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ON SOME SPECTRAL PROPERTIES OF A NONLOCAL BOUNDARY VALUE PROBLEM

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ABSTRACT. This paper is devoted to study the existence of solution of second-order differential equation with a nonlocal conditions. The existence of the maximal and minimal solutions are studied. As an application the existence of eigenvalues and eigenfunctions are also studied.

1. INTRODUCTION

Nonlocal boundary value problems were first considered by Bitsadze and Samarskii [1] and later by Il'in and Moiseev [8],[9]. We refer the reader to ([2],[3] and [12]) for recent results of nonlocal boundary value problems. It is well-known that many topics in mathematical physics require the investigation of the eigenvalues and eigenfunctions of Sturm-Liouville type boundary value problems. Eigenvalue problems with nonlocal conditions are closely linked with boundary problems for differential equations with nonlocal conditions ([10],[11]). Eigenvalue problems for differential operators with nonlocal conditions are considerably less investigated than the classical boundary condition cases.
Recently, in ([4]-[7]) the authors investigated the existence and some asymptotic properties of the eigenvalues and eigenfunctions of the nonlocal boundary value problem of the Sturm-Liouville differential equation.
Consider the nonlocal boundary value problem

$$x''(t) + f(t, x(t)) = 0, \ t \in (0, \pi). \tag{1}$$

with the nonlocal conditions

$$x(\eta) = 0, \ \eta \in [0, \pi), \ x(\xi) = 0, \ \xi \in (0, \pi]. \tag{2}$$

under the assumptions.
Now let f satisfies the assumptions:
(1) $f : [0, \pi] \times R^+ \to R^+$ be Carathèodory function i.e
(i) $x \to f(t, x)$ is continuous for all $t \in [0, \pi]$,
(ii) $t \to f(t, x)$ measurable for each $x \in R$,

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(2) there exists $m \in L^1[0,\pi]$ such that

$$f(t,x) \leq m(t) + \lambda^2 |x|,$$

In this work we study the existence of at least one solution for the nonlocal boundary value problem of the second-order differential equation (1) and (2).

As an application, we study the existence and properties of the eigenvalues and eigenfunctions of the differential equation

$$x''(t) + m(t) = -\lambda^2 x, \ t \in (0,\pi),$$

with the nonlocal boundary conditions (2).

2. Existence of solution

Lemma 1 The solution of the problem (1) and (2) can be represent by the integral equation

$$\begin{aligned} x(t) &= \frac{(\xi - t)}{\xi - \eta}\int_0^\eta (\eta - s) f(s, x(s))ds + \frac{(t-\eta)}{\xi - \eta}\int_0^\xi (\xi - s) f(s,x(s))ds \\ &- \int_0^t (t-s) f(s,x(s))ds. \end{aligned} \tag{3}$$

Proof. Integral the both sides of equation (1) twice, we obtain

$$x(t) + x(0) + tx'(0) = -\int_0^t (t-s) f(s,x(s))ds, \tag{4}$$

when $x(\eta) = 0$ we get

$$x(0) = \int_0^\eta (\eta - s) f(s,x(s))ds - \eta x'(0), \tag{5}$$

also from $x(\xi) = 0$ we have

$$x(0) = \int_0^\xi (\xi - s) f(s,x(s))ds - \xi x'(0), \tag{6}$$

by substituting (5) into (6), we get

$$x'(0) = \frac{1}{\xi - \eta}\left[\int_0^\xi (\xi - s) f(s,x(s))ds - \int_0^\eta (\eta - s) f(s,x(s))ds\right], \tag{7}$$

putting (7) into (5) we get

$$x(0) = \frac{\xi}{\xi - \eta}\int_0^\eta (\eta - s) f(s,x(s))ds - \frac{\eta}{\xi - \eta}\int_0^\xi (\xi - s) f(s,x(s))ds. \tag{8}$$

then we get the required the formula (3).

Definition 1 The function x is called a solution of the integral equation (3), if $x \in C[0,\pi]$ and satisfies (3). For the existence of the solution we have the following theorem.

Theorem 1 Assume f is a Carathèodory function,

$$If \ \lambda^2 < \frac{\xi - \eta}{\pi^2(3\pi - \xi + \eta)}.$$

then the nonlocal boundary value problem (1) and (2) has at least one solution $x \in C[0, \pi]$.

Proof. . Define a subset $S \subset C[0, \pi]$ by

$$S = \{x \in C : \|x\| \leq r\},$$

The set S is nonempty, closed and convex.

Let $x \in S$, we have

$$\begin{aligned}
Fx(t) &= \frac{\xi - t}{\xi - \eta} \int_0^\eta (\eta - s) f(s, x(s)) ds + \frac{t - \eta}{\xi - \eta} \int_0^\xi (\xi - s) f(s, x(s)) ds \\
&- \int_0^t (t - s) f(s, x(s)) ds. \\
|Fx(t)| &= |\frac{\xi - t}{\xi - \eta} \int_0^\eta (\eta - s) f(s, x(s)) ds + \frac{t - \eta}{\xi - \eta} \int_0^\xi (\xi - s) f(s, x(s)) ds \\
&- \int_0^t (t - s) f(s, x(s)) ds| \\
&\leq |\frac{\xi - t}{\xi - \eta}|| \int_0^\eta (\eta - s) f(s, x(s)) ds| + |\frac{t - \eta}{\xi - \eta}|| \int_0^\xi (\xi - s) f(s, x(s)) ds| \\
&+ | \int_0^t (t - s) f(s, x(s)) ds| \\
&\leq \pi |\frac{\xi - t}{\xi - \eta}| \left\{ \int_0^\pi |m(s)| + \lambda^2 |x(s)| ds \right\} + \pi |\frac{t - \eta}{\xi - \eta}| \left\{ \int_0^\pi |m(s)| + \lambda^2 |x(s)| ds \right\} \\
&+ \pi \left\{ \int_0^\pi |m(s)| + \lambda^2 |x(s)| ds \right\} \\
&\leq \frac{[3\pi^2 + \pi(\xi - \eta)]}{\xi - \eta} \left\{ \int_0^\pi |m(s)| + \lambda^2 |x(s)| ds \right\} \\
&\leq \frac{[3\pi^2 + \pi(\xi - \eta)]}{\xi - \eta} \left\{ \|m\| + \lambda^2 \|x\| \pi \right\} \leq r.
\end{aligned}$$

Then $Fx \in S$ and is uniformly bounded in S.

In what follows we show that F is equi-continuous operator.

For $t_1, t_2 \in (0,\pi),\ t_1 < t_2$ such that $|t_2 - t_1| < \delta$ we have

$$\begin{aligned}
Fx(t_2) - Fx(t_1) &= \frac{\xi - t_2}{\xi - \eta}\int_0^{\eta}(\eta - s)f(s,x(s))ds \\
&+ \frac{t_2 - \eta}{\xi - \eta}\int_0^{\xi}(\xi - s)f(s,x(s))ds - \int_0^{t_2}(t_2 - s)f(s,x(s))ds \\
&- \frac{\xi - t_1}{\xi - \eta}\int_0^{\eta}(\eta - s)f(s,x(s))ds - \frac{t_1 - \eta}{\xi - \eta}\int_0^{\xi}(\xi - s)f(s,x(s))ds \\
&+ \int_0^{t_1}(t_1 - s)f(s,x(s))ds, \\
&= \int_0^{t_1}(t_1 - s)f(s,x(s))ds - \int_0^{t_2}(t_2 - s)f(s,x(s))ds \\
&+ \frac{t_2 - t_1}{\xi - \eta}\int_0^{\xi}(\xi - s)f(s,x(s))ds + \frac{t_1 - t_2}{\xi - \eta}\int_0^{\eta}(\eta - s)f(s,x(s))ds, \\
|Fx(t_2) - Fx(t_1)| &= |\int_{t_1}^{t_2}(t_2 - s)f(s,x(s))ds - \frac{t_2 - t_1}{\xi - \eta}\int_0^{\eta}(\eta - s)f(s,x(s))ds \\
&+ \frac{t_2 - t_1}{\xi - \eta}\int_0^{\xi}(\xi - s)f(s,x(s))ds|, \\
|Fx(t_2) - Fx(t_1)| &\leq \int_{t_1}^{t_2}(t_2 - s)\{|m(s)| + \lambda^2|x(s)|\}ds \\
&+ |\frac{t_2 - t_1}{\xi - \eta}|\int_0^{\eta}(\eta - s)\{|m(s)| + \lambda^2|x(s)|\}ds \\
&+ |\frac{t_2 - t_1}{\xi - \eta}|\int_0^{\xi}(\xi - s)\{|m(s)| + \lambda^2|x(s)|\}ds.
\end{aligned}$$

Hence the class of function FS is equi-continuous. By Arzela-Ascolis Theorem, we fined that FS is relatively compact.
Now we prove that $F : S \to S$ is continuous.
Let $x_n \subset S$, such that $x_n \to x_o$,

$$\begin{aligned}
Fx_n(t) &= \frac{\xi - t}{\xi - \eta}\int_0^{\eta}(\eta - s)f(s,x_n(s))ds + \frac{t - \eta}{\xi - \eta}\int_0^{\xi}(\xi - s)f(s,x_n(s))ds \\
&- \int_0^{t}(t - s)f(s,x_n(s))ds.
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} Fx_n(t) &= \lim_{n\to\infty}\{\frac{\xi - t}{\xi - \eta}\int_0^{\eta}(\eta - s)f(s,x_n(s))ds \\
&+ \frac{t - \eta}{\xi - \eta}\int_0^{\xi}(\xi - s)f(s,x_n(s))ds - \int_0^{t}(t - s)f(s,x_n(s))ds\}.
\end{aligned}$$

Now

$$f(s,x_n(s)) \to f(s,x_o(s)).$$

Applying Lebesgue Dominated convergence Theorem, we

$$\begin{aligned} Fx(t) = \lim_{n\to\infty} Fx_n(t) &= \frac{\xi - t}{\xi - \eta}\int_0^{\eta} (\eta - s) \lim_{n\to\infty} f(s, x_n(s))ds \\ &+ \frac{t-\eta}{\xi - \eta}\int_0^{\xi} (\xi - s) \lim_{n\to\infty} f(s, x_n(s))ds \\ &- \int_0^t (t - s) \lim_{n\to\infty} f(s, x_n(s))ds. \end{aligned}$$

$$\begin{aligned} Fx(t) &= \frac{\xi - t}{\xi - \eta}\int_0^{\eta} (\eta - s) f(s, x_o(s))ds + \frac{t-\eta}{\xi - \eta}\int_0^{\xi} (\xi - s) f(s, x_o(s))ds \\ &- \int_0^t (t - s) f(s, x_o(s))ds. \end{aligned}$$

Then $Fx_n(t) \to Fx_o(t)$.
Which means that the operator F is continuous.
Since all conditions of Schauder Theorem are hold, then F has a fixed point in S, integral equation (3) has at least one solution $x \in [0, \pi]$ consequently the nonlocal boundary value problem (1) and (2) has at least one solution $x \in C[0, \pi]$.

3. Maximal and minimal solutions

Here we study the existence of the maximal and minimal solutions of (3).
Definition 2 Let $q(t)$ be a solution of the integral equation(3).
Then $q(t)$ is said to be a maximal solution of (3) if every solution $x(t)$ of (3) satisfies the inequality.

$$x(t) < q(t), \ t \in [0, \pi]. \tag{9}$$

A minimal solution $s(t)$ can be defined by similar way by reversing the above inequality i.e

$$x(t) > s(t), \ t \in [0, \pi] \tag{10}$$

Consider the following lemma
Lemma 2 Let $f(t, x)$ be a continuous function on $[0, \pi]$ satisfying

$$\begin{aligned} x(t) &\le \frac{\xi - t}{\xi - \eta}\int_0^{\eta} (\eta - s) f(s, x(s))ds + \frac{t-\eta}{\xi - \eta}\int_0^{\xi} (\xi - s) f(s, x(s))ds \\ &- \int_0^t (t - s) f(s, x(s))ds, \ t \in [0, \pi] \end{aligned}$$

$$\begin{aligned} y(t) &\ge \frac{\xi - t}{\xi - \eta}\int_0^{\eta} (\eta - s) f(s, x(s))ds + \frac{t-\eta}{\xi - \eta}\int_0^{\xi} (\xi - s) f(s, x(s))ds \\ &- \int_0^t (t - s) f(s, x(s))ds, \ t \in [0, \pi] \end{aligned}$$

and one of them is strict.
If f are monotonic nondecreasing in x, then

$$x(t) < y(t),\ t > 0 \tag{11}$$

Proof. Let the conclusion (11) be false, then there exists t_1 such that

$$x(t_1) = y(t_1),\ t_1 > 0$$

and

$$x(t) < y(t),\ 0 < t < t_1.$$

From the monotonicity of f in x, we get

$$\begin{aligned} x(t) &\leq \frac{\xi - t}{\xi - \eta}\int_0^\eta (\eta - s) f(s, x(s)) ds + \frac{t-\eta}{\xi-\eta}\int_0^\xi (\xi - s) f(s, x(s)) ds \\ &- \int_0^t (t-s) f(s, x(s)) ds,\ t \in [0, \pi] \\ x(t) &< \frac{\xi - t}{\xi - \eta}\int_0^\eta (\eta - s) f(s, x(s)) ds + \frac{t-\eta}{\xi-\eta}\int_0^\xi (\xi - s) f(s, x(s)) ds \\ &- \int_0^t (t-s) f(s, x(s)) ds,\ t \in [0, \pi] \\ x(t_1) &< y(t_1) \end{aligned}$$

which contradicts the fact that $x(t_1) = y(t_1)$.

Then

$$x(t) < y(t).$$

For the existence of the maximal and minimal solutions we have the following theorem.

Theorem 2 Let the assumptions (1) and (2) of Theorem 1 be satisfied. If $f(t, x)$ is monotonic nondecreasing in x for each $t \in [0, \pi]$, then (3) has maximal and minimal solutions.

Proof. Firstly we shall prove the existence of the maximal solution of (3).
Let $\epsilon > 0$ be given, and consider the integral equation

$$\begin{aligned} x_\epsilon(t) \leq \frac{\xi - t}{\xi - \eta}\int_0^\eta (\eta - s) f_\epsilon(s, x_\epsilon(s)) ds + \frac{t-\eta}{\xi-\eta}\int_0^\xi (\xi - s) f_\epsilon(s, x_\epsilon(s)) ds \\ - \int_0^t (t-s) f_\epsilon(s, x_\epsilon(s)) ds,\ t \in [0, \pi], \end{aligned} \tag{12}$$

where

$$f_\epsilon(t, x_\epsilon(t)) = f(t, x_\epsilon(t)) + \epsilon.$$

In the view of Theorem 1, it is clear that equation (12) has at least one solution $x(t) \in C[0, \pi]$. Now, Let $\epsilon_1,\ \epsilon_2$ be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$, then

$$\begin{aligned} x_{\epsilon_2}(t) = \frac{\xi - t}{\xi - \eta}\int_0^\eta (\eta - s)\ f_{\epsilon_2}(s, x_{\epsilon_2}(s))\ ds + \frac{t-\eta}{\xi-\eta}\int_0^\xi (\xi - s)\ f_{\epsilon_2}(s, x_{\epsilon_2}(s))\ ds \\ - \int_0^t (t-s)\ f_{\epsilon_2}(s, x_{\epsilon_2}(s))\ ds \end{aligned}$$

$$= \frac{\xi - t}{\xi - \eta} \int_0^\eta (\eta - s)\ (f_{\epsilon_2}(s, x_{\epsilon_2}(s)) + \epsilon_2)\ ds + \frac{t - \eta}{\xi - \eta} \int_0^\xi (\xi - s)\ (f_{\epsilon_2}(s, x_{\epsilon_2}(s)) + \epsilon_2)\ ds$$
$$- \int_0^t (t - s)\ (f_{\epsilon_2}(s, x_{\epsilon_2}(s)) + \epsilon_2)\ ds \tag{13}$$

also

$$x_{\epsilon_1}(t) = \frac{\xi - t}{\xi - \eta} \int_0^\eta (\eta - s)\ f_{\epsilon_1}(s, x_{\epsilon_1}(s))\ ds + \frac{t - \eta}{\xi - \eta} \int_0^\xi (\xi - s)\ f_{\epsilon_1}(s, x_{\epsilon_1}(s))\ ds$$
$$- \int_0^t (t - s)\ f_{\epsilon_1}(s, x_{\epsilon_1}(s))\ ds$$
$$= \frac{\xi - t}{\xi - \eta} \int_0^\eta (\eta - s)\ (f_{\epsilon_1}(s, x_{\epsilon_1}(s)) + \epsilon_1)\ ds + \frac{t - \eta}{\xi - \eta} \int_0^\xi (\xi - s)\ (f_{\epsilon_1}(s, x_{\epsilon_1}(s)) + \epsilon_1)\ ds$$
$$- \int_0^t (t - s)\ (f_{\epsilon_1}(s, x_{\epsilon_1}(s)) + \epsilon_1)\ ds$$
$$x_{\epsilon_1}(t) > \frac{\xi - t}{\xi - \eta} \int_0^\eta (\eta - s)\ f(s, x_{\epsilon_1}(s))\ ds + \frac{t - \eta}{\xi - \eta} \int_0^\xi (\xi - s)\ f(s, x_{\epsilon_1}(s))\ ds$$
$$- \int_0^t (t - s)\ f(s, x_{\epsilon_1}(s))\ ds \tag{14}$$

Applying Lemma 2 on (13) and (14) we have

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t)\ for\ t \in [0, \pi].$$

As shown before, the family of functions $x_\epsilon(t)$ is equi-continuous and uniformly bounded. Hence, by Arzela-Ascoli Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon_n \to 0$ an $n \to \infty$, and $\lim_{nk \to \pi} x_{\epsilon_n}(t)$ exists uniformly in $[0, \pi]$ and denote this limit by $q(t)$.
From the continuity of the functions $f_\epsilon(t, x_\epsilon(t))$, we get

$$f_\epsilon(t, x_\epsilon(t)) \to f(t, x(t))\ as\ n \to \infty$$

and

$$q(t) = \lim_{n \to \infty} x_{\epsilon_n}(t) = \frac{\xi - t}{\xi - \eta} \int_0^\eta (\eta - s)\ q(s, x(s))\ ds + \frac{t - \eta}{\xi - \eta} \int_0^\xi (\xi - s)\ q(s, x(s))\ ds$$
$$- \int_0^t (t - s)\ q(s, x(s))\ ds$$

which implies that $q(t)$ is a solution of the integral equation (3).
Finally we shall show that $q(t)$ is the maximal solution of (3).

To do this let $x(t)$ be any solution of (3), then

$$x(t) = \frac{\xi - t}{\xi - \eta} \int_0^\eta (\eta - s)\ f(s, \lambda, x(s))\ ds + \frac{t - \eta}{\xi - \eta} \int_0^\xi (\xi - s)\ f(s, x(s))\ ds$$
$$- \int_0^t (t - s)\ f(s, x(s))\ ds \tag{15}$$

also

$$x_\epsilon(t) = \frac{\xi - t}{\xi - \eta} \int_0^\eta (\eta - s)\ f_\epsilon(s, x_\epsilon(s))\ ds + \frac{t - \eta}{\xi - \eta} \int_0^\xi (\xi - s)\ f_\epsilon(s, x_\epsilon(s))\ ds$$

$$- \int_0^t (t-s)\ f_\epsilon(s, x_\epsilon(s))\ ds$$

$$x_\epsilon(t) = \frac{\xi - t}{\xi - \eta} \int_0^\eta (\eta - s)\ (f(s, x_\epsilon(s)) + \epsilon)\ ds + \frac{t-\eta}{\xi - \eta} \int_0^\xi (\xi - s)\ (f(s, x_\epsilon(s)) + \epsilon)\ ds$$
$$- \int_0^t (t-s)\ (f(s, x_\epsilon(s)) + \epsilon)\ ds$$

$$x_\epsilon(t) > \frac{\xi - t}{\xi - \eta} \int_0^\eta (\eta - s)\ f(s, x_\epsilon(s))\ ds + \frac{t-\eta}{\xi - \eta} \int_0^\xi (\xi - s)\ f(s, x_\epsilon(s))\ ds$$
$$- \int_0^t (t-s)\ f(s, x_\epsilon(s))\ ds./ \tag{16}$$

Applying Lemma 2 on (15) and (16) we get

$$x(t)\ <\ x_\epsilon(t),\ for\ t \in\ [0, \pi].$$

From the uniqueness of the maximal solution, it is clear that $x_\epsilon(t)$ tends to $q(t)$ uniformly in $[0, \pi]$ as $\epsilon \to 0$.
By similar way as done above we can prove the existence of the minimal solution.
We set

$$f_\epsilon(t, x_\epsilon(t))\ =\ f(t, x_\epsilon(t)) - \epsilon,$$

which completes the proof of Theorem 2.

4. Eigenvalues and eigenfunctions

Here we study the existence and some general properties of the eigenvalues and eigenfunctions of the second-order differential equation

$$x''(t) + f(t, x(t)) = 0,$$

with

$$f(t, x(t)) = m(t) + \lambda^2 x,$$

where the function $m(t)$ is non-negative real valued function, λ is a spectral parameter. Then we have the the Sturm-Liouville differential equation

$$x'' + m(t) = -\lambda^2 x,\ 0 \le t \le \pi \tag{17}$$

with the nonlocal boundary conditions

$$x(\eta) = 0,\ \eta \in [0, \pi),\ x(\xi) = 0,\ \xi \in (0, \pi]. \tag{18}$$

5. General properties

Here we prove some results concerning the eigenvalues and eigenfunctions of the non-local problem (17)-(18).

Lemma 3 The eigenvalues of the non-local boundary value problem (17) and (18) satisfy the inequality

$$\lambda_0^2\ = \frac{\int_\eta^\xi\ [\ |x_0'|^2\ -\ m(t)\ \bar{x}\ ]\ dt}{\int_\eta^\xi\ |x_0|^2\ dt}.$$

Proof. Let $x_0(t)$ be the eigenfunction that corresponds to the eigenvalue λ_0 of the problem (17) and (18), then

$$x_0'' + m(t) \ = -\lambda_0^2 x_0 \ (0 \leq t \leq \pi), \tag{19}$$

and

$$x_0(\eta) = x_0(\xi) = 0 \tag{20}$$

Multiplying both sides of (19) by $\bar{x_0}$ and then integrating form η to ξ with respect to t, we have

$$\bar{x_0}\ \acute{x_0}\ |_\eta^\xi - \int_\eta^\xi |\acute{x_0}|^2\ dt + \int_\eta^\xi m(t)\ \bar{x} dt = -\lambda_0^2 \int_\eta^\xi |x_0|^2\ dt.$$

using the nonlocal boundary conditions (20), we have

$$\lambda_0^2 \ = \frac{\int_\eta^\xi \ [\ |\acute{x_0}|^2 \ - \ m(t)\ \bar{x}\ ]\ dt}{\int_\eta^\xi \ |x_0|^2\ dt}.$$

Corollary 1 let $m(t) = 0$ then, we have

$$\lambda_0^2 \ = \frac{\int_\eta^\xi \ |\acute{x_0}|^2\ dt}{\int_\eta^\xi \ |x_0|^2\ dt}.$$

Therefore the eigenvalues of the non-local boundary value problem (17) and (18) are real.

Lemma 4 The eigenfunctions that corresponds to two different eigenvalues of the non-local boundary value problem (17) and (18) satisfy the inequality

$$(\lambda_1^2 - \lambda_2^2) \int_\eta^\xi x_1(t)\ \bar{x}_2(t)\ dt \ = \int_\eta^\xi m(t)(x_1(t) - \bar{x}_2(t))dt.$$

Proof. Let $\lambda_1 \neq \lambda_2$ be two different eigenvalues of the non-local boundary value problem (17) and (18). Let $x_1(t), x_2(t)$ be the corresponding eigenfunctions, then

$$x_1'' + m(t) \ = \ -\lambda_1^2\ x_1\ (0 \leq t \leq \pi), \tag{21}$$

$$x_1(\eta) = x_1(\xi) = 0 \tag{22}$$

and

$$x_2'' + m(t) \ = \ -\lambda_2^2\ x_2\ (0 \leq t \leq \pi), \tag{23}$$

$$x_2(\eta) = x_2(\xi) = 0 \tag{24}$$

Multiplying both sides of (21) by $\bar{x}_2$ and integrating with respect to t, we obtain

$$\int_\eta^\xi x''{}_1\ \bar{x}_2\ dt + \int_\eta^\xi m(t)\ \bar{x}_2\ dt \ = \ -\lambda_1^2 \int_\eta^\xi x_1\ \bar{x}_2\ dt. \tag{25}$$

By taking the complex conjugate of (23) and multiply it by x_1 and integrate the resulting expression with respect to t, we have

$$\int_\eta^\xi x_1\ \bar{x}''{}_2\ dt + \int_\eta^\xi m(t)\ x_1\ dt \ = \ -\lambda_2^2 \int_\eta^\xi x_1\ \bar{x}_2\ dt. \tag{26}$$

Subtracting (26) from (25) and using the nonlocal boundary conditions of (22) and (24) we obtain

$$(\lambda_1^2-\lambda_2^2)\int_\eta^\xi x_1(t)\ \bar{x}_2(t)\ dt\ =\int_\eta^\xi m(t)(x_1(t)-\bar{x}_2(t))dt.$$

Corollary 2 Let $m(t)=0$ then, we have

$$(\lambda_1^2-\lambda_2^2)\int_\eta^\xi x_1(t)\ \bar{x}_2(t)dt=0,\ \lambda_1^2\neq\lambda_2^2.$$

Therefore the eigenfunctions that corresponds to two different eigenvalues of the non-local boundary value problem (17) and (18) are orthogonal.

6. The asymptotic formulas for the solution

Here we study the asymptotic formulas for the solutions of problem (17) and (18).
Lemma 3 deals with the nature the eigenvalues. Let be $\phi(t,\lambda)$ the solution of equation (17) and (18) satisfying the initial conditions

$$\phi(\eta,\lambda)=0,\ \phi'(\eta,\lambda)=1 \tag{27}$$

and by $\vartheta(t,\lambda)$ the solution of the same equation, satisfying the initial conditions

$$\vartheta(\eta,\lambda)=1,\ \vartheta'(\eta,\lambda)=0 \tag{28}$$

We notes that $\phi(t,\lambda)$ and $\vartheta(t,\lambda)$ are linearly independent if and only if $\omega(\lambda)\neq 0$.

$$\omega(\lambda)=\phi(t,\lambda)\vartheta'(t,\lambda)-\phi'(t,\lambda)\vartheta(t,\lambda).$$

The characteristic equation will be

$$\omega(\lambda)=\phi(\xi,\lambda) \tag{29}$$

Lemma 5 The solution $\phi(t,\lambda)$ of problem (17) and (18) satisfy the integral equations

$$\phi(t,\lambda)=\frac{\sin\lambda(t-\eta)}{\lambda}+\int_\eta^t\frac{\sin\lambda(t-\tau)}{\lambda}m(\tau)d\tau. \tag{30}$$

Proof. First we obtain formula (30) Indeed,with solution of the form $m(t)=0$. (17) becomes $x''=-\lambda^2x$ by means of variation of parameter method, we have

$$\phi(t,\lambda)=C_1(t,\lambda)\cos\lambda t+C_2(t,\lambda)\sin\lambda t \tag{31}$$

and the direct calculation of $C_1(t,s)$ and $C_2(t,s)$, we have

$$\begin{aligned} C_1(t,\lambda)\ &=-\frac{\sin\lambda\eta}{\lambda}+\int_\eta^t\frac{\sin\lambda\tau}{\lambda}m(\tau)d\tau,\\ C_2(t,\lambda)\ &=\frac{\cos\lambda\eta}{\lambda}-\int_\eta^t\frac{\cos\lambda\tau}{\lambda}m(\tau)d\tau.\end{aligned} \tag{32}$$

substituting from (32) into (31) equation (30) follows. Second we show that the integral representation (30) satisfies the problem (17) and (18). Let $\varphi(t,\lambda)$ be the solution of (17), so that

$$-m(t)\ =\phi''(t,\lambda)+\lambda^2\phi(t,\lambda).$$

We multiply both sides by

$$\frac{\sin\lambda(t-\tau)}{\lambda}$$

and integrating with respect to τ from η to x we obtain

$$-\int_{\eta}^{t}\frac{\sin\lambda(t-\tau)}{\lambda}\ m(\tau)\ d\tau=\int_{\eta}^{t}\frac{\sin\lambda(t-\tau)}{\lambda}\ \phi''(\tau,\lambda)\ d\tau \tag{33}$$

$$+\lambda^{2}\int_{\eta}^{t}\frac{\sin\lambda(t-\tau)}{\lambda}\ \phi(\tau,\lambda)\ d\tau.$$

Integrating by parts twice and using the condition (27), we have

$$\int_{\eta}^{t}\frac{\sin\lambda(t-\tau)}{\lambda}\ \phi''(\tau,\lambda)\ d\tau \tag{34}$$

$$=\phi(t,\lambda)-\frac{\sin\lambda(t-\eta)}{\lambda}-\lambda\int_{\eta}^{t}\sin\lambda(t-\tau)\ \phi(\tau,\lambda)\ d\tau.$$

By substituting from (34) into (33) we get the required formula (30).

Lemma 6 If $\lambda=\sigma+is$. Then the following inequality for the solution $\phi(t,\lambda)$ of the nonlocal boundary value problem (17) and (18) holds true

$$\phi(x,\lambda)=\frac{\sin\lambda(t-\eta)}{\lambda}+O\left(\frac{e^{|Im\lambda|(t-\eta)}}{|\lambda|}\right), \tag{35}$$

Proof. we show that

$$\phi(t,\lambda)=O\left(\frac{e^{|s|(t-\eta)}}{\lambda}\right).$$

where the inequality is uniformly with respect to t. Form the integral equation (30) we have

$$|\phi(t,\lambda)|\leq\frac{|\sin\lambda(t-\eta)|}{|\lambda|}+\frac{1}{|\lambda|}\int_{\eta}^{t}|\sin\lambda(t-\tau)||m(\tau)|\ d\tau. \tag{36}$$

By using the notation $\phi(t,\lambda)e^{-|s|(t-\eta)}=F(t,\lambda)$, equation (36) takes the form

$$|F(t,\lambda)|\leq\frac{1}{|\lambda|}+\frac{1}{|\lambda|}\int_{0}^{\xi}|m(\tau)|e^{-(\tau-\eta)|s|}\ d\tau. \tag{37}$$

Let$\mu=\max\limits_{0\leq t\leq\pi}F(t,\lambda)$, we have from (37) that for $|\lambda|\geq 1$, $t\in[0,\pi]$

$$\mu(t)\ \leq\frac{C}{|\lambda|}, \tag{38}$$

and consequently

$$\phi(t,\lambda)=\ O\left(\frac{e^{|s|(t-\eta)}}{|\lambda|}\right). \tag{39}$$

From (30) and (39) it follows that, $\varphi(t,\lambda)$ has the asymptotic formula (35). Therefore we obtain the following equation for the determination of the eigenvalues: Equation (35) is the characteristic equation which gives roots of λ

$$\lambda_{n}=\frac{n\pi}{(\xi-\eta)},\ n=\pm1,\pm2,... \tag{40}$$

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