,\frac{yt}{(t-1)},\frac{xt}{(t-1)}\right],\tag{35}$$

$$\sum_{n=0}^{\infty}\frac{n!(\lambda I)_n}{(\mu I)_n(I+A)_n}L_n^{(A,B)}(x,y)t^n$$
$$=\frac{1}{(1-t)^{I+B}}F^{(3)}\left[\begin{array}{c}I+B::-;-;\lambda I:\mu I-\lambda I;-;-:\\ \mu I::-;-;-:-;I+B;I+A:\end{array}\ \frac{t}{(t-1)},\frac{yt}{(t-1)},\frac{xt}{(t-1)}\right],\tag{36}$$

$$\sum_{n=0}^{\infty}\frac{n!(\lambda I)_n}{(\mu I)_n(I+A)_n}L_n^{(A,B)}(x,y)t^n=(1-t)^{\mu I-\lambda I-B-I}$$
$$\times F^{(3)}\left[\begin{array}{c}-::-;-;\lambda I,I+B:\mu I-\lambda I,\mu I-B-I;-;-:\\ \mu I::-;-;-:-;I+B;I+A:\end{array}\ t,\frac{yt}{(t-1)},\frac{xt}{(t-1)}\right].\tag{37}$$

3. Recurrence Relations and Summation formulae of $L_n^{(A,B)}(x,y)$

The following recurrence relations hold for Laguerre Matrix polynomials of two variables

$$AL_n^{(A,B)}(x,y)+xD_xL_n^{(A,B)}(x,y)=(A+nI)L_n^{(A-I,B)}(x,y).\tag{38}$$

$$BL_n^{(A,B)}(x,y)+yD_yL_n^{(A,B)}(x,y)=(B+nI)L_n^{(A,B-I)}(x,y).\tag{39}$$

$$\frac{(A+nI)(B+nI)}{nI}L_{n-1}^{(A,B)}(x,y)+(xD_x+yD_y)L_n^{(A,B)}(x,y)=nIL_n^{(A-I,B)}(x,y).\tag{40}$$

Using (38), (39) and (40), we obtain

$$(A+B+nI)L_n^{(A,B)}(x,y)=\frac{(A+nI)(B+nI)}{nI}L_{n-1}^{(A,B)}(x,y)$$
$$+(A+nI)L_n^{(A-I,B)}(x,y)+(B+nI)L_n^{(A,B-I)}(x,y).\tag{41}$$

Proof of (38).

$$AL_n^{(A,B)}(x,y)=(A+nI)L_n^{(A-I,B)}(x,y)-xD_xL_n^{(A,B)}(x,y),$$

making the use of the equation (18) in R.H.S, we get the result.
Similarly, we can prove (39), (40) and (41).
And, the Laguerre Matrix Polynomials of two variables $L_n^{(A,B)}(x,y)$ satisfy the following summation formula, if $m\le n$

$$\sum_{t=0}^{m}\frac{\binom{m}{t}x^t}{(I+A-mI)_t}D_x^tL_n^{(A,B)}(x,y)=\frac{(I+A)_n}{(I+A-mI)_n}L_n^{(A-mI,B)}(x,y),\tag{42}$$

$$\sum_{t=0}^{m}\frac{\binom{m}{t}y^t}{(I+B-mI)_t}D_y^tL_n^{(A,B)}(x,y)=\frac{(I+B)_n}{(I+B-mI)_n}L_n^{(A,B-mI)}(x,y).\tag{43}$$

It may be noted that equations (42) and (43) are generalizations of (38) and (39) respectively because by setting $m = 1$ in (42) and (43) they become (38) and (39).

Proof of (42).

Consider the series

$$\sum_{t=0}^{m} \frac{\binom{m}{t} x^t}{(I + A - mI)_t} D_x^t L_n^{(A,B)}(x, y)$$

$$= \frac{(I + A)_n (I + B)_n}{(n!)^2} \sum_{t=0}^{m} \frac{\binom{m}{t}}{(I + A - mI)_t} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} x^s y^r}{(I + A)_s (I + B)_r r!(s - t)!},$$

$$= \frac{(I + A)_n (I + B)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} x^s y^r}{(I + A)_s (I + B)_r r!(s!)^2} {}_2F_1 \left[\begin{array}{cc} -sI, -mI; & I \\ I + A - mI; & \end{array} \right],$$

making the use of the expression (11), we obtain the required equation.
We can prove (43) on similar lines.
Also, it is easy to derive the following properties of the Laguerre Matrix Polynomials of two variables

$$L_n^{(A,B)}(x, y) = \frac{(I + B)_n}{n!} \sum_{k=0}^{n} \frac{(-1)^k (n - k)!}{(I + B)_{n-k}} L_{n-k}^{(-I-B-A-2nI-kI,B)}(x, -y) L_k^{(-2A-2nI-B-2I)}(x). \tag{44}$$

$$L_n^{(A,B)}(x, y) = \frac{(I + A)_n}{n!} \sum_{k=0}^{n} \frac{(-1)^k (n - k)!}{(I + A)_{n-k}} L_{n-k}^{(A,-I-B-A-2nI-kI)}(-x, y) L_k^{(-2B-2nI-A-2I)}(y). \tag{45}$$

Proof of (44).

Taking equation (22) and using the Kumar's first formula on Matrix polynomials ${}_1F_1[A; B; z] = e^z {}_1F_1[B - A; B; -z]$, we find

$$\sum_{n=0}^{\infty} \frac{n! L_n^{(A-nI,B)}(x, y)}{(I + B)_n} t^n = (1 + t)^{I+B+2A} exp\left(\frac{yt}{1 + t}\right) e^{-xt}$$

$$\times (1 + t)^{-I-B-A} {}_1F_1 \left[\begin{array}{cc} I + B + A; & \frac{-ty}{(1+t)} \\ I + B; & \end{array} \right].$$

Using the equation (22) on the R.H.S of above equation, we get

$$\sum_{n=0}^{\infty} \frac{n! L_n^{(A-nI,B)}(x, y)}{(I + B)_n} t^n = (1 + t)^{I+B+2A} exp\left(\frac{yt}{1 + t}\right) \sum_{n=0}^{\infty} \frac{n! L_n^{(-I-B-A-nI,B)}(x, -y)}{(I + B)_n} t^n.$$

Now, using the generating function

$$(1 - t)^{-I-A} exp\left(\frac{-xt}{1 - t}\right) = \sum_{n=0}^{\infty} L_n^{(A)}(x) t^n,$$

we get

$$\sum_{n=0}^{\infty} \frac{n! L_n^{(A-nI,B)}(x, y)}{(I + B)_n} t^n = \sum_{k=0}^{\infty} L_k^{(-2A-B-2I)}(x)(-t)^k \sum_{n=0}^{\infty} \frac{n! L_n^{(-I-B-A-nI,B)}(x, -y)}{(I + B)_n} t^n,$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k (n-k)!}{(I+B)_{n-k}} L_{n-k}^{(-I-B-A-2nI-kI,B)}(x,-y) L_k^{(-2A-2nI-B-2I)}(x) t^n.$$

Equating the coefficient of t^n from both sides, we get the expression (44).
Similarly, we can prove (45).

4. Series Expansion Formulae and Relations of $L_n^{(A,B)}(x,y)$

We are now in a position to obtain a large number of series expansion formulae involving partial derivatives for Laguerre Matrix Polynomials of two variables, these series expansion formulae are given below

$$\sum_{t=0}^{n} \frac{(w)^t}{t!} D_x^t L_n^{(A,B)}(x,y) = L_n^{(A,B)}(x+w,y). \tag{46}$$

$$\sum_{t=0}^{n} \frac{(w)^t}{t!} D_y^t L_n^{(A,B)}(x,y) = L_n^{(A,B)}(x,y+w). \tag{47}$$

$$\sum_{t=0}^{n} \frac{(-\nu)^t}{t!(I+A)_{n-t}} D_x^t L_n^{(A-tI,B)}(x,y) = \frac{(1+\nu)^{nI}}{(I+A)_n} L_n^{(A,B)}\Big(\frac{x}{1+\nu}, \frac{y}{1+\nu}\Big). \tag{48}$$

$$\sum_{t=0}^{n} \frac{(-\nu)^t}{t!(I+A)_{n-t}} D_y^t L_n^{(A,B-tI)}(x,y) = \frac{(1+\nu)^{nI}}{(I+A)_n} L_n^{(A,B)}\Big(\frac{x}{1+\nu}, \frac{y}{1+\nu}\Big). \tag{49}$$

$$\sum_{t=0}^{n} \frac{(-\nu)^t (n+t)!^2 (-nI)_{-t}}{t!} D_x^t D_y^t L_{n+t}^{(A-tI,B-tI)}(x,y) = n!^2 (1+\nu)^{nI} L_n^{(A,B)}\Big(\frac{x}{1+\nu}, \frac{y}{1+\nu}\Big). \tag{50}$$

Proof of (46). Taking the L.H.S of (46) and substituting the value of $L_n^{(A,B)}(x,y)$ from (18), we get

$$\sum_{t=0}^{n} \frac{(w)^t}{t!} \frac{(I+A)_n (I+B)_n}{(n!)^2} D_x^t \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-nI)_{r+s} x^s y^r}{(I+A)_s (I+B)_r s! r!}$$

$$= \frac{(I+A)_n (I+B)_n}{(n!)^2} \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-nI)_{r+s} x^s y^r}{(I+A)_s (I+B)_r s! r!} \sum_{t=0}^{n} \frac{(-sI)_t (\frac{-w}{x})^t}{t!},$$

$$= \frac{(I+A)_n (I+B)_n}{(n!)^2} \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-nI)_{r+s} (x+w)^s y^r}{(I+A)_s (I+B)_r s! r!},$$

$$= L_n^{(A,B)}(x+w,y).$$

Similarly, we can prove (47).
Proof of (48). Taking the L.H.S of (48), substituting the value of Laguerre Matrix Polynomials of two variables from (18) and differentiating, we get

$$= \frac{(I+B)_n}{(n!)^2} \sum_{t=0}^{n} \frac{(-\nu)^t}{t!(I+A)_{-t}} \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(-nI)_{r+s} x^{s-t} y^r}{(I+A-tI)_s (I+B)_r (s-t)! r!}.$$

Putting $s = k + t$ where k is new parameter of summation and changing the order of summation so that the first summation becomes last,

$$= \frac{(I+B)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{k=0}^{n-r} \frac{(-nI)_{r+s} x^k y^r}{(I+A)_k (I+B)_r k! r!} \sum_{t=0}^{n-r-k} \frac{(-nI+rI+kI)_t (-\nu)^t}{t!},$$

$$= \frac{(I+B)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{k=0}^{n-r} \frac{(-nI)_{r+s} x^k y^r}{(I+A)_k (I+B)_r k! r!} (1+\nu)^{nI-rI-kI},$$

by definition of Laguerre Matrix Polynomials (18), we get

$$= \frac{(1+\nu)^{nI}}{(I+A)_n} L_n^{(A,B)}\Big(\frac{x}{1+\nu}, \frac{y}{1+\nu}\Big).$$

Hence the result (48), similarly, we can obtain (49).

Proof of (50). Taking the L.H.S of (50) and substituting the value of Laguerre Matrix polynomials of two variables from (18), we get

$$\sum_{t=0}^{n} \frac{(-\nu)^t (-nI)_{-t} (I+A-tI)_{n+t} (I+B-tI)_{n+t}}{t!} D_x^t D_y^t$$
$$\times \sum_{r=0}^{n+t} \sum_{s=0}^{n+t-r} \frac{(-nI-tI)_{r+s} x^s y^r}{(I+A-tI)_s (I+B-tI)_r r! s!}.$$

Now, differentiating and substituting $p = s - t$, $q = r - t$, we get

$$(I+A)_n (I+B)_n \sum_{t=0}^{n} \sum_{q=0}^{n-t} \sum_{p=0}^{n-t-q} \frac{(-\nu)^t (-nI)_{p+q+t} x^p y^q}{t! (I+A)_p (I+B)_q p! q!}$$

$$= (I+A)_n (I+B)_n \sum_{q=0}^{n} \sum_{p=0}^{n-q} \frac{(-nI)_{p+q} x^p y^q}{(I+A)_p (I+B)_q p! q!} (1+\nu)^{nI-pI-qI}.$$

Again, using the expression (18), we get the required equation (50).

And, several new result involving not only Laguerre Matrix Polynomials of two variables but also one variable Laguerre Matrix Polynomials are given as follows

$$\sum_{t=0}^{n} \frac{(-x)^t}{t!} D_x^t L_n^{(A,B)}(x,y) = \frac{(I+A)_n}{n!} L_n^{(B)}(y). \tag{51}$$

$$\sum_{t=0}^{n} \frac{(-y)^t}{t!} D_y^t L_n^{(A,B)}(x,y) = \frac{(I+B)_n}{n!} L_n^{(A)}(x). \tag{52}$$

$$\sum_{t=0}^{[\frac{n}{2}]} \frac{(xy)^t}{t!(I+A)_t} D_x^{2t} L_n^{(A)}(x+y) = \frac{n!}{(I+A)_n} L_n^{(A,A)}(x,y). \tag{53}$$

$$\sum_{t=0}^{[\frac{n}{2}]} \frac{(xy)^t}{t!(I+A)_t} D_y^{2t} L_n^{(A)}(x+y) = \frac{n!}{(I+A)_n} L_n^{(A,A)}(x,y). \tag{54}$$

$$\sum_{t=0}^{n} \frac{(xy)^t}{t!(-nI)_t} D_x^t L_n^{(A)}(x) D_y^t L_n^{(B)}(y) = L_n^{(A,B)}(x,y). \tag{55}$$

$$\sum_{t=0}^{n} \frac{(-xy)^t}{t!(I+B+nI)_t} D_x^t L_n^{(A,B+tI)}(x,y) = L_n^{(A)}(x) L_n^{(B)}(y). \quad (56)$$

$$\sum_{t=0}^{n} \frac{(-xy)^t}{t!(I+A+nI)_t} D_y^t L_n^{(A+tI,B)}(x,y) = L_n^{(A)}(x) L_n^{(B)}(y). \quad (57)$$

$$\sum_{t=0}^{n} \frac{(-xy)^t (n+t)!^2 (-nI)_{-t}}{t!(I+A+nI)_t (I+B+nI)_t (n!)^2} D_x^t D_y^t L_{n+t}^{(A,B)}(x,y) = L_n^{(A)}(x) L_n^{(B)}(y). \quad (58)$$

Proof of (51).
By substituting $w = -x$ in (46), we obtain

$$\sum_{t=0}^{n} \frac{(-x)^t}{t!} D_x^t L_n^{(A,B)}(x,y) = L_n^{(A,B)}(0,y).$$

Now, using [28], we get the expression (51).
Similarly, one can prove (52).
Proof of (53).
Consider the L.H.S of (53) and substituting the value of Laguerre Matrix Polynomials from (14), we have

$$\sum_{t=0}^{\left[\frac{n}{2}\right]} \frac{(xy)^t}{t!(I+A)_t} D_x^{2t} \frac{(I+A)_n}{n!} \sum_{k=0}^{n} \frac{(-nI)_k (x+y)^k}{(I+A)_k k!},$$

$$= \sum_{t=0}^{\left[\frac{n}{2}\right]} \frac{(xy)^t (I+A)_n}{t!(I+A)_t n!} \sum_{k=2t}^{n} \sum_{r=0}^{n-k} \frac{(-nI)_{k+r} x^{k-2t} y^r}{(I+A)_{k+r} (k-2t)! r!}.$$

Setting $k = p + t$ and $r = q + t$ where p and q are new parameters of summation and changing the order of summation, we find

$$= \frac{(I+A)_n}{n!} \sum_{p=0}^{n} \sum_{q=0}^{n-p} \frac{(-nI)_{p+q} x^p y^q}{(I+A)_{p+q} p! q!} \sum_{t=0}^{min(p,q)} \frac{(-pI)_t (-qI)_t}{(I+A)_t t!},$$

again, using (11), we get the required equation.
Similarly, one can prove (54).
Proof of (55).
Taking L.H.S of (55) and substituting the value of Laguerre Matrix Polynomials from (14), also differentiating we get

$$\frac{(I+A)_n (I+B)_n}{(n!)^2} \sum_{t=0}^{n} \frac{1}{(-nI)_t t!} \sum_{r=t}^{n} \frac{(-nI)_r x^r}{(I+A)_r (r-t)!} \sum_{s=t}^{n} \frac{(-nI)_s y^s}{(I+B)_s (s-t)!}$$

$$= \frac{(I+A)_n (I+B)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n} \frac{(-nI)_r (-nI)_s x^r y^s}{(I+A)_r (I+B)_s r! s!} \sum_{t=0}^{min(r,s)} \frac{(-rI)_t (-sI)_t}{(-nI)_t t!}.$$

Making the use of (11), we obtain

$$= \frac{(I+A)_n (I+B)_n}{(n!)^2} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-nI)_{r+s} x^r y^s}{(I+A)_r (I+B)_s r! s!}$$

Proof of (56). One can write the L.H.S of (56) as follows

$$\sum_{t=0}^{n} \frac{(-xy)^t (I+A)_n (I+B+tI)_n}{t!(I+B+nI)_t (n!)^2} D_x^t \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-nI)_{r+s} x^r y^s}{(I+A)_s (I+B+tI)_r r! s!}$$

$$= \frac{(I+A)_n (I+B)_n}{(n!)^2} \sum_{t=0}^{n} \frac{1}{t!} \sum_{r=0}^{n} \sum_{s=t}^{n-r} \frac{(-nI)_{r+s} x^s y^{r+t}}{(I+A)_s (I+B)_{r+t} r!(s-t)!}.$$

Putting $r = k - t$ in above expression, where k is a new parameter of summation and changing the order of summation we have

$$= \frac{(I+A)_n (I+B)_n}{(n!)^2} \sum_{k=0}^{n} \sum_{s=0}^{n-k} \frac{(-nI)_{k+s} x^s y^k}{(I+A)_s (I+B)_k s! k!} \sum_{t=0}^{min(k,s)} \frac{(-kI)_t (-sI)_t}{(I+nI-sI-kI)_t t!}.$$

Now using (11), we get

$$= \frac{(I+A)_n (I+B)_n}{(n!)^2} \sum_{k=0}^{n} \sum_{s=0}^{n-k} \frac{(-nI)_{k+s} x^s y^k}{(I+A)_s (I+B)_k s! k!} \frac{(I+nI)_{-s-k}}{(I+nI)_{-s} (I+nI)_{-k}},$$

$$= \frac{(I+A)_n (I+B)_n}{(n!)^2} \sum_{k=0}^{n} \frac{(-nI)_k y^k}{(I+B)_k k!} \sum_{s=0}^{n} \frac{(nI)_s x^s}{(I+A)_s s!},$$

$$= L_n^{(A)}(x) L_n^{(B)}(y).$$

Similarly, we can prove (57) and (58).

5. Generating Functions, Recurrence Relations and Series Expansion of $L_n^{(A,B,C)}(x,y,z)$

For the Laguerre Matrix Polynomials of three variables, which is given by (19), we can easily derive the following generating functions

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(\mu I)_n} L_n^{(A-nI,B-nI,C-nI)}(x,y,z) t^n = F^{(4)} \left[\begin{array}{c} - ::: -C; -; -A; -B :: -; -; -; -; -; - : -; -; -; - \\ - ::: -; -; -; - :: -; -; -; -; -; - : -; -; -; - \\ t, -xt, -yt, -zt \end{array} \right]. \tag{59}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(\mu I)_n} L_n^{(A-nI,B-nI,C-nI)}(x,y,z) t^n = (1+xt)^{\mu I+B+C} F^{(4)} \left[\begin{array}{c} - ::: -A; -; -; - :: -B; -; \mu I+B; -; -C; \mu I+C : -; -; -; - \\ \mu I ::: -; -; -; - :: -; -; -; -; -; - : -; -; \mu I+B; \mu I+C \\ \left(\frac{-t}{xt+1}\right), -xt, -yt, -zt \end{array} \right], \qquad |xt| < 1. \tag{60}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(\mu I)_n}L_n^{(A-nI,B-nI,C-nI)}(x,y,z)t^n=(1+xt)^B$$
$$F^{(4)}\left[\begin{array}{l} -:::-A;-;-;-B::-;-;-;-;-C;\mu I+C:-;-;-;- \\ \mu I:::-;-;-;-::-;-;-;-;-;-:-;\mu I+C;-;- \\ \left(\frac{-t}{xt+1}\right),\left(\frac{-xt}{xt+1}\right),-yt,\left(\frac{-zt}{xt+1}\right) \end{array}\right],\qquad |xt|<1. \tag{61}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(\mu I)_n}L_n^{(A-nI,B-nI,C-nI)}(x,y,z)t^n=(1+xt)^C$$
$$F^{(4)}\left[\begin{array}{l} -:::-A;-;-C;-::-B;-;\mu I+B;-;-;-:-;-;-;- \\ \mu I:::-;-;-;-::-;-;-;-;-;-:-;\mu I+B;-;- \\ \left(\frac{-t}{xt+1}\right),\left(\frac{-xt}{xt+1}\right),\left(\frac{-yt}{xt+1}\right),-zt \end{array}\right],\qquad |xt|<1. \tag{62}$$

where $\mu I, A, B, C$ are matrices in $\mathbb{C}^{N\times N}$ and $F^{(4)}[x,y,z,w]$ denotes a general hypergeometric function of four variables defined by [12]. The special case of (59) for $\mu I = I + A$, the equation (59) reduces to the following form

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(I+A)_n}L_n^{(A,B-nI,C-nI)}(x,y,z)t^n=(1-zt)^B(1-yt)^C$$
$$F^{(2)}\left[\begin{array}{l} -B,-C::-;-: \\ -::-;I+A: \end{array}\left(\frac{t}{(1-zt)(1-yt)}\right),\left(\frac{-xt}{(1-zt)(1-yt)}\right),\right],\qquad |zt|<1,|yt|<1. \tag{63}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(I+B)_n}L_n^{(A-nI,B,C-nI)}(x,y,z)t^n=(1-xt)^C(1-zt)^A$$
$$F^{(2)}\left[\begin{array}{l} -A,-C::-;-: \\ -::-;I+B: \end{array}\left(\frac{t}{(1-zt)(1-xt)}\right),\left(\frac{-yt}{(1-zt)(1-xt)}\right),\right],\qquad |zt|<1,|xt|<1. \tag{64}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(I+C)_n}L_n^{(A-nI,B-nI,C)}(x,y,z)t^n=(1-xt)^B(1-yt)^A$$
$$F^{(2)}\left[\begin{array}{l} -A,-B::-;-: \\ -::-;I+C: \end{array}\left(\frac{t}{(1-yt)(1-xt)}\right),\left(\frac{-yt}{(1-yt)(1-xt)}\right),\right],\qquad |yt|<1,|xt|<1. \tag{65}$$

where $F^{(2)}$ denotes a general hypergeometric function of two variables.
Further, Laguerre Matrix Polynomials of three variables hold the following generating relations

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(I+B)_n(I+C)_n}L_n^{(A-nI,B,C)}(x,y,z)t^n=e^{-xt}(1+t)^A$$
$$\psi_2\left[\begin{array}{l} -A:I+B,I+C: \end{array}\left(\frac{ty}{(t+1)}\right),\left(\frac{tz}{(t+1)}\right)\right]. \tag{66}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(I+A)_n(I+C)_n} L_n^{(A,B-nI,C)}(x,y,z)t^n = e^{-yt}(1+t)^B$$
$$\psi_2 \left[\; -B : I+A, I+C : \quad \left(\tfrac{tx}{(t+1)}\right), \left(\tfrac{tz}{(t+1)}\right) \; \right]. \tag{67}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(I+A)_n(I+C)_n} L_n^{(A,B,C-nI)}(x,y,z)t^n = e^{-zt}(1+t)^C$$
$$\psi_2 \left[\; -C : I+A, I+B : \quad \left(\tfrac{tx}{(t+1)}\right), \left(\tfrac{ty}{(t+1)}\right) \; \right]. \tag{68}$$

where ψ_2 is Humbert's confluent function of two variables, defined by [12]. More over (66), (67) and (68) can be written in the following forms

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(I+B)_n(I+C)_n} L_n^{(A-nI,B,C)}(x,y,z)t^n = e^{-xt}(1+t)^A$$
$$\frac{(A!)^2\Gamma(I+C)\Gamma(I+B)}{\Gamma(I+A+B)\Gamma(I+A+C)} L_A^{(B,C)}\Big(\frac{ty}{1+t}, \frac{tz}{1+t}\Big). \tag{69}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(I+A)_n(I+C)_n} L_n^{(A,B-nI,C)}(x,y,z)t^n = e^{-yt}(1+t)^B$$
$$\frac{(B!)^2\Gamma(I+C)\Gamma(I+A)}{\Gamma(I+A+B)\Gamma(I+B+C)} L_B^{(A,C)}\Big(\frac{tx}{1+t}, \frac{tz}{1+t}\Big). \tag{70}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(I+A)_n(I+B)_n} L_n^{(A,B,C-nI)}(x,y,z)t^n = e^{-zt}(1+t)^C$$
$$\frac{(C!)^2\Gamma(I+A)\Gamma(I+B)}{\Gamma(I+A+C)\Gamma(I+B+C)} L_C^{(A,B)}\Big(\frac{tx}{1+t}, \frac{ty}{1+t}\Big). \tag{71}$$

where $A, B, C \in \{I, 2I, 3I \cdots\}$. Now, we introduce the following three interesting expression for Laguerre Matrix Polynomials of three variables, which can be obtained from the generating functions (66), (67) and (68) respectively

$$L_n^{(A,B,C)}(x+w,y,z) = \frac{(I+B)_n(I+C)_n}{(n!)^2} \sum_{k=0}^{n} \frac{(k!)^2(-wI)^{n-k}}{(n-k)!(I+B)_k(I+C)_k} L_k^{(A+nI-kI,B,C)}(x,y,z). \tag{72}$$

$$L_n^{(A,B,C)}(x,y+w,z) = \frac{(I+A)_n(I+C)_n}{(n!)^2} \sum_{k=0}^{n} \frac{(k!)^2(-wI)^{n-k}}{(n-k)!(I+A)_k(I+C)_k} L_k^{(A,B+nI-kI,C)}(x,y,z). \tag{73}$$

$$L_n^{(A,B,C)}(x,y,z+w) = \frac{(I+A)_n(I+B)_n}{(n!)^2} \sum_{k=0}^{n} \frac{(k!)^2(-wI)^{n-k}}{(n-k)!(I+A)_k(I+B)_k} L_k^{(A,B,C+nI-kI)}(x,y,z). \tag{74}$$

The above expression can also be written in the following forms

$$L_n^{(A,B,C)}(x+w,y,z) = \frac{(I+B)_n(I+C)_n}{(n!)^2} \sum_{k=0}^{n} \frac{(k!)^2(-xI)^{n-k}}{(n-k)!(I+B)_k(I+C)_k} L_k^{(A+nI-kI,B,C)}(w,y,z). \tag{75}$$

$$L_n^{(A,B,C)}(x,y+w,z) = \frac{(I+A)_n(I+C)_n}{(n!)^2}\sum_{k=0}^{n}\frac{(k!)^2(-yI)^{n-k}}{(n-k)!(I+A)_k(I+C)_k}L_k^{(A,B+nI-kI,C)}(x,w,z). \tag{76}$$

$$L_n^{(A,B,C)}(x,y,z+w) = \frac{(I+A)_n(I+B)_n}{(n!)^2}\sum_{k=0}^{n}\frac{(k!)^2(-zI)^{n-k}}{(n-k)!(I+A)_k(I+B)_k}L_k^{(A,B,C+nI-kI)}(x,y,w). \tag{77}$$

Further, by taking the following generating function given by M. A. Khan and A. K. Shukla [23] on three variables Laguerre Matrix Polynomials

$$\sum_{n=0}^{\infty}\frac{(n!)^2L_n^{(A,B,C)}(x,y,z)t^n}{(I+A)_n(I+B)_n(I+C)_n} = e^t{}_0F_1[-;I+A:-xt] \times {}_0F_1[-;I+B:-yt]{}_0F_1[-;I+C:-zt], \tag{78}$$

and expanding the R.H.S of above equation, and equating the coefficient of t^n from both sides we get

$$L_n^{(A,B,C)}(x,y,z) = \frac{(I+A)_n}{(n!)^2}\sum_{k=0}^{n}\frac{(-B)_k(-C)_k(n-k)!(-x)^k}{(I+A)_k k!}L_{n-k}^{(B,C)}(y,z). \tag{79}$$

$$L_n^{(A,B,C)}(x,y,z) = \frac{(I+B)_n}{(n!)^2}\sum_{k=0}^{n}\frac{(-A)_k(-C)_k(n-k)!(-y)^k}{(I+B)_k k!}L_{n-k}^{(A,C)}(x,z). \tag{80}$$

$$L_n^{(A,B,C)}(x,y,z) = \frac{(I+C)_n}{(n!)^2}\sum_{k=0}^{n}\frac{(-A)_k(-B)_k(n-k)!(-z)^k}{(I+C)_k k!}L_{n-k}^{(A,B)}(x,y). \tag{81}$$

Furthermore, if we apply the result

$$\sum_{n=0}^{\infty}{}_1F_1[-nI;I+A:x]\frac{t^n}{n!} = e^t{}_0F_1[-;I+A:-xt],$$

on the R.H.S of generating function (78) and equate the coefficient of t^n form both sides, we get

$$L_n^{(A,B,C)}(x,y,z) = \frac{(I+B)_n(I+C)_n}{(n!)^2}\sum_{k=0}^{n}\sum_{r=0}^{n-k}\frac{(-A-nI)_k y^k z^r}{(I+B)_k(I+C)_r k!r!}L_{n-k-r}^{(A)}(x). \tag{82}$$

$$L_n^{(A,B,C)}(x,y,z) = \frac{(I+A)_n(I+C)_n}{(n!)^2}\sum_{k=0}^{n}\sum_{r=0}^{n-k}\frac{(-B-nI)_k x^k z^r}{(I+A)_k(I+C)_r k!r!}L_{n-k-r}^{(B)}(y). \tag{83}$$

$$L_n^{(A,B,C)}(x,y,z) = \frac{(I+A)_n(I+B)_n}{(n!)^2}\sum_{k=0}^{n}\sum_{r=0}^{n-k}\frac{(-C-nI)_k x^k y^r}{(I+A)_k(I+B)_r k!r!}L_{n-k-r}^{(C)}(z). \tag{84}$$

The Laguerre Matrix Polynomials of three variables satisfy the following recurrence relations:

$$AL_n^{(A,B,C)}(x,y,z) + xD_x L_n^{(A,B,C)}(x,y,z) = (A+nI)L_n^{(A-I,B,C)}(x,y,z). \tag{85}$$

$$BL_n^{(A,B,C)}(x,y,z) + yD_y L_n^{(A,B,C)}(x,y,z) = (B+nI)L_n^{(A,B-I,C)}(x,y,z). \tag{86}$$

$$CL_n^{(A,B,C)}(x,y,z) + zD_z L_n^{(A,B,C)}(x,y,z) = (C+nI)L_n^{(A,B,C-I)}(x,y,z). \tag{87}$$

And, the series expansions involving partial derivative for three variables Laguerre Matrix Polynomials are the following

$$\sum_{t=0}^{n} \frac{(w)^t}{t!} D_x^t L_n^{(A,B,C)}(x,y,z) = L_n^{(A,B,C)}(x+w,y,z). \tag{88}$$

$$\sum_{t=0}^{n} \frac{(w)^t}{t!} D_y^t L_n^{(A,B,C)}(x,y,z) = L_n^{(A,B,C)}(x,y+w,z). \tag{89}$$

$$\sum_{t=0}^{n} \frac{(w)^t}{t!} D_z^t L_n^{(A,B,C)}(x,y,z) = L_n^{(A,B,C)}(x,y,z+w). \tag{90}$$

$$\sum_{t=0}^{n} \frac{(-\nu)^t}{t!(I+A)_{n-t}} D_x^t L_n^{(A-tI,B,C)}(x,y,z) = \frac{(1+\nu)^{nI}}{(I+A)_n} L_n^{(A,B,C)}\Big(\frac{x}{1+\nu}, \frac{y}{1+\nu}, \frac{z}{1+\nu}\Big). \tag{91}$$

$$\sum_{t=0}^{n} \frac{(-\nu)^t}{t!(I+B)_{n-t}} D_y^t L_n^{(A,B-tI,C)}(x,y,z) = \frac{(1+\nu)^{nI}}{(I+B)_n} L_n^{(A,B,C)}\Big(\frac{x}{1+\nu}, \frac{y}{1+\nu}, \frac{z}{1+\nu}\Big). \tag{92}$$

$$\sum_{t=0}^{n} \frac{(-\nu)^t}{t!(I+C)_{n-t}} D_z^t L_n^{(A,B,C-tI)}(x,y,z) = \frac{(1+\nu)^{nI}}{(I+C)_n} L_n^{(A,B,C)}\Big(\frac{x}{1+\nu}, \frac{y}{1+\nu}, \frac{z}{1+\nu}\Big). \tag{93}$$

$$\sum_{t=0}^{n} \frac{(-\nu)^t (n+t)!^3}{(I+nI)_t (I+C+nI)_t t!} D_x^t D_y^t L_{n+t}^{(A-tI,B-tI,C)}(x,y,z) = n!^3 (1+\nu)^{nI} \times L_n^{(A,B,C)}\Big(\frac{x}{1+\nu}, \frac{y}{1+\nu}, \frac{z}{1+\nu}\Big). \tag{94}$$

$$\sum_{t=0}^{n} \frac{(-\nu)^t (n+t)!^3}{(I+nI)_t (I+B+nI)_t t!} D_x^t D_z^t L_{n+t}^{(A-tI,B,C-tI)}(x,y,z) = n!^3 (1+\nu)^{nI} \times L_n^{(A,B,C)}\Big(\frac{x}{1+\nu}, \frac{y}{1+\nu}, \frac{z}{1+\nu}\Big). \tag{95}$$

$$\sum_{t=0}^{n} \frac{(-\nu)^t (n+t)!^3}{(I+nI)_t (I+A+nI)_t t!} D_y^t D_z^t L_{n+t}^{(A,B-tI,C-tI)}(x,y,z) = n!^3 (1+\nu)^{nI} \times L_n^{(A,B,C)}\Big(\frac{x}{1+\nu}, \frac{y}{1+\nu}, \frac{z}{1+\nu}\Big). \tag{96}$$

Also, the summation formulae for $L_n^{(A,B,C)}(x,y,z)$ are as given below, if $m \le n$

$$\sum_{t=0}^{m} \frac{\binom{m}{t} x^t}{(I+A-mI)_t} D_x^t L_n^{(A,B,C)}(x,y,z) = \frac{(I+A)_n}{(I+A-mI)_n} L_n^{(A-mI,B,C)}(x,y,z). \tag{97}$$

$$\sum_{t=0}^{m}\frac{\binom{m}{t}y^t}{(I+B-mI)_t}D_y^tL_n^{(A,B,C)}(x,y,z)=\frac{(I+B)_n}{(I+B-mI)_n}L_n^{(A,B-mI,C)}(x,y,z). \tag{98}$$

$$\sum_{t=0}^{m}\frac{\binom{m}{t}z^t}{(I+C-mI)_t}D_z^tL_n^{(A,B,C)}(x,y,z)=\frac{(I+C)_n}{(I+C-mI)_n}L_n^{(A,B,C-mI)}(x,y,z). \tag{99}$$

It may be noted that results (97), (98) and (99) are generalizations of (85), (86) and (87) respectively because by setting $m=1$ in (97), (98) and (99), they become (85), (86) and (87). At last, several new expression involving not only Laguerre Matrix Polynomials of three variables but also Laguerre Matrix Polynomials of two and four variables are given below

$$\sum_{t=0}^{n}\frac{(-x)^t}{t!}D_x^tL_n^{(A,B,C)}(x,y,z)=\frac{(I+A)_n}{n!}L_n^{(B,C)}(y,z). \tag{100}$$

$$\sum_{t=0}^{n}\frac{(-y)^t}{t!}D_y^tL_n^{(A,B,C)}(x,y,z)=\frac{(I+B)_n}{n!}L_n^{(A,C)}(x,z). \tag{101}$$

$$\sum_{t=0}^{n}\frac{(-z)^t}{t!}D_z^tL_n^{(A,B,C)}(x,y,z)=\frac{(I+C)_n}{n!}L_n^{(A,B)}(x,y). \tag{102}$$

$$\sum_{t=0}^{[\frac{n}{2}]}\frac{(xy)^t}{t!(I+A)_t}D_x^{2t}L_n^{(A,B)}(x+y,z)=\frac{n!}{(I+A)_n}L_n^{(A,A,B)}(x,y,z). \tag{103}$$

$$\sum_{t=0}^{[\frac{n}{2}]}\frac{(xy)^t}{t!(I+A)_t}D_y^{2t}L_n^{(A,B)}(x+y,z)=\frac{n!}{(I+A)_n}L_n^{(A,A,B)}(x,y,z). \tag{104}$$

$$\sum_{t=0}^{[\frac{n}{2}]}\frac{(yz)^t}{t!(I+B)_t}D_y^{2t}L_n^{(A,B)}(x,y+z)=\frac{n!}{(I+B)_n}L_n^{(A,B,B)}(x,y,z). \tag{105}$$

$$\sum_{t=0}^{[\frac{n}{2}]}\frac{(yz)^t}{t!(I+B)_t}D_z^{2t}L_n^{(A,B)}(x,y+z)=\frac{n!}{(I+B)_n}L_n^{(A,B,B)}(x,y,z). \tag{106}$$

$$\sum_{t=0}^{[\frac{n}{2}]}\frac{(yz)^t}{t!(I+B)_t}D_y^{2t}L_n^{(A,B,C)}(x,y+z,w)=\frac{n!}{(I+B)_n}L_n^{(A,B,B,C)}(x,y,z,w). \tag{107}$$

$$\sum_{t=0}^{[\frac{n}{2}]}\frac{(zw)^t}{t!(I+C)_t}D_z^{2t}L_n^{(A,B,C)}(x,y,z+w)=\frac{n!}{(I+C)_n}L_n^{(A,B,C,C)}(x,y,z,w). \tag{108}$$

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References

[1] A. G. Constanttine, Some non-central distribution problem in multivariate analysis, Ann. Math. Statis., 34, 1270-1285, 1963.

[2] A. G. Constanttine, R. J. Muirhead, Partial differential equations for hypergeometric functions of two argument matrix, J. Multivariate Anal., 3, 332-338, 1972.

[3] A. J. Durán, Markov's Theorem for orthogonal matrix polynomials, Can. J. Math., 48, 1180-1195, 1996.

[4] A. J. Durán, P. Lopez-Rodriguez, Orthogonal matrix polynomials: zeroes and Blumenthal's Theorem, J. Apporx. Theory, 84, 96-118, 1996.

[5] A. J. Durán, W. Van Assche, Orthogonal matrix polynomials and higher order recurrence relations, Linear Algebra and its Applications, 219, 261-280, 1996.

[6] A. Sinap, W. Van Assche, Polynomials interpolation and Gaussian quadrature for matrix valued functions, Linear Algebra and its Applications, 207, 71-114, 1994.

[7] A. Sinap, W. Van Assche, Orthogonal matrix polynomials and applications, J. Comp. Appl. Math., 66, 27-52, 1996.

[8] A. Terras, Special functions for the symmetric space of positive matrices, SIAM J. Math. Anal., 16, 620-640, 1985.

[9] A. T. James, Special functions of matrix and single argument in statistics, in Theory and Applications of Special Functions, Ed. R. A. Askey, Academic Press, 497-520, 1975.

[10] E. Defez, L. Jódar, Some application of the Hermite matrix polynomials series expansions, J. Comp. Appl. Math., 99, 105-117, 1998.

[11] E.D. Rainville, Special Functions, Macmillan, New Yark; Reprented by Chelsea Publ.co., Bronx, New Yark, 1997.

[12] H.M. Srivastava, H.L. Manocha, A Treatise on Generating Functions, Chichester, New Yark; Ellis Horwood Limited, John Wiley and Sons, New Yark, 1984.

[13] I. Gohberg, P. Lancaster, L. Rodman, Matrix polynomials, Academic Press, New York, 1982.

[14] K. A. M. Sayyed, M. S. Metwally, R. S. Batahan, Gegenbauer matrix polynomials and second order matrix differential equations, Divulg. Mate., 12, 2, 101-115, 2004.

[15] K. A. M. Sayyed, R. S. Batahan, On generalized Hermite matrix polynomials, accepted in Electronic Journal of Linear Algebra.

[16] L. Jódar, E. Defez, A connection between Laguerre's and Hermite's matrix polynomials, Appl. Math. Lett., 11, 1, 13-17, 1998.

[17] L. Jódar, J. C. Cortés, On the hypergeometric matrix function, J. Comp. Appl. Math., 99, 205-217, 1998.

[18] L. Jódar, J. C. Cortés, Closed form general solution of the hypergeometric matrix differential equation, Mathematical and Computer Modelling. 32, 1017-1028, 2000.

[19] L. Jódar, J. Sastre, The growth of Laguerre matrix polynomials on bounded intervals, Appl. Math. Lett., 13, 21-26, 2000.

[20] L. Jódar, R. Company, Hermite matrix polynomials and second order matrix differential equations, J. Approx. Theory Appl., 12, 2, 20-30, 1996.

[21] L. Jódar, R. Company, E. Navarro, Laguerre matrix polynomials and system of second-order differential equations, Appl. Num. Math., 15, 53-63, 1994.

[22] L. Rodman, Orthogonal matrix polynomials, In Orthogonal Polynomials: Theory and Peactice, P. Nevai, Ed., 294 of NATO ASI Series C. Kluwer, Dordrecht, 345-362, 1990.

[23] M.A. Khan, A.K. Shukla, On Laguerre polynomials of several variables, Bull. Cal. Math. Soc., 89, 155-164, 1997.

[24] M. S. Metwally, M. T. Mohamed, A. Shehata, On Chebyshey matrix polynomials, matrix differential equations and their properties, Afr. Math., 26, 1037-1047, 2015.

[25] R. J. Muirhead, Systems of partial differential equations for hypergeometric functions of matrix argument, Ann. Math. Statist., 41, 991-1001, 1970.

[26] S.F. Ragab, On Laguerre polynomials of two variables $L_n^(\alpha, \beta)(x, y)$, Bull. Cal. Math. Soc., 83, 253-262, 1990.

[27] S. Khan, N.A.M. Hassan, 2variables Laguerre matrix polynomials and Lie-algebraic techniques, J. Phys. A. Math. Theor., 43, 2010.

[28] S.K. Chatterjea, A note on Laguerre polynomials of two variables, Bull. Cal. Math. Soc., 82, 263-266, 1991.

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