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EXISTENCE OF SOLUTIONS FOR A HIGHER-ORDER BOUNDARY VALUE PROBLEM ON THE HALF-LINE VIA MONOTONE THEORY

M. BRIKI AND T. MOUSSAOUI

Abstract. In this paper, a higher-order boundary value problem on the half-line is considered and existence of solutions is proved by using the Minty-Browder monotone theorem.

1. Introduction

We consider the following higher-order boundary value problem posed on the half-line

$$\begin{cases} \displaystyle\sum_{k=0}^{m}(-1)^k u^{(2k)}(t) = f(t,u(t)), \quad t\in(0,+\infty), \\ u^{(2i)}(0)=u^{(2i)}(+\infty) = 0, \quad i\in\{0,1,.....,m-1\}, \end{cases} \tag{1.1}$$

where $f\in C([0,+\infty)\times\mathbb{R},\ \mathbb{R})$ and $m\in\mathbb{N}^\star$.
Boundary value problems appears in many mathematical models of physical, mathematical and biological phenomena, see ([7], [2], [9]). We refer to [6] in which using variational methods and critical point theory the existence of solutions for a class of Kirchhoff-type second-order impulsive differential equations on the half-line was discussed. In this paper, we will study the existence of solutions for a higher-order boundary value problem set on the half-line by using monotone theory.
We endow the following Sobolev space

$$H_0^m(0,+\infty)=\{u\in L^2 \mid u^{(i)}\in L^2 \text{ for } i\in\{1,..,m\},\ u^{(j)}(0)=0 \text{ for } j\in\{1,..,m-1\}\}$$

with its natural norm

$$\|u\|=\Big(\sum_{i=0}^{m}\|u^{(i)}\|^2_{L^2}\Big)^{\frac{1}{2}}.$$

Note that if $u\in H_0^m(0,+\infty)$, then $u^{(i)}(+\infty)=0$, for $i\in\{1,...,m-1\}$, (see [1]).
Let $p:[0,+\infty)\longrightarrow(0,+\infty)$ be a continuously differentiable and bounded function

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with

$$M = \max(\|p\|_{L^2}, \|p'\|_{L^2}) < +\infty.$$

We also consider the following Banach space

$$C_{l,p}[0,+\infty) = \{u \in C([0,+\infty),\mathbb{R}) : \lim_{t\to+\infty} p(t)u(t) \text{ exists } \}$$

endowed with the norm

$$\|u\|_{\infty,p} = \sup_{t\in[0,+\infty)} p(t)|u(t)|.$$

Now we recall some information from the literature needed in this paper.

Definition 1.1. [10] Let X be a Banach space. An operator $A : X \longrightarrow X^*$ which satisfies

$$\langle Au - Av, u - v\rangle \geq 0 \tag{1.2}$$

for any $u, v \in X$ is called a monotone operator. An operator A is called strictly monotone if for $u \neq v$ strict inequality holds in (1.2). An operator A is called strongly monotone if there exists $C > 0$ such that

$$\langle Au - Av, u - v\rangle \geq C\|u - v\|^2$$

for any $u, v \in X$. It is clear that a strongly monotone operator is strictly monotone.

Definition 1.2. [10] Let $A : X \longrightarrow X^*$ be an operator on the real Banach space X.

(a) A is said to be demicontinuous if

$$u_n \longrightarrow u \text{ as } n \longrightarrow +\infty \quad \textit{implies } Au_n \rightharpoonup Au \text{ as } n \longrightarrow +\infty.$$

(b) A is said to be hemicontinous if the real function

$$t \mapsto \langle A(u + tv), w\rangle \text{ is continuous on } [0, 1] \text{ for all } u, v, w \in X.$$

(c) A is said to be coercive if

$$\lim_{\|u\|\longrightarrow+\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty\cdot$$

Remark 1.3. [5] It is easy to see that for monotone operator $A : X \longrightarrow X^*$ with $Dom(A) = X$, demicontinuity and hemicontinuity are equivalent.

Finally in this section we recall the Minty-Browder Theorem.

Theorem 1.4. [8]*(Minty-Browder) Let X be a reflexive real Banach space. Let $A : X \longrightarrow X^*$ be an operator which is bounded, hemicontinous, coercive and monotone on the space X. Then, the equation $Au = f$ has at least one solution for each $f \in X^*$. If A is strictly monotone then the solution is unique.*

2. Variational setting and embedding results

Take $v \in H_0^m(0, +\infty)$, and multiply the equation in problem (1.1) by v and integrate over $(0, +\infty)$, so we get

$$\int_0^{+\infty} \big(\sum_{k=0}^{m}(-1)^k u^{(2k)}(t)\big)v(t)dt = \int_0^{+\infty} f(t, u(t))v(t)dt.$$

Hence

$$\sum_{k=0}^{m}\int_0^{+\infty} u^{(k)}(t)v^{(k)}(t)dt = \int_0^{+\infty} f(t, u(t))v(t)dt.$$

This leads to the natural concept of a weak solution for problem (1.1).

Definition 2.1. We say that a function $u \in H_0^m(0,+\infty)$ is a weak solution of problem (1.1) if

$$\sum_{k=0}^{m}\int_0^{+\infty} u^{(k)}(t)v^{(k)}(t)dt = \int_0^{+\infty} f(t,u(t))v(t)dt,$$

for all $v \in H_0^m(0,+\infty)$.?

We consider the following space

$$H_0^1(0,+\infty) = \{u \in L^2(0,+\infty) \mid u' \in L^2(0,+\infty), u(0) = 0\}$$

endowed with the norm

$$\|u\|_0 = \left(\|u\|_{L^2}^2 + \|u'\|_{L^2}^2\right)^{\frac{1}{2}}.$$

Lemma 2.2. [3, 4] *The embedding* $H_0^1(0,+\infty) \hookrightarrow C_{l,p}[0,+\infty)$ *is continuous, i.e*

$$\exists M \geq 0, \quad \|u\|_{p,\infty} \leq M\|u\|_0, \;\; \forall u \in H_0^1(0,+\infty).$$

Lemma 2.3. [3, 4] *The embedding* $H_0^1(0,+\infty) \hookrightarrow C_{l,p}[0,+\infty)$ *is compact.*

Lemma 2.4. *The embedding* $H_0^m(0,+\infty) \hookrightarrow C_{l,p}[0,+\infty)$ *is compact.*

Proof. We have the embedding $H_0^m(0,+\infty) \hookrightarrow H_0^1(0,+\infty)$ is continuous; also the embedding $H_0^1(0,+\infty) \hookrightarrow C_{l,p}[0,+\infty)$ is compact (Lemma 2.3), then the embedding $H_0^m(0,+\infty) \hookrightarrow C_{l,p}[0,+\infty)$ is compact. □

3. Main result

Suppose the following conditions hold:
$(H1)$ there exist functions a, b and there exists a constant $\sigma \in (0,1)$ with $\frac{a}{p^\sigma} \in L^2(0,+\infty), b \in L^2(0,+\infty)$ such that

$$|f(t,x)| \leq a(t)|x|^\sigma + b(t), \quad \forall\, t \in [0,+\infty), \forall\, x \in \mathbb{R}.$$

$(H2)$ $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is decreasing with respect to the second variable, i.e.,

$$f(t,x_1) \leq f(t,x_2) \quad \text{for a.e. } t \in [0,+\infty) \quad \text{and } x_1, x_2 \in \mathbb{R}, x_1 \geq x_2.$$

Let A be the operator defined from $H_0^m(0,+\infty)$ into $(H_0^m(0,+\infty))^*$ by

$$A = I - F,$$

where

$$\langle I(u), v\rangle = \sum_{k=0}^{m}\int_0^{+\infty} u^{(k)}(t)v^{(k)}(t)dt$$

and

$$\langle F(u), v\rangle = \int_0^{+\infty} f(t,u(t))v(t)dt.$$

We search for a weak solution of problem (1.1) which is a function $u \in H_0^m(0,+\infty)$ that satisfies the operator equation $Au = 0$.

Theorem 3.1. *Assume that f satisfies the hypotheses $(H1)$ and $(H2)$. Then problem (1.1) has a unique weak solution.*

Proof. We divide our proof into five steps.

Step 1: A is bounded.

Note that the functional

$$\psi(u)=\frac{1}{2}\Big(\sum_{i=0}^{m}\|u^{(i)}\|_{L^2}^2\Big)=\frac{1}{2}\|u\|^2$$

is of class C^1 and I is the derivative operator of ψ in the weak sense, so I is continuous.

Let $u\in H_0^m(0,+\infty)$ be such that $\|u\|\le R$. Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\|I(u)\|_{(H_0^m(0,+\infty))^*} &= \sup_{\|v\|\le 1}\Big|\langle I(u),v\rangle\Big| \\
&= \sup_{\|v\|\le 1}\Big|\Big(\sum_{k=0}^{m}\int_0^{+\infty}u^{(k)}(t)v^{(k)}(t)dt\Big)\Big| \\
&\le \sup_{\|v\|\le 1}\big(\sum_{k=0}^{m}(\|u^{(k)}\|_{L^2}\|v^{(k)}\|_{L^2}\big) \\
&\le (m+1)\|u\|\sup_{\|v\|\le 1}\big(\|v\|\big) \\
&\le (m+1)\|u\|\le (m+1)R,
\end{aligned}$$

and

$$\begin{aligned}
\|F(u)\|_{(H_0^m(0,+\infty))^*} &= \sup_{\|v\|\le 1}\Big|\langle F(u),v\rangle\Big| = \sup_{\|v\|\le 1}\Big|\int_0^{+\infty}f(t,u(t))v(t)dt\Big| \\
&\le \sup_{\|v\|\le 1}\Big(\int_0^{+\infty}a(t)|u(t)|^{\sigma}|v(t)|dt+\int_0^{+\infty}b(t)|v(t)|dt\Big) \\
&= \sup_{\|v\|\le 1}\Big(\int_0^{+\infty}\frac{a(t)}{p^{\sigma}(t)}p^{\sigma}(t)|u(t)|^{\sigma}|v(t)|dt+\int_0^{+\infty}b(t)|v(t)|dt\Big) \\
&\le \sup_{\|v\|\le 1}\Big[\|u\|_{p,\infty}^{\sigma}\Big(\int_0^{+\infty}\frac{a(t)}{p^{\sigma}(t)}|v(t)|dt\Big)+\Big(\int_0^{+\infty}b(t)|v(t)|dt\Big)\Big] \\
&\le \sup_{\|v\|\le 1}\Big[M^{\sigma}\|u\|^{\sigma}\Big(\int_0^{+\infty}\frac{a^2(t)}{p^{2\sigma}(t)}dt\Big)^{\frac{1}{2}}\Big(\int_0^{+\infty}|v(t)|^2dt\Big)^{\frac{1}{2}} \\
&+ \Big(\int_0^{+\infty}b^2(t)dt)\Big)^{\frac{1}{2}}\Big(\int_0^{+\infty}|v(t)|^2dt\Big)^{\frac{1}{2}}\Big]
\end{aligned}$$

$$\begin{aligned}
&\le \sup_{\|v\|\le 1}\|v\|\Big[M^{\sigma}\|u\|^{\sigma}\Big(\int_0^{+\infty}\frac{a^2(t)}{p^{2\sigma}(t)}dt\Big)^{\frac{1}{2}}+\Big(\int_0^{+\infty}b^2(t)dt)\Big)^{\frac{1}{2}}\Big] \\
&\le M^{\sigma}\|u\|^{\sigma}\|\frac{a}{p^{\sigma}}\|_{L^2}+\|b\|_{L^2} \\
&\le M^{\sigma}R^{\sigma}\|\frac{a}{p^{\sigma}}\|_{L^2}+\|b\|_{L^2}.
\end{aligned}$$

Hence A is bounded.

Step 2 : A is demicontinuous.

We prove that F is strongly continuous, that is, if $u_n\rightharpoonup u$ then $F(u_n)\longrightarrow F(u)$.

Let (u_n) be such that $u_n \rightharpoonup u$ in $H_0^m(0,+\infty)$. Now (u_n) is bounded in $H_0^m(0,+\infty)$ and by Lemma 2.4, we have that (u_n) is bounded in $C_{l,p}[0,+\infty)$. By Lemma 2.4, $u_n \longrightarrow u$ in $C_{l,p}[0,+\infty)$. We have

$$\begin{aligned}
\|F(u)-F(u_n)\|_{(H_0^m(0,+\infty))^*} &= \sup_{\|v\|\le 1}\Big|\Big\langle F(u)-F(u_n),v\Big\rangle\Big| \\
&= \sup_{\|v\|\le 1}\Big|\int_0^{+\infty}\big(f(t,u(t))-f(t,u_n(t))\big)v(t)dt\Big| \\
&\le \sup_{\|v\|\le 1}\Big(\int_0^{+\infty}|f(t,u(t))v(t)|dt\Big) \\
&+ \sup_{\|v\|\le 1}\Big(\int_0^{+\infty}|f(t,u_n(t))v(t)|dt\Big) \\
&\le \sup_{\|v\|\le 1}\Big(\int_0^{+\infty}(a(t)|u(t)|^\sigma+b(t))|v(t)|dt\Big) \\
&+ \sup_{\|v\|\le 1}\Big(\int_0^{+\infty}(a(t)|u_n(t)|^\sigma+b(t))|v(t)|dt\Big) \\
&\le M^\sigma\|\frac{a}{p^\sigma}\|_{L^2}\big(\|u\|^\sigma+\|u_n\|^\sigma\big)+2\|b\|_{L^2} \\
&\le M^\sigma\|\frac{a}{p^\sigma}\|_{L^2}C+2\|b\|_{L^2},
\end{aligned}$$

for some constant $C>0$. Since $u_n \longrightarrow u$ in $C_{l,p}[0,+\infty)$, we obtain

$$\int_0^{+\infty}(f(t,u(t))-f(t,u_n(t)))v(t)dt \longrightarrow 0 \ \text{ as } \ n\longrightarrow +\infty.$$

Thus F is strongly continuous and therefore it is continuous. From the fact that I is continuous, we deduce that the operator A is continuous. Thus A is demicontinuous.

Step 3 : A is monotone.

Note that

$$\langle I(u)-I(v),u-v\rangle = \|u-v\|^2, \tag{3.1}$$

so, I is strongly monotone.

Also, since f is decreasing with respect to the second variable,

$$\langle F(u)-F(v),u-v\rangle = \int_0^{+\infty}(f(t,u(t))-f(t,v(t)))(u(t)-v(t))dt \le 0, \tag{3.2}$$

so A is strongly monotone.

Step 4 : A is a coercive operator.

We have

$$\begin{aligned}
\frac{1}{\|u\|}\langle A(u),u\rangle &= \frac{1}{\|u\|}\Big[\|u\|^2-\int_0^{+\infty} f(t,u(t))u(t)dt\ \Big]\\
&\geq \frac{1}{\|u\|}\Big[\|u\|^2-\Big(\int_0^{+\infty} a^2(t)|u(t)|^{2\sigma}dt\Big)^{\frac{1}{2}}\Big(\int_0^{+\infty}|u(t)|^2dt\Big)^{\frac{1}{2}}\\
&- \int_0^{+\infty} b(t)|u(t)|dt\Big]\\
&\geq \frac{1}{\|u\|}\Big[\|u\|^2-\Big(\int_0^{+\infty}\frac{a^2(t)}{p^{2\sigma}(t)}p^{2\sigma}(t)|u(t)|^{2\sigma}dt\Big)^{\frac{1}{2}}\|u\|_{L^2}\\
&- \|b\|_{L^2}\|u\|_{L^2}\Big]\\
&\geq \frac{1}{\|u\|}\Big[\|u\|^2-\|\frac{a}{p^\sigma}\|_{L^2}\|u^\sigma\|_{\infty,p}\|u\|-\|b\|_{L^2}\|u\|\Big]\\
&\geq \frac{1}{\|u\|}\Big[\|u\|^2-M^\sigma\|\frac{a}{p^{\,\sigma}}\|_{L^2}\|u\|^{\sigma+1}-\|b\|_{L^2}\|u\|\Big]\\
&= \|u\|-M^\sigma\|\frac{a}{p^{\,\sigma}}\|_{L^2}\|u\|^{\sigma}-\|b\|_{L^2},
\end{aligned}$$

so A is coercive.
Theorem 1.4 guarantees that problem (1.1) has a weak solution.

Step 5 : Uniqueness.
Let $u,v\in H_0^m(0,+\infty)$ be such that $u\neq v$. From (3.1) and (3.2), it follows that

$$\langle A(u)-A(v),u-v\rangle\geq\|u-v\|^2>0,$$

so A is strictly monotone. □

Example 3.2. Consider the higher-order boundary value problem

$$\begin{cases}\displaystyle\sum_{k=0}^{m}(-1)^k u^{(2k)}(t) = -3u^{\frac{1}{3}}(t)e^{-t}+\tfrac{1}{1+4t}, \quad t\in(0,+\infty),\\ u^{(2i)}(0)=u^{(2i)}(+\infty) = 0, \quad i\in\{0,1,.....,m-1\}.\end{cases}\tag{3.3}$$

All conditions of Theorem 3.1 are satisfied with $f(t,x)=-3x^{\frac{1}{3}}e^{-t}+\frac{1}{1+4t}$, $\sigma=1/3$, $a(t)=3e^{-t}$, $b(t)=\frac{1}{1+4t}$, $p(t)=e^{-t}$. Therefore problem (3.3) has a unique weak solution.

Next, we consider the limit case $\sigma=1$.

Theorem 3.3. *Assume that $(H2)$ holds both with*
$(H'1)$ there exist functions a, b with $\frac{a}{p}\in L^2(0,+\infty), b\in L^2(0,+\infty)$ such that

$$|f(t,x)|\leq a(t)|x|+b(t),\quad \forall\ t\in[0,+\infty),\forall\ x\in\mathbb{R}$$

with

$$M\|\frac{a}{p}\|_{L^2}<1.$$

Then problem (1.1) has a unique weak solution.

Proof. Arguing as in the proof of Theorem (3.1), one can check that A is a coercive operator. Indeed, under $(H'1)$, we have the estimates:

$$\begin{aligned}
\frac{1}{\|u\|}\langle A(u),u\rangle &= \frac{1}{\|u\|}\Big[\|u\|^2-\int_0^{+\infty} f(t,u(t))u(t)dt\ \Big]\\
&\geq \frac{1}{\|u\|}\Big[\|u\|^2-\Big(\int_0^{+\infty} a^2(t)|u(t)|^2dt\Big)^{\frac{1}{2}}\Big(\int_0^{+\infty}|u(t)|^2dt\Big)^{\frac{1}{2}}\\
&- \int_0^{+\infty} b(t)|u(t)|dt\Big]\\
&\geq \frac{1}{\|u\|}\Big[\|u\|^2-\Big(\int_0^{+\infty}\frac{a^2(t)}{p^2(t)}\cdot p^2(t)|u(t)|^2dt\Big)^{\frac{1}{2}}\|u\|_{L^2}\\
&- \|b\|_{L^2}\|u\|_{L^2}\Big]\\
&\geq \frac{1}{\|u\|}\Big[\|u\|^2-\|\frac{a}{p}\|_{L^2}\|u\|_{\infty,p}\|u\|-\|b\|_{L^2}\|u\|\Big]\\
&\geq \frac{1}{\|u\|}\Big[\|u\|^2-M\|\frac{a}{p}\|_{L^2}\|u\|^2-\|b\|_{L^2}\|u\|\Big]\\
&= \Big(1-M\|\frac{a}{p}\|_{L^2}\Big)\|u\|-\|b\|_{L^2},
\end{aligned}$$

so A is coercive.
Theorem 1.4 guarantees that problem (1.1) has a unique weak solution. □

Example 3.4. Consider the higher-order boundary value problem

$$\begin{cases}
\displaystyle\sum_{k=0}^{m}(-1)^k u^{(2k)}(t) = -e^{-2t}\sqrt{|u(t)|}+\tfrac{1}{1+3t}, & t\in(0,+\infty),\\
u^{(2i)}(0)=u^{(2i)}(+\infty) = 0, & i\in\{0,1,.....,m-1\}.
\end{cases}\tag{3.4}$$

All conditions of Theorem 3.3 are satisfied with $f(t,x)=-e^{-2t}\sqrt{|x|}+\frac{1}{1+3t}$, $a(t)=e^{-2t}$, $b(t)=\frac{1}{1+3t}$, $p(t)=e^{-t}, M=\frac{1}{\sqrt{2}}$. Therefore problem (3.4) has a unique weak solution.

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M. Briki
Laboratory of Fixed Point Theory and Applications, Department of Mathematics, E.N.S., Kouba, Algiers, Algeria
E-mail address: mabroukbriki@yahoo.fr

T. Moussaoui
Laboratory of Fixed Point Theory and Applications, Department of Mathematics, E.N.S., Kouba, Algiers, Algeria
E-mail address: moussaoui@ens-kouba.dz