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A STUDY ON SEQUENCES IN A RANDOM METRIC SPACE VIA THE CONCEPT OF AN IDEAL

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Abstract. The distance between any two points in a random metric (RM) space is a nonnegative random variable. Such spaces naturally arise in random functional analysis. The aim of this paper is to contribute to the mathematical analysis of RM spaces. In this context, we introduce the concepts of ideal convergent ($\mathcal{I}-$convergent) sequence, $\mathcal{I}-$Cauchy sequence and $\mathcal{I}-$bounded sequence in an RM space endowed with the $(\varepsilon, \lambda)-$topology, and establish some basic facts. $\mathcal{I}-$convergence and related properties of a sequence presenting random deviations in an RM space could provide a general mathematical setting to model the behaviour of the sequence, although it may not be convergent in the ordinary sense. We also consider certain $\mathcal{I}-$convergence properties of a sequence of functions defined on RM spaces.

1. INTRODUCTION

In 1942, Menger [31] introduced the concept of a statistical metric space which is now called a probabilistic metric (PM) space [37], as a generalization of an ordinary metric space. In the theory of PM spaces, a probability distribution function F_{pq} is assigned as a distance between any arbitrary points p and q of a nonempty abstract set S. It is known that many papers have been published in the theory of PM spaces up to now, and it is nearly impossible to list all of them here. However, a clear and detailed history of the subject can be found in the books [5] and [37].

While the theory of PM spaces was developing, a related theory, namely, the theory of random metric (RM) spaces was put forward by Špaček [41, 42] and further studied by many others (see, for instance, [2], [3], [14], [15], [16], [34] and [39]). To obtain an RM space, one should consider the concept of a PM space from the point of view of the standard measure-theoretic model of probability theory, that is, an RM space is mainly based on the theory of random variables. To be more concise, in the setting of RM spaces, a random variable X_{pq} is directly assigned as a distance function to a pair of points $(p, q) \in S \times S$, rather than its probability distribution function F_{pq} as in the theory of PM spaces, where S is a nonempty

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set. On the other hand, we see that, in the modern theory of RM spaces, random variables are replaced with more general measurable functions satisfying certain properties and RM spaces are much more related to random functional analysis (see [15]). For a detailed history and the development of RM theory, we refer to [14], [15], [16], [37] and the references therein.

As for the concept of ideal convergence ($\mathcal{I}-$convergence, briefly), it is a generalization of the well known concept of statistical convergence (see, e.g. [9], [36] and [43] for the notion of statistical convergence; and see also [46] for an earlier concept, namely, the notion of almost convergence). $\mathcal{I}-$convergence is thus a generalization of ordinary convergence, and it is defined via the concept of an ideal $\mathcal{I}$ of subsets of the set $\mathbb{N}$ of all positive integers. It was first introduced for a sequence in a metric space by Kostyrko et al. [27], and since then, it has been investigated by many others (see, e.g. [1], [4], [7], [35] and [40]). Note that, a similar concept, called the filter convergence was introduced in [33] for a sequence of numbers (see also [26]).

The aim of this paper is to introduce the $\mathcal{I}-$convergence and related concepts in an RM space endowed with the $(\varepsilon, \lambda)-$topology, and to obtain basic results. Since the study of convergence of a sequence in any abstract space is essential for a mathematical analysis of the space, the concept of $\mathcal{I}-$convergence in an RM space could provide a more general framework for the study of sequences in RM spaces.

Our paper is organized as follows. In the second section, for an easy reading of the paper, we recall some preliminary notations, definitions and results related to RM spaces and $\mathcal{I}-$convergence. The main results of the paper are presented in the third section.

2. PRELIMINARIES

First, we recall some of the basic concepts related to the theory of RM spaces, and we refer to [15] for more details in the following.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathbb{R}$ be the real number field. Then $L_0(\Omega, \mathbb{R})$ will denote the set of all real valued random variables on Ω, and $L_0^+(\Omega)$ the set of all such random variables ξ on Ω that $\xi(\omega) \geq 0$ almost surely (a.s.), where $\omega \in \Omega$. Now we are ready for

Definition 2.1 [15, 37] An ordered pair $(S, \mathcal{X})$ is called a *random metric space* (briefly, an *RM space*) with base $(\Omega, \mathcal{A}, P)$ if S is a nonempty set and $\mathcal{X}$ is a mapping from $S \times S$ to $L_0^+(\Omega)$ such that, denoting $\mathcal{X}(p, q)$ by X_{pq}, for all p, q, r in S, the following hold:
(RM 1) $X_{pq}(\omega) = 0$ a.s. if $p = q$,
(RM 2) $X_{pq}(\omega) = X_{qp}(\omega)$ a.s.,
(RM 3) $X_{pq}(\omega) = 0$ a.s. implies $p = q$,
(RM 4) $X_{pr}(\omega) \leq X_{pq}(\omega) + X_{qr}(\omega)$ a.s.
Moreover, X_{pq} is said to be the *random distance* between p and q. If the axiom (RM 3) is dropped, then $(S, \mathcal{X})$ is called a *random pseudometric space* (briefly, an *RPM space*).

Remark 2.1 The function $\mathcal{X}$ given in Definition 2.1 is called a *strong stochastic metric* in [2], and the RM space $(S, \mathcal{X})$ is called an $F-$*random metric space* in [15]. However, in this paper, the pair $(S, \mathcal{X})$ will be called a *random metric space* as in [37], $\mathcal{X}$ will be called a *random metric*, and we will not explicitly mention the base space $(\Omega, \mathcal{A}, P)$ unless necessary. Moreover, we would like to point out that; in [14], another definition of an RM space, namely, the concept of an $E-$*random metric space* was introduced (see also [15]). In this space, the distance between any two points is an equivalence class of a nonnegative random variable. In this context, one can have a notion of an $E-$*random normed space*. The advantage of this approach is that an important notion of a *random normed module* can be introduced and its *random conjugate space* can be deeply developed as can be seen by the recent advances in [13], [17], [18], [19], [20], [21], [22] and [23].

Now let us cast a glance at the topological structure of RM spaces. Let $(S, \mathcal{X})$ be an RPM space with base $(\Omega, \mathcal{A}, P)$. For $\varepsilon > 0$ and $0 < \lambda < 1$, let

$$\begin{aligned} U(\varepsilon, \lambda) &= \{(p, q) \in S \times S : P\{\omega \in \Omega : X_{pq}(\omega) < \varepsilon\} > 1 - \lambda\} \\ &= \{(p, q) \in S \times S : P(X_{pq} < \varepsilon) > 1 - \lambda\}. \end{aligned}$$

Then $\{U(\varepsilon, \lambda) : \varepsilon > 0,\ 0 < \lambda < 1\}$ forms a subbase for some pseudometrizable uniformity on S, which determines a pseudometrizable topology, called the $(\varepsilon, \lambda)-$*topology.* Thus, given an RM space $(S, \mathcal{X})$ and a point p in S, we will call the the set

$$\mathcal{N}_p(\varepsilon, \lambda) = \{q \in S : P(X_{pq} < \varepsilon) > 1 - \lambda\}$$

the $(\varepsilon, \lambda)-$*neighborhood* of p.

Throughout the rest of the paper, when we speak about an RM space $(S, \mathcal{X})$, we always assume that S is endowed with the $(\varepsilon, \lambda)-$topology.

Definition 2.2 [2] Let $(S, \mathcal{X})$ be an RM space. Then, $\mathcal{X}$ is *uniformly continuous* means that, given $\varepsilon > 0$ and $0 < \lambda < 1$, there are numbers $\varepsilon' > 0$ and $0 < \lambda' < 1$ such that

$$P(|X_{pq} - X_{p'q'}| < \varepsilon) > 1 - \lambda$$

whenever $P(X_{pp'} < \varepsilon') > 1 - \lambda'$ and $P(X_{qq'} < \varepsilon') > 1 - \lambda'$, where $p, p', q, q' \in S$.

According to [2], it is enough to choose $\varepsilon' = \frac{\varepsilon}{2}$ and $\lambda' = \frac{\lambda}{2}$ in Definition 2.2 to prove the following result which is essential for all our purposes.

Lemma 2.1 [2] If $(S, \mathcal{X})$ is an RM space, then the random metric $\mathcal{X}$ is uniformly continuous.

Note 2.1 Since $\mathcal{X}$ is uniformly continuous, as a particular case of Definition 2.2 and as a consequence of Lemma 2.1 we can say that, given $\varepsilon > 0$ and $0 < \lambda < 1$ we have $P(X_{pr} < \varepsilon) > 1 - \lambda$ whenever $P\left(X_{pq} < \frac{\varepsilon}{2}\right) > 1 - \frac{\lambda}{2}$ and $P\left(X_{qr} < \frac{\varepsilon}{2}\right) > 1 - \frac{\lambda}{2}$ where $p, q, r \in S$.

Finally, we list some of the basic concepts related to the theory of $\mathcal{I}-$convergence in a metric space, and we refer to [27] for more details.

Let Y be a non-empty set and $\mathcal{P}(Y)$ be its power set. Then a family $\mathcal{I} \subset \mathcal{P}(Y)$ is an *ideal* if and only if for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and each $B \subset A$ we have $B \in \mathcal{I}$. A non-empty family $\mathcal{F} \subset \mathcal{P}(Y)$

is a *filter* on Y if and only if $\emptyset \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, and for each $A \in \mathcal{F}$ and each $B \supset A$ we have $B \in \mathcal{F}$. An ideal $\mathcal{I}$ is called *non-trivial* if $\mathcal{I} \neq \emptyset$ and $Y \notin \mathcal{I}$. An $\mathcal{I} \subset \mathcal{P}(Y)$ is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{Y \setminus A : A \in \mathcal{I}\}$ is a filter on Y. A non-trivial ideal $\mathcal{I} \subset \mathcal{P}(Y)$ is called *admissible* if and only if $\mathcal{I} \supset \{\{y\} : y \in Y\}$. For instance, the collection $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$ is an admissible ideal in $\mathcal{P}(\mathbb{N})$.

Another important admissible ideal is obtained as follows: Let $A \subset \mathbb{N}$. Then, the limit given by

$$\lim_{n\to\infty} \frac{1}{n} \, \# \{j \in A : j \leq n\}$$

(whenever exists) is called the *natural density* of A and it is denoted by $\delta(A)$, where $\# \{j \in A : j \leq n\}$ denotes the number of elements of A not exceeding n. Note that for a finite subset A of $\mathbb{N}$, we have $\delta(A) = 0$. Moreover, if $\delta(A)$ exists, then $\delta(A^c) = 1 - \delta(A)$ where $A^c = \mathbb{N} \diagdown A$ (see also [32]). Thus the collection $\{A \subset \mathbb{N} : \delta(A) = 0\}$ is an admissible ideal in $\mathcal{P}(\mathbb{N})$, and we will denote it by $\mathcal{I}_\delta$.

Definition 2.3 [27] Let $\mathcal{I}$ be an ideal in $\mathcal{P}(\mathbb{N})$, (x_n) be a sequence in a metric space (X, ρ) and $x \in X$. If, for each $\varepsilon > 0$ we have

$$\{n \in \mathbb{N} : \rho(x_n, x) \geq \varepsilon\} \in \mathcal{I},$$

then we say that (x_n) is $\mathcal{I}-$*convergent* to x, and we write

$$\mathcal{I} - \lim \ x_n = x.$$

Definition 2.4 [27] Let $\mathcal{I}$ be an ideal in $\mathcal{P}(\mathbb{N})$, (x_n) be a sequence in a metric space (X, ρ) and $x \in X$. Then (x_n) is said to be $\mathcal{I}^*-$*convergent to* x, provided that there exists a set $M \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N} \setminus M \in \mathcal{I}$) such that

$$M = \{m_k : m_1 < m_2 < ... < m_k < ...\}$$

and $x_{m_k} \to x$ as $k \to \infty$. In this case, we write $\mathcal{I}^* - \lim \ x_n = x$.

Lemma 2.2 [27] Let $\mathcal{I}$ be an admissible ideal in $\mathcal{P}(\mathbb{N})$, (x_n) be a sequence in a metric space (X, ρ) and $x \in X$. If (x_n) is $\mathcal{I}^*-$convergent to x, then (x_n) is $\mathcal{I}-$convergent to x.

3. MAIN RESULTS

First of all, following [27], [28] and [40], we will introduce the concept of an $\mathcal{I}-$convergent sequence in an RM space $(S, \mathcal{X})$, and present some main results. Throughout the rest of the paper, $\mathcal{I}$ will denote an admissible ideal in $\mathcal{P}(\mathbb{N})$.

Definition 3.1 Let $(S, \mathcal{X})$ be an RM space. We say that a sequence (p_n) in S is $\mathcal{I}-$*convergent in the* $(\varepsilon, \lambda)-$*topology* (or $\mathcal{I}-$*convergent in the random metric*) to a point p in S (or briefly, (p_n) is $\mathcal{I}-$*convergent*) and we write $p_n \overset{\mathcal{I}-RM}{\longrightarrow} p$, provided that for each $\varepsilon > 0$ and $0 < \lambda < 1$ we have

$$\{n \in \mathbb{N} : p_n \notin \mathcal{N}_p(\varepsilon, \lambda)\} = \{n \in \mathbb{N} : P(X_{p_n p} < \varepsilon) \leq 1 - \lambda\} \in \mathcal{I}.$$

That is, $p_n \overset{\mathcal{I}-RM}{\longrightarrow} p$ iff for each $\varepsilon > 0$ and $0 < \lambda < 1$ we have

$$\{n \in \mathbb{N} : P(X_{p_n p} < \varepsilon) > 1 - \lambda\} \in \mathcal{F}(\mathcal{I}).$$

In this case, we call p the $\mathcal{I}-$*RM limit* of (p_n).

Remark 3.1 (a) In [1], an important concept, called the $\mathcal{I}-$*convergence in measure* was introduced as follows. Given a finite measure space $(\Omega, \mathcal{A}, \mu)$, let us denote by $\mathcal{L}^0$ the space of real valued measurable functions defined almost everywhere on Ω. If (f_n) is a sequence in $\mathcal{L}^0$ and $f \in \mathcal{L}^0$, then we say that (f_n) is $\mathcal{I}-$*convergent in measure* to f provided that

$$\mathcal{I}-\lim\ \mu\left\{\omega \in \Omega : \left|f_n(\omega) - f(\omega)\right| \geq \varepsilon\right\} = 0$$

for every $\varepsilon > 0$. We will denote this situation by $(\mu) - \mathcal{I}-\lim\ f_n = f$. Such a convergence was first considered in [9] as a special case, under the name of *asymptotic statistical convergence.* If, in particular, we choose our measure space as a probability space $(\Omega, \mathcal{A}, P)$, then the $\mathcal{I}-$convergence in measure will be called the $\mathcal{I}-$*convergence in probability.* In this case, we can say that a sequence (f_n) of random variables is $\mathcal{I}-$*convergent in probability* to f provided that

$$\mathcal{I}-\lim\ P\left\{\omega \in \Omega : \left|f_n(\omega) - f(\omega)\right| \geq \varepsilon\right\} = 0$$

for every $\varepsilon > 0$, or equivalently,

$$\mathcal{I}-\lim\ P\left\{\omega \in \Omega : \left|f_n(\omega) - f(\omega)\right| < \varepsilon\right\} = 1$$

for every $\varepsilon > 0$ (see also [12] and [24]). We will denote this situation by

$$(P) - \mathcal{I}-\lim\ f_n = f.$$

(b) As for our context, since the base $(\Omega, \mathcal{A}, P)$ of an RM space $(S, \mathcal{X})$ is a probability space, we can say that, for a sequence (p_n) in $(S, \mathcal{X})$ we have $p_n \overset{\mathcal{I}-RM}{\longrightarrow} p$ iff

$$\mathcal{I}-\lim\ P\left\{\omega \in \Omega : \left|X_{p_np}(\omega)\right| < \varepsilon\right\} = \mathcal{I}-\lim\ P\left(X_{p_np} < \varepsilon\right) = 1$$

for each $\varepsilon > 0$, since $\lambda \in (0, 1)$ can be made arbitrarily small according to Definition 3.1. This means that $p_n \overset{\mathcal{I}-RM}{\longrightarrow} p$ iff the sequence (X_{p_np}) of random variables is $\mathcal{I}-$convergent in probability to the zero random variable Θ defined by $\Theta(\omega) = 0$ a.s., namely,

$$p_n \overset{\mathcal{I}-RM}{\longrightarrow} p \Longleftrightarrow (P) - \mathcal{I}-\lim\ X_{p_np} = \Theta.$$

(c) Finally, we would like to point out that, in [25], as a special case of $\mathcal{I}-$convergence, the statistical convergence in a Šerstnev type of probabilistic normed space endowed with the $(\varepsilon, \lambda)-$topology (see [38] for such spaces) is defined by using probability distribution functions. Also, Şençimen and Pehlivan [44] introduced the strong $\mathcal{I}-$convergence in a general PM space endowed with the strong topology by using probability distribution functions. However, as stated in the introductory part of our work, probabilistic metric/normed spaces are essentially different from RM spaces in their fundamental structures. Therefore, our definitions presented here will be formulated differently from those of [25] and [44], because we will base ourselves on the measure theoretical properties of random variables directly, not the probability distribution functions.

Now we continue our work with certain properties of the $\mathcal{I}-$convergence in an RM space $(S, \mathcal{X})$. First, since the collection $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$ is an admissible ideal in $\mathcal{P}(\mathbb{N})$; we can say that a convergent sequence (p_n) in $(S, \mathcal{X})$ with respect to the $(\varepsilon, \lambda)-$topology is $\mathcal{I}_f-$convergent in the same topology. Now let us consider the following

Example 3.1 Let $S=\mathbb{R}^2$ and given $p=(a_1,a_2)\,,\ q=(b_1,b_2)\in S$, let

$$d(p,q)=\left[(a_1-b_1)^2+(a_2-b_2)^2\right]^{\frac{1}{2}},$$

namely, the Euclidean metric. Moreover, let us given the probability space $(I,\mathcal{A},m)$ where $I=[0,1]\,,\ \mathcal{A}$ is the $\sigma-$algebra of all Lebesgue measurable subsets of I and m is the Lebesgue measure. Now let us define a random variable $\xi:I\longrightarrow\mathbb{R}$ by $\xi(\omega)=\omega$. Now consider the mapping

$$\mathcal{X}:S\times S\longrightarrow L_0^+(I)$$

defined by

$$X_{pq}(\omega)=d(p,q)\ \xi(\omega),$$

where $p,q\in S$. Under these conditions, the pair $(S,\mathcal{X})$ is an RM space with base $(I,\mathcal{A},m)$ (see Chp. 9 of [37], for such a construction of an RM space). Now let (p_n) be a sequence in $(S,\mathcal{X})$ defined by

$$p_n=\begin{cases}(1,1), & \text{if } n \text{ is perfect square}\\ \left(\frac{1}{n},\frac{1}{n}\right), & \text{otherwise}\end{cases},$$

where $n\in\mathbb{N}$, and let $\theta=(0,0)\in S$. If we consider the sequence $(X_{p_n\theta})$ of random variables defined by

$$X_{p_n\theta}(\omega)=d(p_n,\theta)\ \xi(\omega)=\begin{cases}\sqrt{2}\omega, & \text{if } n \text{ is perfect square}\\ \frac{\sqrt{2}\omega}{n}, & \text{otherwise}\end{cases},$$

then we see that

$$P\left(X_{p_n\theta}<\varepsilon\right)=\begin{cases}\frac{\varepsilon}{\sqrt{2}}, & \text{if } 0<\varepsilon<\sqrt{2}\\ 1, & \text{if } \varepsilon\geq\sqrt{2}\end{cases}$$

when n is a perfect square, and

$$P\left(X_{p_n\theta}<\varepsilon\right)=\begin{cases}\frac{n\varepsilon}{\sqrt{2}}, & \text{if } 0<\varepsilon<\frac{\sqrt{2}}{n}\\ 1, & \text{if } \varepsilon\geq\frac{\sqrt{2}}{n}\end{cases}$$

when n is not a perfect square. Note that

$$\lim_{k\to\infty}P\left(X_{p_{n_k}\theta}<\varepsilon\right)=1$$

for each $\varepsilon>0$, where $M=\{n_k:k\in\mathbb{N}\}$ is the set of positive integers which are not perfect square, $\delta(M)=1$ and hence $M\in\mathcal{F}(\mathcal{I}_\delta)$. Thus we have

$$\mathcal{I}_\delta^*-\lim\ P\left(X_{p_n\theta}<\varepsilon\right)=1$$

for each $\varepsilon>0$ by Definition 2.4, which yields

$$\mathcal{I}_\delta-\lim\ P\left(X_{p_n\theta}<\varepsilon\right)=1$$

for each $\varepsilon>0$ by Lemma 2.2. Thus, in view of Remark 3.1-(b), we get

$$(P)-\mathcal{I}_\delta-\lim\ X_{p_n\theta}=\Theta.$$

Hence we have $p_n\overset{\mathcal{I}_\delta-RM}{\longrightarrow}\theta$ (This also means that (p_n) is statistically convergent in the $(\varepsilon,\lambda)-$topology). Note that (p_n) is not convergent in the ordinary sense with respect to the $(\varepsilon,\lambda)-$topology. Moreover, the subsequence $(p_n)_{n\in M^c}$ of the sequence (p_n) is $\mathcal{I}_\delta-$convergent in the $(\varepsilon,\lambda)-$topology to the point $p=(1,1)\in S$.

Thus a subsequence of an $\mathcal{I}-$convergent sequence need not $\mathcal{I}-$converge to the $\mathcal{I}-$RM limit of the sequence in an RM space.

Now we continue with a

Lemma 3.1 Let $(S,\mathcal{X})$ be an RM space with base $(\Omega,\mathcal{A},P)$. Then, given $\varepsilon>0$ and $0<\lambda<1$ we have $P\left(X_{ps}<\varepsilon\right)>1-\lambda$ whenever $P\left(X_{pq}<\frac{\varepsilon}{3}\right)>1-\frac{\lambda}{3}$, $P\left(X_{qr}<\frac{\varepsilon}{3}\right)>1-\frac{\lambda}{3}$and $P\left(X_{rs}<\frac{\varepsilon}{3}\right)>1-\frac{\lambda}{3}$ where $p,q,r,s\in S$.

Proof. Let $\varepsilon>0$ and $0<\lambda<1$. Moreover, let $A=\left\{\omega\in\Omega:X_{pq}(\omega)<\frac{\varepsilon}{3}\right\}$, $B=\left\{\omega\in\Omega:X_{qr}(\omega)<\frac{\varepsilon}{3}\right\}$ and $C=\left\{\omega\in\Omega:X_{pr}(\omega)<\frac{2\varepsilon}{3}\right\}$. Then, by the axiom (RM 4) of Definition 2.1, we have

$$P(C)\geq P(A\cap B)>1-\frac{\lambda}{3}+1-\frac{\lambda}{3}-1=1-\frac{2\lambda}{3}.$$

Now let $D=\{\omega\in\Omega:X_{ps}(\omega)<\varepsilon\}$ and $E=\left\{\omega\in\Omega:X_{rs}(\omega)<\frac{\varepsilon}{3}\right\}$. Similarly, we have

$$P(D)\geq P(C\cap E)>1-\frac{2\lambda}{3}+1-\frac{\lambda}{3}-1=1-\lambda,$$

which completes the proof. ■

Theorem 3.1 Let $(S,\mathcal{X})$ be an RM space. If (p_n) and (q_n) are sequences in S such that $p_n\overset{\mathcal{I}-RM}{\longrightarrow}p$ and $q_n\overset{\mathcal{I}-RM}{\longrightarrow}q$, then we have

$$(P)-\mathcal{I}-\lim\ X_{p_nq_n}=X_{pq}.$$

Proof. First, let us recall that $\mathcal{X}$ is a uniformly continuous mapping by Lemma 2.1. Now let $\varepsilon>0$ and $0<\lambda<1$ be given. Thus we can say that

$$P\left(\left|X_{p_nq_n}-X_{pq}\right|<\varepsilon\right)>1-\lambda$$

whenever

$$P\left(X_{p_np}<\frac{\varepsilon}{2}\right)>1-\frac{\lambda}{2}$$

and

$$P\left(X_{q_nq}<\frac{\varepsilon}{2}\right)>1-\frac{\lambda}{2}.$$

Now let us consider the following sets:

$$\begin{aligned}A_1&=\left\{n\in\mathbb{N}:P\left(\left|X_{p_nq_n}-X_{pq}\right|<\varepsilon\right)\leq1-\lambda\right\},\\A_2&=\left\{n\in\mathbb{N}:P\left(X_{p_np}<\frac{\varepsilon}{2}\right)\leq1-\frac{\lambda}{2}\right\},\\A_3&=\left\{n\in\mathbb{N}:P\left(X_{q_nq}<\frac{\varepsilon}{2}\right)\leq1-\frac{\lambda}{2}\right\}.\end{aligned}$$

Thus we see that $A_1\subset(A_2\cup A_3)$. Moreover, by Definition 3.1, we have $A_2,A_3\in\mathcal{I}$ since $p_n\overset{\mathcal{I}-RM}{\longrightarrow}p$ and $q_n\overset{\mathcal{I}-RM}{\longrightarrow}q$, respectively. Hence we get $A_1\in\mathcal{I}$, and thus $A_1^c\in\mathcal{F}(\mathcal{I})$. This means that

$$\left\{n\in\mathbb{N}:P\left(\left|X_{p_nq_n}-X_{pq}\right|<\varepsilon\right)>1-\lambda\right\}\in\mathcal{F}(\mathcal{I}),$$

and since $\lambda\in(0,1)$ can be made arbitrarily small, we have

$$\mathcal{I}-\lim\ P\left(\left|X_{p_nq_n}-X_{pq}\right|<\varepsilon\right)=1,$$

which completes the proof. ■

At this stage, note that, in [27], the concept of an $\mathcal{I}-$*convergence preserving function* on a metric space was introduced. Thus, by using the language of [27], we can also say that the random metric $\mathcal{X}$ preserves $\mathcal{I}-$convergence, by Theorem 3.1.

In what follows, based on the concept of real statistically Cauchy sequence defined in [10] and the concept of $\mathcal{I}-$Cauchy sequence (a generalization of statistically Cauchy sequence) in a metric space (see [7]), we will introduce the concept of an $\mathcal{I}-$Cauchy sequence in an RM space and prove some basic facts. We would like to point out that, in [29], as a very general case, $\mathcal{I}-$convergence of nets in a topological space was introduced. Also in [6], the concept of a $\mathcal{I}-$Cauchy net in a uniform space was introduced and studied its basic properties. Since $(S, \mathcal{X})$ is a uniform space, our results which will be given by Theorems 3.2 - 3.3 and Corollary 3.1 may be considered as special cases of the ones given in [6]. However, to make our paper self-contained and to see in detail what is happening in the measure-theoretical setting of random variables, we will prove these results by using the tools of RM spaces.

Definition 3.2 Let $(S, \mathcal{X})$ be an RM space. We say that a sequence (p_n) in S is $\mathcal{I}-$*Cauchy* provided that, for every $\varepsilon > 0$ and $0 < \lambda < 1$, there exists a number $N = N(\varepsilon, \lambda) \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : P\left(X_{p_n p_N} < \varepsilon\right) \leq 1 - \lambda\} \in \mathcal{I}.$$

Theorem 3.2 In an RM space $(S, \mathcal{X})$, if a sequence is $\mathcal{I}-$convergent, then it is also $\mathcal{I}-$Cauchy.

Proof. Let (p_n) be a sequence in S such that $p_n \overset{\mathcal{I}-RM}{\longrightarrow} p$. Moreover, let any $\varepsilon > 0$ and $0 < \lambda < 1$ be given. Since $p_n \overset{\mathcal{I}-RM}{\longrightarrow} p$, we have

$$C\left(\varepsilon, \lambda\right) = \left\{n \in \mathbb{N} : P\left(X_{p_n p} < \frac{\varepsilon}{2}\right) > 1 - \frac{\lambda}{2}\right\} \in \mathcal{F}(\mathcal{I}).$$

Now choose an $N \in C\left(\varepsilon, \lambda\right)$. Thus we have $P\left(X_{p_N p} < \frac{\varepsilon}{2}\right) > 1 - \frac{\lambda}{2}$. Then, by Lemma 2.1, we have $P\left(X_{p_n p_N} < \varepsilon\right) > 1 - \lambda$ whenever $n \in C\left(\varepsilon, \lambda\right)$. Namely,

$$\{n \in \mathbb{N} : P\left(X_{p_n p_N} < \varepsilon\right) > 1 - \lambda\} \in \mathcal{F}(\mathcal{I}).$$

This shows that (p_n) is $\mathcal{I}-$Cauchy. ■

Theorem 3.3 Let (p_n) be a sequence in an RM space $(S, \mathcal{X})$. If (p_n) is $\mathcal{I}-$Cauchy, then for every $\varepsilon > 0$ and $0 < \lambda < 1$, there exists a set $A = A(\varepsilon, \lambda) \in \mathcal{I}$ such that $P\left(X_{p_m p_n} < \varepsilon\right) > 1 - \lambda$ for any $m, n \notin A$.

Proof. Let (p_n) be an $\mathcal{I}-$Cauchy sequence in $(S, \mathcal{X})$. Moreover, let any $\varepsilon > 0$ and $0 < \lambda < 1$ be given. Then there exists an $N = N(\varepsilon, \lambda) \in \mathbb{N}$ such that

$$\left\{n \in \mathbb{N} : P\left(X_{p_n p_N} < \frac{\varepsilon}{2}\right) \leq 1 - \frac{\lambda}{2}\right\} \in \mathcal{I}.$$

Now, let us write

$$A = A(\varepsilon, \lambda) = \left\{n \in \mathbb{N} : P\left(X_{p_n p_N} < \frac{\varepsilon}{2}\right) \leq 1 - \frac{\lambda}{2}\right\}.$$

Thus we have $P\left(X_{p_m p_N} < \frac{\varepsilon}{2}\right) > 1 - \frac{\lambda}{2}$ and $P\left(X_{p_n p_N} < \frac{\varepsilon}{2}\right) > 1 - \frac{\lambda}{2}$ for any $m, n \notin A$. Now Lemma 2.1 yields that

$$P\left(X_{p_m p_n} < \varepsilon\right) > 1 - \lambda$$

for any $m, n \notin A$. ■

Corollary 3.1 If (p_n) is an $\mathcal{I}-$Cauchy sequence in an RM space $(S, \mathcal{X})$, then for every $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a set $B = B(\varepsilon, \lambda) \in \mathcal{F}(\mathcal{I})$ such that $P\left(X_{p_m p_n} < \varepsilon\right) > 1 - \lambda$ for any $m, n \in B$.

Theorem 3.4 Let $(S, \mathcal{X})$ be an RM space. If (p_n) and (q_n) are $\mathcal{I}-$Cauchy sequences in S, then, given any $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a set $D = D(\varepsilon, \lambda) \in \mathcal{F}(\mathcal{I})$ such that

$$P\left(\left|X_{p_m q_m} - X_{p_n q_n}\right| < \varepsilon\right) > 1 - \lambda$$

whenever $m, n \in D$.

Proof. Let (p_n) and (q_n) be $\mathcal{I}-$Cauchy sequences in S. Moreover, let any $\varepsilon > 0$ and $0 < \lambda < 1$ be given. Then, by Corollary 3.1, there exist

$$B = B(\varepsilon, \lambda),\ C = C(\varepsilon, \lambda) \subset \mathbb{N} \text{ with } B, C \in \mathcal{F}(\mathcal{I})$$

such that

$$P\left(X_{p_i p_j} < \frac{\varepsilon}{2}\right) > 1 - \frac{\lambda}{2}$$

holds for any $i, j \in B$, and

$$P\left(X_{q_k q_l} < \frac{\varepsilon}{2}\right) > 1 - \frac{\lambda}{2}$$

holds for any $k, l \in C$. Now consider the set $B \cap C = D \in \mathcal{F}(\mathcal{I})$. Thus we can say that for every $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a set $D = D(\varepsilon, \lambda) \in \mathcal{F}(\mathcal{I})$ such that $P\left(X_{p_m p_n} < \frac{\varepsilon}{2}\right) > 1 - \frac{\lambda}{2}$ and $P\left(X_{q_m q_n} < \frac{\varepsilon}{2}\right) > 1 - \frac{\lambda}{2}$ for any $m, n \in D$. Hence we have

$$P\left(\left|X_{p_m q_m} - X_{p_n q_n}\right| < \varepsilon\right) > 1 - \lambda$$

for any $m, n \in D$ by Lemma 2.1, which completes the proof. ■

Now we will mention the concept of $\mathcal{I}-$boundedness in an RM space. In [11] and [45], the concept of a real statistically bounded sequence was introduced. Similarly, in [35], the concept of a real $\mathcal{I}-$bounded sequence was given. Now an analogous concept in an RM space will be introduced by the following

Definition 3.3 Let $(S, \mathcal{X})$ be an RM space. We say that a sequence (p_n) in S is $\mathcal{I}-$*bounded* provided that, there exist a $p_0 \in S$, an $\varepsilon_0 > 0$ and a $\lambda_0 \in (0, 1)$ such that

$$\{n \in \mathbb{N} : P\left(X_{p_n p_0} < \varepsilon_0\right) \leq 1 - \lambda_0\} \in \mathcal{I},$$

namely,

$$\{n \in \mathbb{N} : P\left(X_{p_n p_0} < \varepsilon_0\right) > 1 - \lambda_0\} \in \mathcal{F}(\mathcal{I}).$$

Note that there are different types of bounded sets in the theory of PM spaces (see [37]) and RM spaces (see [14]). However, an $\mathcal{I}-$bounded sequence introduced above is based on the concept of boundedness of an arbitrary set in a uniform space (see, e.g. [30]).

Now the following result is immediate from the definitions presented in this section.

Theorem 3.5 Every $\mathcal{I}-$Cauchy sequence (and hence every $\mathcal{I}-$convergent sequence) in an RM space $(S,\mathcal{X})$ is $\mathcal{I}-$bounded.

Finally, we would like to mention some properties of a sequence of functions defined between RM spaces. In what follows, we have inspired from [1] and [8]. In [1], certain types of $\mathcal{I}-$convergent sequences of metric space-valued functions are considered. In [8], $\mu-$statistically convergent sequences of real functions are investigated. We generalize their results to the setting of RM spaces. First, we begin with a

Definition 3.4 Let $(S,\mathcal{X})$ and $\left(\tilde{S},\tilde{\mathcal{X}}\right)$ be RM spaces and (f_n) be a sequence of functions each defined from S into $\tilde{S}$. We say that (f_n) is $\mathcal{I}-$*pointwise convergent* to a function $f : S \longrightarrow \tilde{S}$ provided that; for each $p \in S$, $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a set $A \in \mathcal{I}$ such that $f_n(p) \in \mathcal{N}_{f(p)}(\varepsilon,\lambda)$, that is,

$$P\left(\tilde{X}_{f_n(p)f(p)} < \varepsilon\right) > 1-\lambda$$

for all $n \notin A$. We say that (f_n) is $\mathcal{I}-$*uniformly convergent* to f provided that; for each $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a set $A \in \mathcal{I}$ such that for all $n \notin A$ and $p \in S$ we have $f_n(p) \in \mathcal{N}_{f(p)}(\varepsilon,\lambda)$, that is,

$$P\left(\tilde{X}_{f_n(p)f(p)} < \varepsilon\right) > 1-\lambda.$$

Theorem 3.6 Let $f_n : S \longrightarrow \tilde{S}$ and each f_n be continuous. If (f_n) is $\mathcal{I}-$uniformly convergent to a function $f : S \longrightarrow \tilde{S}$, then f is continuous on S.

Proof. By hypothesis, for each $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a set $A = A(\varepsilon,\lambda) \in \mathcal{I}$ such that $P\left(\tilde{X}_{f_n(p)f(p)} < \frac{\varepsilon}{3}\right) > 1-\frac{\lambda}{3}$ for all $p \in S$ and $n \in A^c$. Now pick an $N \in A^c$. Thus we have

$$P\left(\tilde{X}_{f_N(p)f(p)} < \frac{\varepsilon}{3}\right) > 1-\frac{\lambda}{3} \tag{3.1}$$

for all $p \in S$. Now let $p_0 \in S$. Hence we can write

$$P\left(\tilde{X}_{f_N(p_0)f(p_0)} < \frac{\varepsilon}{3}\right) > 1-\frac{\lambda}{3}. \tag{3.2}$$

Since f_N is continuous at p_0, there exists a $\gamma > 0$ and an $h \in (0,1)$ such that $P\left(X_{p_0p} < \gamma\right) > 1-h$ implies

$$P\left(\tilde{X}_{f_N(p)f_N(p_0)} < \frac{\varepsilon}{3}\right) > 1-\frac{\lambda}{3}. \tag{3.3}$$

Now whenever $p \in S$ with $P\left(X_{p_0p} < \gamma\right) > 1-h$, if we combine (3.1)-(3.3), we get

$$P\left(\tilde{X}_{f(p)f(p_0)} < \varepsilon\right) > 1-\lambda,$$

by Lemma 3.1. Since p_0 is arbitrary, f is continuous on S. ■

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