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# NUMERICAL SOLUTION FOR A FREE SURFACE FLOW OVER MULTIPLE OBSTACLES

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Abstract. Two-dimensional free-surface flows of an inviscid and incompressible fluid over multiple obstacles is considered. The flow is assumed to be steady and irrotational. Both gravity and surface tension are included in the dynamic boundary conditions. Far upstream, the flow is assumed to be uniform. Multiples obstructions are located at the channel bottom. In this paper, the fully nonlinear problem is formulated by using a boundary integral equation technique. The resulting integro-differential equations are solved iteratively by Newton's method. When surface tension and gravity are included, there are two additional parameters in the problem known as the Weber number and Froude number. Finally, solution diagrams for all flow regimes are presented.

## 1. Introduction

Fully nonlinear flow over obstacles has increased considerably in the last decade. Much of this work has concentrated on finding flow solutions over a single obstacle and most often this obstacle has had a special shape. For example, Forbes [3] considered flow over a semi-elliptical obstacle, Forbes and Schwartz [4] considered flow over a semi-circular obstruction, Dias and Vanden-Broeck [2] studied flow over a triangular weir. We may also mention in particular the work of Abd-el-Malek. M.B [1], Hanna. S.N [6]. The advantage of using these special shapes is that a conformal mapping can be used to cater for singularities that occur on the bottom surface. This paper is concerned with the numerical calculation of 2-D fluid flow in an open channel. The flow is uniform far upstream and is disturbed by an obstruction on the bottom of the channel, which are two triangles. The effects of surface tension and the gravity are included in the boundary conditions and the problem is solved numerically by a boundary element method. An integral equation is constructed from the related boundary value problem by firstly introducing a hodograph variable, and then transforming the flow domain into an artificial one which is a half complex plane. The integral equation expresses the hodograph variable on the boundary of the artificial plane. Numerically, this equation can be solved by discretizing the domain of integration to construct a system of nonlinear

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equations, and the Newton iteration method is applied. In presenting this paper, we organize the sections as follows. In section 2 and 3 we formulate the problem into the integral equation as described above. This is followed by presenting the numerical procedure in solving the integral equation that is given in section 4. The result of the calculations is then discussed in section 5.

## 2. Mathematical Formulation

We consider two-dimensional flow over two triangles of an inviscid and incompressible fluid. The flow is assumed to be steady and irrotational. The fluid domain is bounded below by the rigid streamline $AE$, the two triangles $BCD$, $B'C'D'$ forming the angle $\gamma$ (respectively $\beta$ ), with the horizontal, where $0 < |\gamma| < \frac{\pi}{2}$ and $0 < |\beta| < \frac{\pi}{2}$, and above by the free surface $MN$ (see Figure 1)

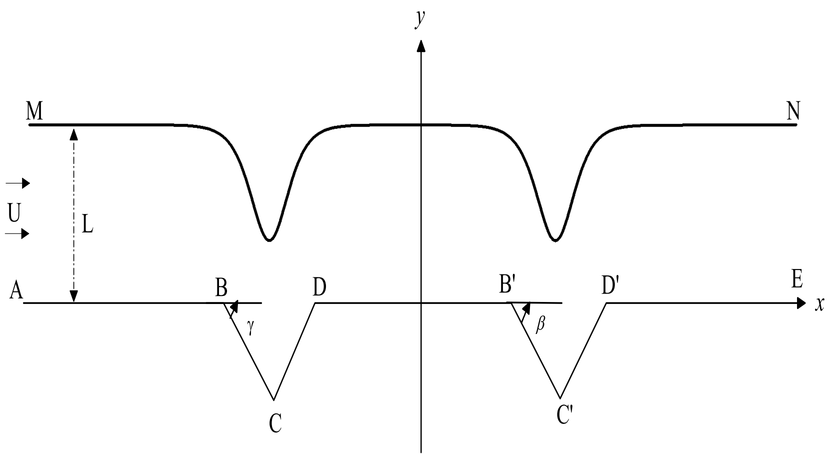


Figure 1. Sketch of the flow and the coordinate in the physical plane $z = x + iy$

Let us introduce cartesian coordinates with the $x-$axis along the bottom and the $y-$axis directed vertically upwards. As $x \longrightarrow \infty$, the flow is assumed to approach a uniform stream with constant velocity $U$ and constant length $L$. It is convenient to define dimensionless variables by taking $U$ as the unit velocity and $L$ as the unit length. The dimensionless parameters in the problem are the Froude number $Fr = \dfrac{U}{\sqrt{gL}}$ and the inverse Weber number $\delta = \dfrac{T}{\rho U^2 L}$ Here $T$ is the surface tension, $g$ is gravity and $\rho$ is the fluid density.
Let's introduce the velocity potential $\phi(x, y)$ and the stream function $\psi(x, y)$, by defining the complex potential function

$$f(x, y) = \phi(x, y) + i\psi(x, y) \tag{2.1}$$

The complex velocity $\omega$ can be written as

$$\omega = \frac{df}{dz} = u - iv \tag{2.2}$$

where $u$ and $v$ are the velocity components in the $x$ and $y$ directions, and $z = x+iy$.
We choose $\psi = 0$ along the free surface $MN$.
Then $\psi = -1$ and $-\infty < \phi < +\infty$ along the bottom streamline $ABCDB'C'D'E$.
The flow configuration in the $f-$plane is sketched in figure 2.

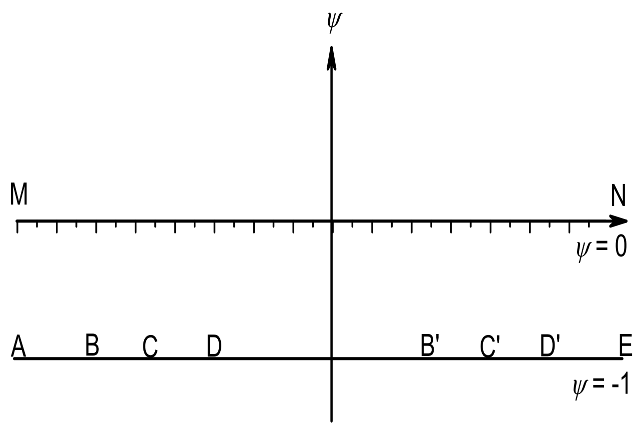


FIGURE 2. Sketch of the flow in the potential f-plane $f = \phi + i\psi$

The mathematical problem can be formulated in terms of the potential function $\phi$ satisfying the Laplace's equation

$$\Delta\phi = 0 \text{ in the fluid domain}$$

On the free surface, the Bernoulli equation has to be satisfied

$$\frac{1}{2}\left(u^{*2} + v^{*2}\right) + \frac{p^*}{\rho^*} + g^*y^* = C = \frac{1}{2}U^2 + \frac{p_0}{\rho} + gL \tag{2.3}$$

It is noted that the right-hand-side of Equation (2.3) is evaluated at the free surface far upstream. Using the capillary Laplace's equation defined by

$$p^* - p_0 = \frac{T}{R} = KT \tag{2.4}$$

where $K = \dfrac{1}{R}$ is the curvature, $p^*$ is the fluid pressure, $p_0$ is the atmospheric pressure, $\rho^*$ is the fluid density and $g$ is the gravitational constant.
Substituting (2.4) into (2.3) and in terms of the dimensionless variables, Bernoulli's equation on the free surface becomes

$$\frac{1}{2}\left(u^2 + v^2\right) + \frac{KT}{\rho L U^2} + \frac{1}{Fr^2}(y-1) = \frac{1}{2} \tag{2.5}$$

The kinematic boundary conditions are

$$\begin{cases} v = 0 \text{ on } -\infty < \phi < \phi_B \ , \phi_D < \phi < \phi_{B'} \ , \phi_{D'} < \phi < +\infty \text{ and } \psi = -1 \\ v = u \tan|\gamma| \text{ on } \phi_B < \phi < \phi_C \ , \phi_C < \phi < \phi_D \text{ and } \psi = -1 \\ v = u \tan|\beta| \text{ on } \phi_{B'} < \phi < \phi_{C'} \ , \phi_{C'} < \phi < \phi_{D'} \text{ and } \psi = -1 \end{cases} \tag{2.6}$$

Now we reformulate the problem as an integral equation. We define the function $\tau - i\theta$ by

$$\omega = u - iv = e^{\tau - i\theta} \tag{2.7}$$

We map the flow domain onto the upper half of the $\zeta$-plane by the transformation

$$\zeta = \alpha + i\beta = e^{-\pi f} = e^{-\pi\phi}\left(\cos\pi\psi - i\sin\pi\psi\right) \tag{2.8}$$

as shown in figure 3.

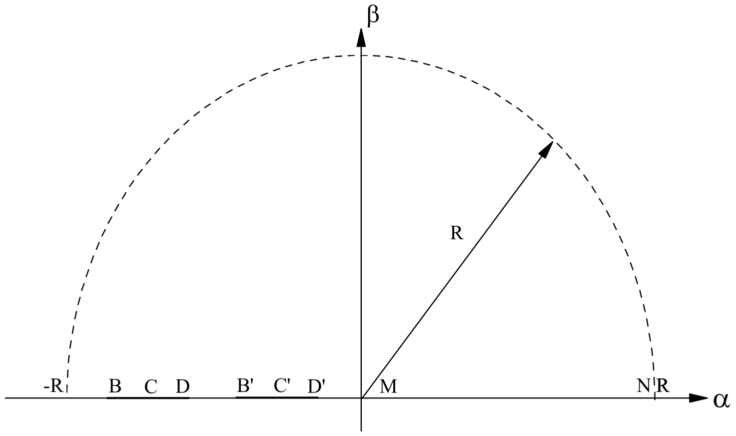


FIGURE 3. The upper half $\zeta - plane$ $\zeta = \alpha + i\beta$

The velocity terms, first, become

$$u^2 + v^2 = e^{2\tau} \tag{2.9}$$

Secondly, the curvature $K$ of a streamline, in terms of $\theta$, is given by

$$K = -e^{\tau}\left|\frac{\partial\theta}{\partial\phi}\right| \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.5), gives the final form of Bernoulli's equation that is needed for the numerical calculation. This is

$$\frac{1}{2}e^{2\tau} - \delta e^{\tau}\left|\frac{\partial\theta}{\partial\phi}\right| + \frac{1}{Fr^2}(y-1) = \frac{1}{2} \quad \text{on} \quad MN \tag{2.11}$$

Using (2.8), the kinematic boundary conditions (2.6) in the $\zeta-$plane, become

$$\begin{cases} \theta = 0 \quad \text{for} -\infty < \alpha < \alpha_B,\, \alpha_D < \alpha < \alpha_{B'},\, \alpha_{D'} < \alpha \leq 0 \\ \theta = |\gamma| \quad \text{for } \alpha_B < \alpha < \alpha_C, \alpha_C < \alpha < \alpha_D \\ \theta = |\beta| \quad \text{for } \alpha_{B'} < \alpha < \alpha_{C'}, \alpha_{C'} < \alpha < \alpha_{D'} \\ \theta = \text{unknown} \quad 0 < \alpha < +\infty \end{cases} \tag{2.12}$$

## 3. Boundary integral techniques

Next we apply the Cauchy integral formula to the function $\tau - i\theta$ in the complex $\zeta$-plane, we choose a contour consisting of the real axis and a half-circle of arbitrary large radius in the upper half-plane.

$$\tau(\zeta_0) - i\theta(\zeta_0) = \frac{1}{i\pi} \int_\Gamma \frac{\tau(\zeta) - i\theta(\zeta)}{\zeta - \zeta_0} d\zeta \tag{3.1}$$

The contour $\Gamma$ is a simple, where $\zeta_0$ is an image point of a point on the free surface, i.e. $\zeta_0 \in MN$. The path $\Gamma$ consists of a large semi-circular arc of radius $R$ centred at the origin. Taking the limit of (3.1), with this particular contour, as $R \longrightarrow \infty$ gives

$$\tau(\zeta_0) - i\theta(\zeta_0) = \frac{1}{i\pi} \int_{-\infty}^{+\infty} \frac{\tau(\zeta) - i\theta(\zeta)}{\zeta - \zeta_0} d\zeta \tag{3.2}$$

This integral is a Cauchy principle value. By considering only the real part of (3.2), gives an integral equation for $\tau$ , which is

$$\tau(\alpha_0) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\theta(\alpha)}{\alpha - \alpha_0} d\alpha \tag{3.3}$$

Where $\zeta_0 = \alpha_0 + i\beta_0$, $\tau(\alpha_0) = \tau(\alpha_0, 0)$ and $\theta(\alpha) = \theta(\alpha, 0)$ are used to simplify the notation. By using the boundary conditions (2.12), the integral is separated into eight parts to give

$$\tau(\alpha_0) = -\frac{1}{\pi} \begin{bmatrix} \int_{-\infty}^{\alpha_B} \frac{\theta(\alpha)}{\alpha-\alpha_0} d\alpha + \int_{\alpha_B}^{\alpha_C} \frac{\theta(\alpha)}{\alpha-\alpha_0} d\alpha + \int_{\alpha_C}^{\alpha_D} \frac{\theta(\alpha)}{\alpha-\alpha_0} d\alpha + \\ \int_{\alpha_D}^{\alpha_{B'}} \frac{\theta(\alpha)}{\alpha-\alpha_0} d\alpha + \int_{\alpha_{B'}}^{\alpha_{C'}} \frac{\theta(\alpha)}{\alpha-\alpha_0} d\alpha + \int_{\alpha_{C'}}^{\alpha_{D'}} \frac{\theta(\alpha)}{\alpha-\alpha_0} d\alpha + \\ \int_{\alpha_{D'}}^{0} \frac{\theta(\alpha)}{\alpha-\alpha_0} d\alpha + \int_{0}^{+\infty} \frac{\theta(\alpha)}{\alpha-\alpha_0} d\alpha \end{bmatrix} \tag{3.4}$$

Using the conditions (2.12), equation (3.4) can be simplified as

$$\begin{aligned} \tau(\alpha_0) = {} & \tfrac{\gamma}{\pi} \log \left| \tfrac{\alpha_C - \alpha_0}{\alpha_B - \alpha_0} \right| - \tfrac{\gamma}{\pi} \log \left| \tfrac{\alpha_D - \alpha_0}{\alpha_C - \alpha_0} \right| + \tfrac{\beta}{\pi} \log \left| \tfrac{\alpha_{C'} - \alpha_0}{\alpha_{B'} - \alpha_0} \right| - \\ & \tfrac{\beta}{\pi} \log \left| \tfrac{\alpha_{C'} - \alpha_0}{\alpha_{D'} - \alpha_0} \right| - \tfrac{1}{\pi} \int_0^{+\infty} \tfrac{\theta(\alpha)}{\alpha - \alpha_0} d\alpha \end{aligned} \tag{3.5}$$

Using (2.8), with $\psi = 0$ gives

$$\alpha = e^{-\pi\phi}, \alpha_0 = e^{-\pi\phi_0} \tag{3.6}$$

This equation holds along the free surface.

Substituting (3.6) into (3.5), yields

$$\begin{aligned} \tau'(\phi_0) = {} & \tfrac{\gamma}{\pi} \log \left| \tfrac{e^{-\pi\phi_C} + e^{-\pi\phi_0}}{e^{-\pi\phi_B} + e^{-\pi\phi_0}} \right| - \tfrac{\gamma}{\pi} \log \left| \tfrac{e^{-\pi\phi_D} + e^{-\pi\phi_0}}{e^{-\pi\phi_C} + e^{-\pi\phi_0}} \right| + \tfrac{\beta}{\pi} \log \left| \tfrac{e^{-\pi\phi_{C'}} + e^{-\pi\phi_0}}{e^{-\pi\phi_{B'}} + e^{-\pi\phi_0}} \right| \\ & - \tfrac{\beta}{\pi} \log \left| \tfrac{e^{-\pi\phi_{C'}} + e^{-\pi\phi_0}}{e^{-\pi\phi_{D'}} + e^{-\pi\phi_0}} \right| + \int_{-\infty}^{\phi} \tfrac{\theta'(\phi) e^{-\pi\phi}}{e^{-\pi\phi} - e^{-\pi\phi_0}} d\phi \end{aligned} \tag{3.7}$$

for $-\infty < \phi < +\infty$

where $\tau'(\phi_0) = \tau\left(e^{-\pi\phi_0}\right)$ and $\theta'(\phi) = \theta\left(e^{-\pi\phi}\right)$. The equation (2.11) is now rewritten in terms of $\tau'$ and $\theta'$ as

$$\frac{1}{2}e^{2\tau'} - \delta e^{\tau'}\left|\frac{\partial\theta'}{\partial\phi}\right| + \frac{1}{Fr^2}(y-1) = \frac{1}{2} \quad \text{on } MN \tag{3.8}$$

Next we evaluate the values of $y$ on the free surface by using (2.7) and integrating the identity

$$\frac{d(x+iy)}{df} = \omega^{-1} \tag{3.9}$$

This gives

$$\begin{cases} x(\alpha) = x_\infty - \frac{1}{\pi}\int_0^\alpha \frac{e^{-\tau(\alpha_0)}\cos\theta(\alpha_0)}{\alpha_0}d\alpha_0 \\ y(\alpha) = 1 - \frac{1}{\pi}\int_0^\alpha \frac{e^{-\tau(\alpha_0)}\sin\theta(\alpha_0)}{\alpha_0}d\alpha_0 \qquad \text{for } 0 \le \alpha < +\infty \end{cases} \tag{3.10}$$

By using (3.6), we rewrite (3.10) as

$$\begin{cases} x'(\phi) = x_\infty + \int_{-\infty}^{\phi} e^{-\tau'(\phi_0)}\cos\theta'(\phi_0)d\phi_0 \\ y'(\phi) = 1 + \int_{-\infty}^{\phi} e^{-\tau'(\phi_0)}\sin\theta'(\phi_0)d\phi_0 \qquad \text{for } -\infty < \phi < +\infty \end{cases} \tag{3.11}$$

By substituting (3.7) and (3.11) into (3.8), an integro-differential equation is created and this is solved numerically in the following section.

## 4. Numerical procedure

The above system of nonlinear equations is solved numerically by using equally spaced points in the potential function $\phi$. We introduce equally spaced mesh points in the potential function $\phi$ by

$$\phi_I = \left[\frac{-(N-1)}{2} + (I-1)\right]\Delta, I = 1, ..., N, -\infty < \phi < +\infty \tag{4.1}$$

where $\Delta > 0$, is the uniform increment between consecutive mesh points. If these are the points used in calculating, then a problem arises with the integral part

$$\int_{-\infty}^{+\infty} \frac{\theta'(\phi)\, e^{-\pi\phi}}{e^{-\pi\phi} - e^{-\pi\phi_0}}d\phi \tag{4.2}$$

of (3.7). It can be seen in (4.2), that for each mesh point, there would be a singularity in the integrand, where $\phi_I = \phi_0$. The integral (4.2) is evaluated by the trapezoidal rule, with a summation over $\phi_I$ such that $\phi_0$ is the midpoint of one sub-interval, defined as follows

$$\phi_M = \frac{\phi_{I+1} + \phi_I}{2}, I = 1, ..., N-1 \tag{4.3}$$

Firstly, (3.7) is rewritten as

$$\tau_M=\tau'(\phi_M)=\frac{\gamma}{\pi}\log\left|\frac{e^{-\pi\phi_C}+e^{-\pi\phi_M}}{e^{-\pi\phi_B}+e^{-\pi\phi_M}}\right|-\frac{\gamma}{\pi}\log\left|\frac{e^{-\pi\phi_D}+e^{-\pi\phi_M}}{e^{-\pi\phi_C}+e^{-\pi\phi_M}}\right|+ \frac{\beta}{\pi}\log\left|\frac{e^{-\pi\phi_{C'}}+e^{-\pi\phi_M}}{e^{-\pi\phi_{B'}}+e^{-\pi\phi_M}}\right|-\frac{\beta}{\pi}\log\left|\frac{e^{-\pi\phi_{C'}}+e^{-\pi\phi_M}}{e^{-\pi\phi_{D'}}+e^{-\pi\phi_M}}\right|+\int_{-\infty}^{+\infty}\frac{\theta'(\phi)\,e^{-\pi\phi}}{e^{-\pi\phi}-e^{-\pi\phi_M}}d\phi \tag{4.4}$$

where $\tau_M=\tau'(\phi_M)$ has been introduced to ease notation. Then, using the trapezoidal rule gives

$$\tau_M=\frac{\gamma}{\pi}\log\left|\frac{e^{-\pi\phi_C}+e^{-\pi\phi_M}}{e^{-\pi\phi_B}+e^{-\pi\phi_M}}\right|-\frac{\gamma}{\pi}\log\left|\frac{e^{-\pi\phi_D}+e^{-\pi\phi_M}}{e^{-\pi\phi_C}+e^{-\pi\phi_M}}\right|+ \frac{\beta}{\pi}\log\left|\frac{e^{-\pi\phi_{C'}}+e^{-\pi\phi_M}}{e^{-\pi\phi_{B'}}+e^{-\pi\phi_M}}\right|-\frac{\beta}{\pi}\log\left|\frac{e^{-\pi\phi_{C'}}+e^{-\pi\phi_M}}{e^{-\pi\phi_{D'}}+e^{-\pi\phi_M}}\right|+\sum_{j=1}^{N}\frac{\theta_j e^{-\pi\phi_j}\Delta w_j}{e^{-\pi\phi_j}-e^{-\pi\phi_M}} \tag{4.5}$$

where $I=1,...,N-1$ and $w_j$ is the weighting function such that

$$w_j=\begin{cases}\frac{1}{2}: j=1,N\\ 1: otherwise\end{cases}$$

and

$$\theta_j=\theta'(\phi_j)$$

By substituting (4.5) into (3.8), evaluated at the mesh points (4.1), a system of $N$ nonlinear equations in $N$ unknowns is obtained. These are $\theta_I$ for $I=1,...,N$. We evaluate $y_I=y'(\phi_I)$ by applying the trapezoidal rule to (4.5) and by using (3.9). This yields

$$\begin{aligned}&y_1=1\\&y_{I+1}=y_I+\Delta e^{-\tau_M}\sin\theta_M, I=1,...,N-1\end{aligned}$$

and

$$\begin{aligned}&x_1=x_\infty\\&x_{I+1}=x_I+\Delta e^{-\tau_M}\cos\theta_M, I=1,...,N-1\end{aligned}$$

Here $\theta_M=\dfrac{\theta_{I+1}+\theta_I}{2}$. We now satisfy (3.8) at the midpoints (4.3). This yields $N$ nonlinear algebraic equations for the $N$ unknowns $\theta_I, I=1,...,N$. The derivative, $\dfrac{\partial\theta'}{\partial\phi}$, at the mesh points (4.1), is approximated by a finite difference, whereby

$$\frac{\partial\theta'}{\partial\phi}\approx\frac{\theta_{I+1}-\theta_I}{\Delta}, I=1,...,N-1$$

For a given value of $\delta$ and $Fr$, this system of $N$ equations with $N$ unknowns is solved by Newton's method.

## 5. Discussion of the results

In order to illustrate the results obtained from this technique, numerical calculations have been performed for two triangular depressions for different values of the Froude number $Fr$ and for various values of the inverse Weber number $\delta$. The effect of varying the Froude number which determines the nature of the fluid flow, namely $Fr > 1$ ; supercritical, $Fr < 1$ subcritical. Accuracy of numerical solutions depends on the grid spacing $\Delta$ and the domain truncations $N$. The calculations in this article are obtained with $\gamma = \beta = -\dfrac{\pi}{4}$ and performed with $N = 301$ and $\Delta = 0.15$. Figures **4** $-$ **6** illustrates the scheme of our calculations.

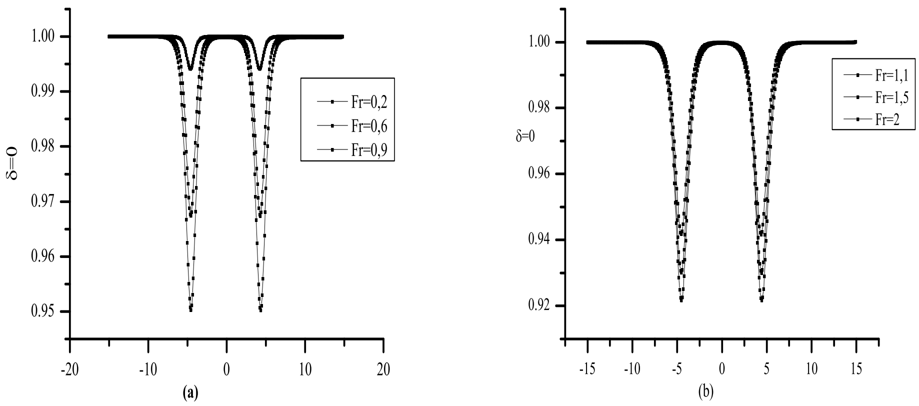


Figure 4. Typical free-surface profiles of flow for $\delta = 0$ (without surface tension) and various Froude number $F$

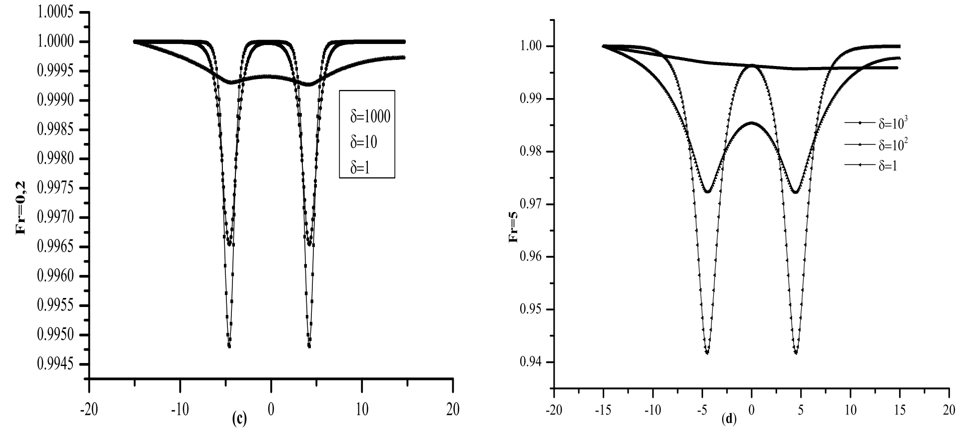


FIGURE 5. Typical free-surface profiles of subcritical and supercritical flow for (c)Fr=0.2 and various $\delta$ (d)Fr=5 and various $\delta$

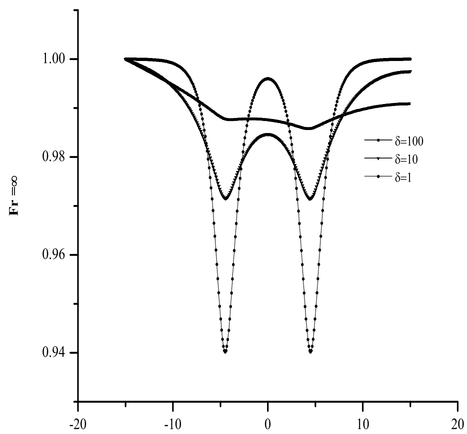


FIGURE 6. Typical free-surface profiles of flow for $Fr = \infty$ (without gravity) and various Weber number $\delta$

## 6. CONCLUSION

In this work, a boundary integral technique has been developed for 2D, inviscid, steady, irrotational free surface fluid flow under the influence of gravity and surface tension over two triangular depressions. The non-linear integral equations have been solved using a boundary integral method. The results obtained shows that for each fixed value $Fr$ and as the inverse Weber number increases, the shape of the free surface flattens and tends to a straight line.

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