Electronic Journal of Mathematical Analysis and Applications
Vol. 6(1) Jan. 2018, pp. 253-265.
ISSN: 2090-729X(online)
http://fcag-egypt.com/Journals/EJMAA/

ON THE EXPONENTIAL STUDY OF SOLUTIONS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH TIME LAG

CEMIL TUNC AND OSMAN TUNC

Abstract. We are concerned with three nonlinear Volterra integro-differential equations (NVIDEs) with constant time lag. The variational of parameters inequalities, that is, the boundedness of the solutions, to that (NVIDEs) is investigated by the Lyapunov functionals (LFs). The results obtained here improve and complement a sample of works found in the literature. In fact, the novelty and originality of this article are that it improves and extends earlier results from very simple cases without time lag to the more general and non-linear cases with time lag.

1. Introduction

Mathematical models by functional differential equations (FDEs), (VIDEs), Volterra integral equations (VIEs), integral equations (IEs) and integro-differential equations (IDEs) have attracted the attention of scientists due to their useful applications to day life problems in various scientific fields like sciences, engineering and many other areas (see Burton [6], Burton and Mahfoud [10], Corduneanu [13], Gripenberg et al. [26], Lakshmikantham and Rama Mohan Rao [38], Peschel and Mende [48], Staffans [59], Wazwaz [76]). Therefore, due to this reality, qualitative properties of solutions of various models of the mentioned equations have been widel investigated by different authors (Adivar and Raffoul [1], Becker ([2], [3], [4]), Burton ([5], [7]), Burton and Haddock [8], Burton and Mahfoud ([9], [11]), Chang and Wang [12], Diamandescu[14], Dung ([15], [16]), Eloe et al. [17], Engler [18], Funakubo et al. [19], Furumochi and Matsuoka [20], Grace and Akin [21], Graef and Tunç [22], Graef et al. [23], Grimmer and Seifert [24], Grimmer and Zeman [25], Grossman and Miller [27], Hara et al. ([28],[29],[30]), Hino and Murakami [31], Islam [32], Islam and Al-Eid [33], Islam and Raffoul ([34], [35]), Jin and Luo [36], Lakshmikantham and Rama Mohan Rao [37], Mahfoud ([39],[40],[41]), Mesmouli et al. [42], Martinez [43], Miller [44], Murakami [45], Ngoc [46], Napoles Valdes [47], Raffoul ([49],[50], [51],[52]), Raffoul and Rai [53], Raffoul and Ren [54], Raffoul and Sanbo

2010 *Mathematics Subject Classification.* 34D05, 34K20, 45J05.
Key words and phrases. Variational of parameters inequality, (NVIDE), first order, (LF)..
Submitted March 19, 2016.

[55], Raffoul and nal [56], Rama Mohana Rao and Raghavendra [57], Rama Mohana Rao and Srinivas [59], Talpalaru [60], Tunç ([61], [62], [63], [64], [65]), Tunç and Ayhan [66], Tunç and Mohammed ([67], [68]), Vanualailai [69],Vanualailai and Nakagiri ([70], [71]), Wang ([72], [73], [74]), Wang et al. [75], Zhang [77], Da Zhang [78]). As distinguished line from these facts, the following article is notable: In 2009, Raffoul [51] considered the (FDE)

$$\frac{dx}{dt}\Phi(t,x(s);0\leq s\leq t):=\Phi(t,x(.)), \tag{1}$$

in which x is an n-dimensional vector, Φ is given a continuous function in t and $x(.)$ such that $\Phi(t,0)=0$.

A stereotype of (FDE) (1) is the (VIDE) given by

$$\frac{dx}{dt}=h(x)+\int_0^t B(t,s)f(x(s))ds+g(t).$$

Let $t_0\geq 0$. Then, for each continuous function $\phi:[0,t_0]\to\Re^n$, at least, there exists a continuous function $x(t)=x(t,t_0,\phi)$ on $[t_0,I]$, which is a solution of (FDE) (1) for $0\leq t_0\leq t\leq I$ so that $x(t,t_0,\phi)=\phi$ for $0\leq t\leq t_0$ (see Raffoul [51]). In [51], Raffoul investigated sufficient conditions to guarantee that all solutions of (FDE) (1) satisfy specific variational of parameters inequalities by means of (LFs), and the author gave examples for illustrations. Indeed, Raffoul [51] considered the below (VIDEs) to show applicability of the obtained results:

$$\frac{dx}{dt}=\sigma(t)x+\int_0^t B(t,s)x(s)ds+g(t), \tag{2}$$

$$\frac{dx}{dt}=\sigma(t)x+\int_0^t B(t,s)x^{\frac{2}{3}}(s)ds+g(t) \tag{3}$$

and

$$\frac{dx}{dt}=\sigma(t)x^3+\int_0^t B(t,s)x^{\frac{1}{3}}(s)ds+g(t). \tag{4}$$

Motivated by the results of Raffoul [51], which are related to (VIDEs) (2)-(4), in this paper, we consider the following (NVIDEs) with constant time lag:

$$\frac{dx}{dt}=-f(t,x)+\int_{t-\tau}^t H(t,s)K(x(s))ds+F(t), \tag{5}$$

$$\frac{dx}{dt}=-p(t,x)+\int_{t-\tau}^t H(t,s)q^{\frac{2}{3}}(x(s))ds+F(t) \tag{6}$$

and

$$\frac{dx}{dt}=-r^3(t,x)+\int_{t-\tau}^t H(t,s)h^{\frac{1}{3}}(x(s))ds+F(t), \tag{7}$$

respectively, where $x(t)=\phi(t)$, $0\leq t\leq t_0$, and $\phi(t)$ is a given bounded continuous initial function, $t-\tau\geq 0$, $\tau\in\Re$, $\tau>0$, fixed constant time lag, $x\in\Re$, $F:\Re^+\to\Re$, $\Re^+=[0,\infty)$, and $f,p,r:\Re^+\times\Re\to\Re$ are continuous functions with $f(t,0)=p(t,0)=r(t,0)=0$, $K,q,h:\Re\to\Re$ are continuous functions with $K(0)=q(0)=h(0)=0$, and $H(t,s)$ is a continuous function for $0\leq s\leq t<\infty$. It is also assumed that the derivatives $\frac{\partial}{\partial x}f(t,x)=f_x(t,x)$, $\frac{\partial}{\partial x}p(t,x)=p_x(t,x)$ and $\frac{\partial}{\partial x}r(t,x)=r_x(t,x)$ exist and are continuous for all (t,x).

Let us define

$$f_1(t,x)=\begin{cases}\frac{f(t,x)}{x}, & x\neq 0\\ f_x(t,0), & x=0\end{cases}$$

$$p_1(t,x)=\begin{cases}\frac{p(t,x)}{x}, & x\neq 0\\ p_x(t,0), & x=0\end{cases}$$

$$r_1(t,x)=\begin{cases}\frac{r(t,x)}{x}, & x\neq 0\\ r_x(t,0), & x=0.\end{cases}$$

Hence, we can write from (NVIDEs) (5)-(7) that

$$x'(t)=f_1(t,x)x+\int_{t-\tau}^{t}H(t,s)K(x(s))ds+F(t), \tag{8}$$

$$x'(t)=-p_1(t,x)x+\int_{t-\tau}^{t}H(t,s)q^{\frac{2}{3}}(x(s))ds+F(t) \tag{9}$$

and

$$x'(t)=-r_1^3(t,x)x^3+\int_{t-\tau}^{t}H(t,s)h^{\frac{1}{3}}(x(s))ds+F(t), \tag{10}$$

respectively.

It is notable form the given information that Raffoul [51] considered one linear and two non-linear (VIDEs) without time lag. However, we pay attention to certain three new (NVIDEs) with constant time lag. In fact, when we choose $f(t,x)=-\sigma(t)x, K(x)=x, F(t)=g(t)$; $p(t,x)=-\sigma(t)x$, $q^{\frac{2}{3}}(x)=x^{\frac{2}{3}}$, $F(t)=g(t)$, and $r(t,x)=-\sigma^{\frac{1}{3}}(t)x$, $h^{\frac{1}{3}}(x)=x^{\frac{1}{3}}$, $F(t)=g(t)$, respectively, and put zero "0" instead of the term $t-\tau$, then (NVIDEs) (5)-(7) reduce to (VIDE) (2)-(4) investigated by Raffoul [51]. That is, (NVIDEs) (5)-(7) include and extend (VIDEs) (2)-(4) studied by Raffoul [51].

The purpose of this paper is to establish new sufficient conditions to guarantee some new variational of parameters inequalities so that the solutions (NVIDEs) (5)-(7) are bounded. By this way, we would like to do a contribution to the subject and the related literature.

Let $\varphi:[0,t_0]\to\Re^n$ be a continuous function and define $|\varphi|=\sup\{\|\varphi(s)\|:0\leq s\leq t_0\}$.

Definition ([51]). The solutions of (FDE) (1) are said bounded if any solution $x(t,t_0,\varphi)$ of equation (1) fulfills

$$\|x(t,t_0,\eta)\|\leq C(\|\eta\|,t_0), t\geq t_0, \tag{11}$$

where $C:\Re^+\times\Re^+\to\Re^+,\Re^+=[0,\infty)$, C is here two-parameters a positive function, which depends on φ and t_0, and φ is a given bounded and continuous initial function.

If the C in inequality (11) is independent of t_0 , then it is said that the solutions of (FDE) (1) are uniformly bounded.

Throughout this paper, the below basic theorem is needed for our results, and when we need x represents $x(t)$.

Theorem A ([51]). Assume that $D\subset\Re^n$ contains the origin and there exists a continuously differentiable (LF) $W:[0,\infty)\times D\to[0,\infty)$ such that the below assumptions hold for all $(t,x)\in[0,\infty)\times D$:

- **(A1)** $W_1(\|x\|)\leq W(t,x)\leq W_2(\|x\|)+\int_0^t\phi_1(t,s)W_3(|x(s)|)ds$,
- **(A2)** $\dot{W}(t,x(.))\leq-\alpha_1(t)W_4(\|x\|)-\alpha_2(t)\int_0^t\phi_2(t,s)W_5(|x(s)|)ds+F(t)$

for positive and continuous functions $\alpha_1(t)$, $\alpha_2(t)$ and $F(t)$, where $\phi_1(t,s) \geq 0$ and $\phi_2(t,s) \geq 0$ are scalar valued and continuous functions for $0 \leq s \leq t < \infty$. If the condition

(A3) $W_2(\|x\|) - W_4(\|x\|) + \int_0^t [\phi_1(t,s)W_3|x(s)| - \phi_2(t,s)W_5|x(s)|]ds \leq \gamma$

holds for some constant $\gamma \geq 0$, then all solutions of (FDE) (1) that start in D satisfy the variational of parameters inequality

$$\|x\| \leq W_1^{-1}\{W(t_0,\|\phi\|)\exp(-\int_{t_0}^t \alpha(s)ds) + \int_{t_0}^t [(\gamma\alpha(u)+F(u))\exp(-\int_u^t \alpha(s)ds)]du\}$$

for all $t \geq t_0$, where $\alpha(t) = \max\{\alpha_1(t), \alpha_2(t)\}$.

Proof. See Raffoul [51].

2. Boundedness of solutions

We introduce some assumptions for (VIDE) (5).

A. Assumptions

Let the below assumptions be true.

(C1) $f(t,0) = H(0) = 0, f_1(t,x) \geq f_0(t) \ for \ t \in \Re^+, x \in \Re,$

(C2) $\omega(t) = 2f_0(t) - \int_{t-\tau}^t K_0|H(t,s)|ds - \sigma\int_{t-\tau}^\infty K_0|H(u+\tau,t)|du - 1 > 0,$

in which the function f_0 is positive, bounded and continuous,

(C3) $(\sigma-1)|H(t,s)| \geq \sigma\int_{t-\tau}^\infty |H(u,s)|du, \int_{t-\tau}^t |H(t,s)|ds < \infty$ and $\int_{t-\tau}^\infty |H(u+\tau,t)|du < \infty$

with $0 \leq s \leq t \leq u < \infty$.

The first outcome of the current work is given below.

Theorem 1. If assumptions (C1)-(C3) hold, then all solutions $x(t)$ of (NVIDE) (5) start in D satisfy the variational of parameters inequality

$$|x| \leq \{\Omega_1(t_0,\|\phi\|)\exp(-\int_{t_0}^t \omega(s)ds) + \int_{t_0}^t [(\gamma\alpha(u)+G(u))\exp(-\int_u^t \omega(s)ds)]du\} \tag{12}$$

with

$$\omega(t) = \max\{2f_0(t) - \int_{t-\tau}^t K_0|H(t,s)|ds - \sigma\int_{t-\tau}^\infty K_0|H(u+\tau,t)|du - 1, 1\}$$

$$\gamma = 0, G(t) = F^2(t), \ and$$

$$\Omega_1(t_0,\phi) = \phi^2 + \sigma\int_0^{t_0}\int_{t_0-\tau}^\infty K_0|H(u+\tau,s)|du\phi^2(s)ds.$$

Proof. Define a (LF) $\Omega_1 = \Omega_1(t,x(t))$ by

$$\Omega_1 = x^2 + \sigma\int_0^t\int_{t-\tau}^\infty K_0|H(u+\tau,s)|dux^2(s)ds, \tag{13}$$

where $\sigma, K_0 \in \Re$, $K_0 > 0$, $\sigma > 0$, we fix the constant σ in the proof later.

It can easily be seen from (13) that Ω_1 is positive definite, since

$$\Omega_1(t,0) = 0 \ and \ \Omega_1(t,x) \geq x^2.$$

By a clear calculation from (13) and (NVIDEs) (5) and (8), we obtain

$$\begin{aligned}\Omega_1' &= 2xx' + \sigma x^2 \int_{t-\tau}^{\infty} K_0|H(u+\tau,t)|du - \sigma \int_0^t K_0|H(t,s)|x^2(s)ds \\ &= -2f_1(t,x)x^2 + 2x\int_{t-\tau}^{t} H(t,s)K(x(s))ds + 2xF(t) \\ &\quad + \sigma x^2 \int_{t-\tau}^{\infty} K_0|H(u+\tau,t)|du - \sigma \int_{t-\tau}^{\infty} K_0|H(t,s)|x^2(s)ds.\end{aligned}$$

By an elementary inequality together with the assumption $f_1(t,x) \geq f_0(t)$, it can be enable that

$$\begin{aligned}\Omega_1' &\leq -2f_0(t)x^2 + 2|x|\int_{t-\tau}^{t} K_0|H(t,s)||x(s)|ds \\ &+ \sigma x^2 \int_{t-\tau}^{\infty} K_0|H(u+\tau,t)|du - \sigma \int_{t-\tau}^{\infty} K_0|H(t,s)|x^2(s)ds \\ &+ 2|x||F(t)| \\ &\leq -[2f_0(t) - \int_{t-\tau}^{t} K_0|H(t,s)|ds - \sigma \int_{t-\tau}^{\infty} K_0|H(u+\tau,t)|du - 1]x^2 \\ &+ (1-\sigma)K_0\int_{t-\tau}^{t} H(t,s)x^2(s)ds + F^2(t).\end{aligned}$$

Let $G(t) = F^2(t)$ and $\sigma > 1$. Then

$$\Omega_1' \leq -\omega(t)x^2 - (\sigma-1)K_0\int_{t-\tau}^{t} H(t,s)x^2(s)ds + G(t)$$

by assumption $(C2)$

Let us take that

$$\begin{aligned}&\alpha_1(t) = \omega(t), \alpha_2(t) = 1, \\ &W_1(.) = W_2(.) = W_4(.) = x^2(t), \\ &W_3(.) = W_5(.) = x^2(s), \\ &\phi_1(t,s) = \sigma\int_{t-\tau}^{\infty} |H(u+\tau,s)|du\end{aligned}$$

and

$$\phi_2(t,s) = (\sigma-1)K_0H(t,s).$$

Then we can conclude that all assumptions (A1)-(A3) of Theorem A hold. Hence, one can show that all solutions $x(t)$ of (VIDE) (5) satisfy relation (12).

B.Assumpitons

Let α_1 be a positive constant such that the following assumptions hold.

(D1) $p(t,0) = q(0) = 0$, $p_1(t,x) \geq \alpha(t), t \in \Re^+, x \in \Re, |q(x)| \leq \alpha_1^{\frac{1}{2}}|x|$, $0 < \alpha_1 < 1$,

(D2) $2\alpha(t) - \int_{t-\tau}^{t} |H(t,s)|ds - \int_{t-\tau}^{\infty} |H(u+t,s)|du - 1 > 0$,

in which the function α is positive, bounded and continuous,

(D3) $\int_{t-\tau}^{t} |H(t,s)|ds < \infty, \int_{t-\tau}^{\infty} |H(u+\tau,t)|du < \infty$ and $\frac{1}{3}|H(t,s)|ds \geq \int_{t-\tau}^{\infty} |H(u+\tau,s)|du$

with $0 \le s \le t \le u < \infty$.

The second outcome of the current work is given below.

Theorem 2. If assumptions (D1)-(D3) hold, then all solutions of (NVIDE) (6) start in D satisfy the variational of parameters inequality

$$|x| \le \{\Omega_2(t_0, \|\phi\|)\exp(-\int_{t_0}^{t}\beta(s)ds) + \int_{t_0}^{t}[(\gamma\alpha(u)+G(u))\exp(-\int_{u}^{t}\omega(s)ds)]du\}$$

with

$$\beta(t) = \max\{2\alpha_1(t) - \int_{t-\tau}^{t} K_0|H(t,s)|ds - \lambda\int_{t-\tau}^{\infty} K_0|H(u+\tau,t)|du - 1, 1\}$$

$$\gamma = 0, G(t) = F^2(t), \ and$$

$$\Omega_2(t_0,\phi) = \phi^2 + \lambda\int_{0}^{t_0}\int_{t_0-\tau}^{\infty} K_0|H(u+\tau,s)|du\phi^2(s)ds.$$

Proof. We describe a (LF) $\Omega_2 = \Omega_2(t, x(t))$ by

$$\Omega_2 = x^2 + \lambda\int_{0}^{t}\int_{t-\tau}^{\infty}|H(u+\tau,s)|dux^2(s)ds, \tag{14}$$

where $\lambda \in \Re$,$\lambda > 0$, we fix the constant in the proof.

It is clear from (14) that Ω_2 is positive definite, since

$$\Omega_2(t,0) = 0 \ and \ \Omega_2(t,x) \ge x^2.$$

An easy computation from (14) and (NVIDEs) (6) and (9) shows that

$$\begin{aligned}\Omega_2' &= 2xx' + \lambda x^2\int_{t-\tau}^{\infty}|H(u+\tau,t)|du - \lambda\int_{0}^{t}|H(t,s)|x^2(s)ds \\ &= -2p_1(t,x)x^2 + 2x\int_{t-\tau}^{t}H(t,s)q^{\frac{2}{3}}(x(s))ds + 2xF(t) \\ &\quad + \lambda x^2\int_{t-\tau}^{\infty}|H(u+\tau,t)|du - \lambda\int_{t-\tau}^{\infty}|H(t,s)|x^2(s)ds.\end{aligned}$$

An elementary inequality together with the assumption $p_1(t,x) \ge \alpha(t)$ make enable that

$$\begin{aligned}\Omega_2' \le &-2\alpha(t)x^2 + x^2\int_{t-\tau}^{t}|H(t,s)|x^2(s)ds + \int_{t-\tau}^{t}|H(t,s)|q^{\frac{4}{3}}(x(s))ds \\ &+ \lambda x^2\int_{t-\tau}^{\infty}|H(u+\tau,t)|du - \lambda\int_{t-\tau}^{\infty}|H(t,s)|x^2(s)ds \\ &+ x^2 + F^2(t).\end{aligned}$$

Consider the term

$$\int_{t-\tau}^{t}|H(t,s)|q^{\frac{4}{3}}(x(s))ds, \tag{15}$$

which is involved in (15).

Let $a = \frac{3}{2}$ and $b = 3$. By using the Young's inequality,

$$mn \le a^{-1}m^a + b^{-1}n^b, a^{-1} + b^{-1} = 1,$$

and assumption (D1), we get

$$\begin{aligned}\int_{t-\tau}^{t}|H(t,s)|q^{\frac{4}{3}}(x(s))ds &= \int_{t-\tau}^{t}|H(t,s)|^{\frac{1}{3}}|H(t,s)|^{\frac{2}{3}}q^{\frac{4}{3}}(x(s))ds \\ &\le \frac{1}{3}\int_{t-\tau}^{t}|H(t,s)|ds + \frac{2}{3}\int_{t-\tau}^{t}|H(t,s)|q^2(x(s))ds \\ &\le \frac{1}{3}\int_{t-\tau}^{t}|H(t,s)|ds + \frac{2}{3}\alpha_1\int_{t-\tau}^{t}|H(t,s)|x^2(s)ds.\end{aligned}$$

On gathering the former inequality into (15), we have

$$\begin{aligned}\Omega_2' \le & -[2\alpha(t) - \int_{t-\tau}^{t}|H(t,s)|ds - \lambda\int_{t_0-\tau}^{\infty}|H(u+t,s)|du]x^2 \\ & + \frac{1}{3}\int_{t-\tau}^{t}|H(t,s)|ds + \frac{2}{3}\alpha_1\int_{t-\tau}^{t}|H(t,s)|x^2(s)ds \\ & - \lambda\int_{t-\tau}^{\infty}|H(t,s)|x^2(s)ds + x^2 + F^2(t).\end{aligned}$$

Let $\lambda = 1$, $G(t) = F^2(t)$ and $L = \frac{1}{3}\int_{t-\tau}^{t}|H(t,s)|ds$. Hence

$$\begin{aligned}\Omega_2' \le & -[2\alpha(t) - \int_{t-\tau}^{t}|H(t,s)|ds - \int_{t_0-\tau}^{\infty}|H(u+t,s)|du - 1]x^2 \\ & + (-1 + \frac{2}{3}\alpha_1)\int_{t-\tau}^{t}|H(t,s)|x^2(s)ds + G(t).\end{aligned}$$

The former inequality together with (D2) yields that

$$\begin{aligned}\Omega_2' \le & -[2\alpha(t) - \int_{t-\tau}^{t}|H(t,s)|ds - \int_{t-\tau}^{\infty}|H(u+t,s)|du - 1]x^2 \\ & + (-1 + \frac{2}{3}\alpha_1)\int_{t-\tau}^{t}|H(t,s)|x^2(s)ds + G(t) + L.\end{aligned}$$

Let

$$\beta(t) = \max\{2\alpha(t) - \int_{t-\tau}^{t}|H(t,s)|ds - \int_{t-\tau}^{\infty}|H(u+t,s)|du - 1, 1\}.$$

Then

$$\Omega_2' \le -\beta(t)x^2 - (1 - \frac{2}{3}\alpha_1)\int_{t-\tau}^{t}|H(t,s)|x^2(s)ds + G(t).$$

When we choose

$$\begin{aligned}&\alpha_1(t) = \beta(t), \alpha_2(t) = 1 - \frac{2}{3}\alpha_1, \\ &W_1(.) = W_2(.) = W_4(.) = x^2(t), \\ &W_3(.) = W_5(.) = x^2(s), \\ &\phi_1(t,s) = \lambda\int_{t-\tau}^{\infty}|H(u+\tau,s)|du\end{aligned}$$

and

$$\phi_2(t,s) = H(t,s),$$

then we can conclude that all assumptions (A1)-(A3) of Theorem A hold. Hence, one can easily conclude that all solutions $x(t)$ of (NVIDE) (6) satisfy the desired result.

We now state some assumptions on the functions appearing in (NVIDE) (7).

C.Assumptions

We accept the following assumptions hold.

(H1) $r(t,0) = h(0) = 0, r_1(t,x) \geq \vartheta(t) > 0, |h(x)| \leq \delta^{\frac{1}{4}}|x|, 0 < \delta < 1, t \in \Re^+, x \in \Re$,

(H2) $\gamma(t) = 2\vartheta^3(t) - \frac{1}{2}\int_{t-\tau}^{t} |H(t,s)|ds - \mu \int_{t-\tau}^{\infty} |H(u+\tau,t)|du - \frac{1}{2} > 0$

with $0 \leq s \leq t \leq u < \infty$, where γ is a positive function, which is bounded and continuous,

(H3) $\int_{t-\tau}^{t} |H(t,s)|^{\frac{1}{2}} ds < \infty$, $\int_{t-\tau}^{\infty} |H(u+\tau,t)|du < \infty$

and

$$\frac{5}{6}|H(t,s)| \geq \int_{t}^{\infty} |H(u,s)|du.$$

Set

$$D = \{x \in \Re : \|x\| \geq 1\}.$$

Let $\phi(t)$ be an initial function. We also suppose that this function is bounded and continuous, and $\|\phi\| = 1$ for $0 \leq t \leq t_0$.

The last outcome of the current work is given below.

Theorem 3. If assumptions (H1)-(H3) hold, then all solutions of (NVIDE) (7) initiating in D satisfy the inequality

$$\|x\| \leq W^{-1}\{\Omega_3(t_0, \|\phi\|)\exp(-\int_{t_0}^{t}\gamma(u)du) + \int_{t_0}^{t}[(\alpha\gamma(u)+F(u))\exp(-\int_{u}^{t}\gamma(s)ds)]du\},$$

for all $t \geq t_0$, where $\gamma(t) = \max\{\gamma_1(t), \gamma_2(t)\}$.

Remark. It is worth mentioning that in Theorem 3, W is a continuous function from $\Re^+$ to $\Re^+$ with $W(0) = 0, W(s) > 0$ if $s > 0$ and W is strictly increasing, and it is called a wedge.

Proof. We describe a (LF) $\Omega_3 = \Omega_3(t, x(t))$ by

$$\Omega_3 = x^2 + \mu \int_0^t \int_{t-\tau}^{\infty} |H(u+\tau,s)|du x^4(s)ds, \tag{16}$$

where $\mu \in \Re$, $\mu > 0$, we fix that constant in the proof.

We have from (16) that

$$\Omega_3(t,0) = 0 \ and \ \Omega_3(t,x) \leq x^2.$$

Thus, we see that the (LF) Ω_3 is positive definite.

Differentiating the (LF) Ω_3 with respect to t, along solutions of (NVIDE) (7), we obtain from (16) and (NVIDE) (7) that

$$\begin{aligned}\Omega_3' &= 2xx' + \mu x^4 \int_{t-\tau}^{\infty} |H(u+\tau,t)|du - \mu \int_0^t |H(t,s)|x^4(s)ds \\ &= -2r_1(t,x)x^4 + 2x\int_{t-\tau}^{t} H(t,s)h^{\frac{1}{3}}(x(s))ds + 2xF(t) \\ &\quad + \mu x^4 \int_{t-\tau}^{\infty} |H(u+\tau,t)|du - \mu \int_{t-\tau}^{t} |H(t,s)|x^4(s)ds.\end{aligned}$$

By the use of the inequality $2|x(t)||x(s)|^{\frac{1}{3}} \leq x^2(t) + x^{\frac{2}{3}}(s)$ and the assumption $r_1(t,x) \geq \vartheta(t) > 0$, we get

$$\begin{aligned}\Omega_3' &\leq -2\vartheta(t)x^4 + \int_{t-\tau}^{t} |H(t,s)|[x^2(t) + h^{\frac{2}{3}}(x(s))ds] + (x^2(t) + F^2(t)) \\ &+ \mu x^4 \int_{t-\tau}^{\infty} |H(u+\tau,t)|du - \mu \int_{t-\tau}^{t} |H(t,s)|x^4(s)ds \\ &= -2\vartheta^3(t)x^4 + \int_{t-\tau}^{t} |H(t,s)|x^2(t)ds + \int_{t-\tau}^{t} h^{\frac{2}{3}}(x(s))ds \\ &+ x^2 + F^2(t) + \mu x^4 \int_{t-\tau}^{\infty} |H(u+\tau,t)|du - \mu \int_{t-\tau}^{t} |H(t,s)|x^4(s)ds. \qquad (17)\end{aligned}$$

Let $a = 2, b = 2$ and $a = 6, b = \frac{6}{5}$ respectively. From the Young's inequality,

$$mn \leq \frac{1}{a}m^a + \frac{1}{b}n^b, \frac{1}{a} + \frac{1}{b} = 1$$

and assumption (H1), we get the following relations, respectively:

$$\begin{aligned}\int_{t-\tau}^{t} |H(t,s)|x^2(t)ds &= \int_{t-\tau}^{t} |H(t,s)|^{\frac{1}{2}}|H(t,s)|^{\frac{1}{2}}x^2(t)ds \\ &\leq \frac{1}{2}\int_{t-\tau}^{t} |H(t,s)|^{\frac{3}{2}}ds + \frac{1}{2}\int_{t-\tau}^{t} |H(t,s)|^{\frac{1}{2}}x^4(t)ds\end{aligned}$$

and

$$\begin{aligned}\int_{t-\tau}^{t} |H(t,s)|h^{\frac{2}{3}}(x(s))ds &= \int_{t-\tau}^{t} |H(t,s)|^{\frac{5}{6}}|H(t,s)|^{\frac{1}{6}}|h^{\frac{2}{3}}(x(s))ds \\ &\leq \frac{5}{6}\int_{t-\tau}^{t} |H(t,s)|ds + \frac{\delta}{6}\int_{t-\tau}^{t} |H(t,s)|x^4(s)ds.\end{aligned}$$

In addition, we have

$$x^2 \leq \frac{1}{2}x^4 + \frac{1}{2}.$$

Substituting the previous inequalities into (17), we obtain

$$\begin{aligned}\Omega_3' &\leq -[2\vartheta^3(t) - \frac{1}{2}\int_{t-\tau}^{t} |H(t,s)|^{\frac{1}{2}}ds - \mu\int_{t-\tau}^{\infty} |H(u+\tau,t)|du - \frac{1}{2}]x^4 \\ &+ \frac{1}{2}\int_{t-\tau}^{t} |H(t,s)|^{\frac{3}{2}}ds + \frac{5}{6}\int_{t-\tau}^{t} |H(t,s)|ds \\ &+ (\frac{\delta}{6} - \mu)\int_{t-\tau}^{t} |H(t,s)|x^4(s)ds + 1 + F^2(t).\end{aligned}$$

Let

$$\begin{aligned}&G(t) = F^2(t), \frac{5}{6}|H(t,s)| \geq \int_{t}^{\infty} |H(u,s)|du, \\ &\mu = \delta, L = \frac{1}{2}\int_{t-\tau}^{t} |H(t,s)|^{\frac{3}{2}}ds + \frac{5}{6}\int_{t-\tau}^{t} |H(t,s)|ds\end{aligned}$$

and

$$\rho(t) = \max\{\gamma(t), 1\}.$$

Thereafter, using assumption (H2) of Theorem 3, we conclude that

$$\Omega_3' \leq -\rho(t)x^4.$$

The remaining of the proof can be done by following a similar way that shown in the proof of Theorem 1 or 2. Therefore, we omit the details.

3. Conclusion

We investigate three (NVIDEs) with constant time lag. Three variational of parameters inequalities are obtained so that the all solutions of the considered (NVIDEs) remain bounded. We benefited from the (LFs). The obtained results have a contribution to the related literature, and they improve and extend the results in [51] from the cases of without time lag to that general non-linear cases with time lag. The obtained results are compared with that found in [51] and that in the references of this paper.

4. Acknowledgement

The authors of this paper would like to express their sincere appreciations to the main editor and the anonymous referees for their valuable comments and suggestions which have led to an improvement in the presentation of the paper.

References

[1] M. Adivar, Y. N. Raffoul, Inequalities and exponential stability and instability in finite delay Volterra integro-differential equations. Rend. Circ. Mat. Palermo (2) 61 (2012), no. 3, 321-330.
[2] L. C. Becker, Principal matrix solutions and variation of parameters for a Volterra integro-differential equation and its adjoint. Electron. J. Qual. Theory Differ. Equ. 2006, no. 14, 22 pp. (electronic).
[3] L. C. Becker, Function bounds for solutions of Volterra equations and exponential asymptotic stability. Nonlinear Anal. 67 (2007), no. 2, 382-397.
[4] L. C. Becker, Uniformly continuous L^1 solutions of Volterra equations and global asymptotic stability. Cubo 11 (2009), no. 3, 1-24.
[5] T. A. Burton, Boundedness and periodicity in integral and integro-differential equations. Differential Equations Dynam. Systems 1 (1993), no. 2, 161-172.
[6] T. A. Burton, Volterra integral and differential equations. Second edition. Mathematics in Science and Engineering, 202. Elsevier B. V., Amsterdam, 2005.
[7] T. A. Burton, A Liapunov functional for a linear integral equation. Electron. J. Qual. Theory Differ. Equ. 2010, No. 10, 10 pp.
[8] T. A. Burton and J. R. Haddock, Qualitative properties of solutions of integral equations. Nonlinear Anal. 71 (2009), no. 11, 5712-5723.
[9] T. A. Burton, W. E. Mahfoud, Stability criteria for Volterra equations. Trans. Amer. Math. Soc. 279 (1983), no. 1, 143-174.
[10] T. A. Burton, W. E. Mahfoud, Instability and stability in Volterra equations. Trends in theory and practice of nonlinear differential equations (Arlington, Tex., 1982), 99-104, Lecture Notes in Pure and Appl. Math., 90, Dekker, New York, 1984.
[11] T. A. Burton, W. E. Mahfoud, Stability by decompositions for Volterra equations. Tohoku Math. J. (2) 37 (1985), no. 4, 489-511.
[12] X. Chang, R. Wang, Stability of perturbed n -dimensional Volterra differential equations. Nonlinear Anal. 74 (2011), no. 5, 1672-1675.
[13] C. Corduneanu, Principles of differential and integral equations. Second edition. Chelsea Publishing Co., Bronx, N. Y., 1977.
[14] A. Diamandescu, On the strong stability of a nonlinear Volterra integro-differential system. Acta Math. Univ. Comenian. (N.S.) 75 (2006), no. 2, 153-162.
[15] N. T. Dung New stability conditions for mixed linear Levin-Nohel integro-differential equations. J. Math. Phys. 54 (2013), no. 8, 082705, 11 pp.

[16] N. T. Dung, On exponential stability of linear Levin-Nohel integro-differential equations. J. Math. Phys. 56 (2015), no. 2, 022702, 10 pp.
[17] P. Eloe, M. Islam, B. Zhang, Uniform asymptotic stability in linear Volterra integro-differential equations with application to delay systems. Dynam. Systems Appl. 9 (2000), no. 3, 331-344.
[18] H. Engler, Asymptotic properties of solutions of nonlinear Volterra integro-differential equations.Results Math. 13 (1988), no. 1-2, 65-80.
[19] M. Funakubo, T. Hara, S. Sakata, On the uniform asymptotic stability for a linear integro-differential equation of Volterra type. J. Math. Anal. Appl. 324 (2006), no. 2, 1036-1049.
[20] T. Furumochi, S. Matsuoka, Stability and boundedness in Volterra integro-differential equations. Mem. Fac. Sci. Eng. Shimane Univ. Ser. B Math. Sci. 32 (1999), 25-40.
[21] S. Grace, E. Akin, Asymptotic behavior of certain integro-differential equations. Discrete Dyn. Nat. Soc. 2016, Art. ID 4231050, 6 pp.
[22] J. R. Graef, C. Tunç, Continuability and boundedness of multi-delay functional integro-differential equations of the second order. Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM 109 (2015), no. 1, 169-173.
[23] J. R. Graef, C. Tunç, S. Sevgin, Behavior of solutions of non-linear functional Voltera integro -differential equations with multiple delays. Dynam. Systems Appl. 25 (2016), no. 1-2, 39-46.
[24] R. Grimmer, G. Seifert, Stability properties of Volterra integrodifferential equations. J. Differential Equations 19 (1975), no. 1, 142-166.
[25] R. Grimmer, M. Zeman, Nonlinear Volterra integro-differential equations in a Banach space. Israel J. Math. 42 (1982), no. 1-2, 162-176.
[26] G. Gripenberg, S. Q. Londen, O. Staffans, Volterra integral and functional equations. Encyclopedia of Mathematics and its Applications, 34. Cambridge University Press, Cambridge, 1990.
[27] S. I. Grossman, R. K. Miller, Perturbation theory for Volterra integro-differential systems. J. Differential Equations 8 (1970) 457-474.
[28] T. Hara, T. Yoneyama, T. Itoh, On the characterization of stability concepts of Volterra integro-differential equations. J. Math. Anal. Appl. 142 (1989), no. 2, 558-572.
[29] T. Hara, T. Yoneyama, T. Itoh, Asymptotic stability criteria for nonlinear Volterra integro-differential equations. Funkcial. Ekvac. 33 (1990), no. 1, 39-57.
[30] T. Hara, T. Yoneyama, R. Miyazaki, Volterra integro-differential inequality and asymptotic criteria. Differential Integral Equations 5 (1992), no. 1, 201-212.
[31] Y. Hino, S. Murakami, Stability properties of linear Volterra integro-differential equations in a Banach space. Funkcial. Ekvac. 48 (2005), no. 3, 367-392
[32] M. Islam, Asymptotically stable solutions of a system of nonlinear differential equations. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 22 (2015), no. 4, 303-312
[33] M. N. Islam, M. M. G. Al-Eid, Boundedness and stability in nonlinear Volterra integro-differential equations. PanAmer. Math. J. 14 (2004), no. 3, 49-63.
[34] M. N. Islam, Y. Raffoul, Stability properties of linear Volterra integrodifferential equations with nonlinear perturbation. Commun. Appl. Anal. 7 (2003), no. 2-3, 405-416.
[35] M. N. Islam, Y. Raffoul, Stability in linear Volterra integrodifferential equations with nonlinear perturbation. J. Integral Equations Appl. 17 (2005), no. 3, 259-276.
[36] C. Jin, J. Luo, Stability of an integro-differential equation. Comput. Math. Appl. 57 (2009), no. 7, 1080-1088.
[37] V. Lakshmikantham, M. Rama Mohan Rao, Stability in variation for nonlinear integro-differential equations. Applicable Analysis 24 (1987), no. 3, 165-173.
[38] V. Lakshmikantham, M. Rama Mohana Rao, Theory of integro-differential equations. Stability and Control: Theory, Methods and Applications, 1. Gordon and Breach Science Publishers, Lausanne, 1995.
[39] W. E. Mahfoud, Stability theorems for an integro-differential equation. Arabian J. Sci. Engrg. 9 (1984), no. 2, 119-123.
[40] W. E. Mahfoud, Stability criteria for linear integro-differential equations. Ordinary and partial differential equations (Dundee, 1984), 243251, Lecture Notes in Math., 1151, Springer, Berlin, 1985.
[41] W. E. Mahfoud, Boundedness properties in Volterra integro-differential systems. Proc. Amer. Math. Soc. 100 (1987), no. 1, 37-45.

[42] M.B. Mesmouli, A. Ardjouni, A. Djoudi, Stability in nonlinear Levin-Nohel integro-differential equations. Nonlinear Stud. 22 (2015), no. 4, 705-718.
[43] C. Martinez, Bounded solutions of a forced nonlinear integro-differential equation. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 9 (2002), no. 1, 35-42.
[44] R. K. Miller, Asymptotic stability properties of linear Volterra integro-differential equations. J. Differential Equations 10 (1971) 485-506.
[45] S. Murakami, Exponential asymptotic stability for scalar linear Volterra equations. Differential Integral Equations 4 (1991), no. 3, 519-525.
[46] P. H. A. Ngoc, On stability of a class of integro-differential equations. Taiwanese J. Math. 17 (2013), no. 2, 407-425.
[47] Juan E. Napoles Valdes, A note on the boundedness of an integro-differential equation. Quaest. Math. 24 (2001), no. 2, 213-216.
[48] M. Peschel, W. Mende, The predator-prey model: do we live in a Volterra world? Springer-Verlag, Vienna, 1986.
[49] Y. Raffoul, Boundedness in nonlinear functional differential equations with applications to Volterra integro-differential equations. J. Integral Equations Appl. 16 (2004), no. 4, 375-388.
[50] Y. Raffoul, Construction of Lyapunov functionals in functional differential equations with applications to exponential stability in Volterra integro-differential equations. Aust. J. Math. Anal. Appl. 4 (2007), no. 2, Art. 9, 13 pp.
[51] Y. Raffoul, Exponential analysis of solutions of functional differential equations with unbounded terms. Banach J. Math. Anal. 3 (2009), no. 2, 28-41.
[52] Y. Raffoul, Exponential stability and instability in finite delay nonlinear Volterra integro-differential equations. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 20 (2013), no. 1, 95-106.
[53] Y. Raffoul, H. Rai, Uniform stability in nonlinear infinite delay Volterra integro-differential equations using Lyapunov functionals. Nonauton. Dyn. Syst. 3 (2016), 14-23.
[54] Y. Raffoul, D. Ren, Theorems on boundedness of solutions to stochastic delay differential equations. Electron. J. Differential Equations 2016, Paper No. 194, 14 pp.
[55] Y. Raffoul, A. Sanbo, Boundedness and stability results for the finite delay nonlinear Volterra discrete system. Nonlinear Stud. 23 (2016), no. 1, 87-94.
[56] Y. Raffoul, M. nal, Stability in nonlinear delay Volterra integro-differential systems. J. Nonlinear Sci. Appl. 7 (2014), no. 6, 422-428.
[57] M. Rama Mohana Rao, V. Raghavendra, Asymptotic stability properties of Volterra integro-differential equations. Nonlinear Anal. 11 (1987), no. 4, 475-480.
[58] M. Rama Mohana Rao, P. Srinivas, Asymptotic behavior of solutions of Volterra integro-differential equations. Proc. Amer. Math. Soc. 94 (1985), no. 1, 55-60.
[59] O. J. Staffans, A direct Lyapunov approach to Volterra integro-differential equations. SIAM J. Math. Anal. 19 (1988), no. 4, 879-901.
[60] P. Talpalaru, Stability criteria for Volterra integro-differential equations. An. Ştiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) 46 (2000), no. 2, 349-358 (2001).
[61] C. Tunç, A note on the qualitative behaviors of non-linear Volterra integro-differential equation. J. Egyptian Math. Soc. 24 (2016), no. 2, 187-192.
[62] C. Tunç, New stability and boundedness results to Volterra integro-differential equations with delay. J. Egyptian Math. Soc. 24 (2016), no. 2, 210-213.
[63] C. Tunç, Properties of solutions to Volterra integro-differential equations with delay. Appl. Math. Inf. Sci. 10 (2016), no. 5, 1775-1780.
[64] C. Tunç, Qualitative properties in nonlinear Volterra integro-differential equations with delay. Journal of Taibah University for Science. 11 (2017), no.2, 309-314.
[65] C. Tunç, Asymptotic stability and boundedness criteria for nonlinear retarded Volterra integro-differential equations. Journal of King Saud University - Science, (2017), (in press).
[66] C. Tunç, T. Ayhan, On the global existence and boundedness of solutions to a nonlinear integro-differential equations of second order. J. Interpolat. Approx. Sci. Comput. (2015), no.1, 1-14.
[67] C. Tunç, S.A. Mohammed, On the stability and instability of functional Volterra integro-differential equations of first order. Bull. Math. Anal. Appl. 9(2017), no.1, 151-160.
[68] C. Tunç, S.A. Mohammed, New results on exponential stability of nonlinear Volterra integro-differential equations with constant time-lag, Proyecciones, (2017), (in press).

[69] J. Vanualailai, Some stability and boundedness criteria for a class of Volterra integro-differential systems. Electron. J. Qual. Theory Differ. Equ. 2002, No. 12, 20 pp.
[70] J. Vanualailai, S. Nakagiri, Some stability criteria for a class of Volterra integro-differential systems. Dynamics of functional equations and related topics (Kyoto, 2001). Surikaisekikenkyusho Kokyuroku No. 1254 (2002), 73-81.
[71] J. Vanualailai, S. Nakagiri, Stability of a system of Volterra integro-differential equations. J. Math. Anal. Appl. 281 (2003), no. 2, 602-619.
[72] Q. Wang, The stability of a class of functional differential equations with infinite delays. Ann. Differential Equations 16 (2000), no. 1, 89-97.
[73] T. Wang, Inequalities of solutions of Volterra integral and differential equations. Electron. J. Qual. Theory Differ. Equ. 2009, Special Edition I, No. 28, 10 pp.
[74] T. Wang, Lower and upper bounds of solutions of functional differential equations. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 20 (2013), no. 1, 131-141.
[75] Zhi Cheng Wang, Zhi Xiang Li, Jian Hong Wu, Stability properties of solutions of linear Volterra integro-differential equations. Tohoku Math. J. (2) 37 (1985), no. 4, 455-462.
[76] A. M. Wazwaz, Linear and nonlinear integral equations. Methods and applications. Higher Education Press, Beijing; Springer, Heidelberg; 2011.
[77] Bo Zhang, Necessary and sufficient conditions for stability in Volterra equations of non-convolution type. Dynam. Systems Appl. 14 (2005), no. 3-4, 525-549.
[78] Zong Da Zhang, Asymptotic stability of Volterra integro-differential equations. J. Harbin Inst. Tech. 1990, no. 4, 11-19.

Cemil TUNC
Department of Mathematics, Faculty of Sciences, Yuzuncu Yil University, 65080, Van-Turkey
E-mail address: cemtunc@yahoo.com

Osman TUNC
Department of Mathematics, Faculty of Sciences, Yuzuncu Yil University, 65080, Van-Turkey
E-mail address: osmantunc89@gmail.com