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SOME INEQUALITIES FOR THE RATIO OF CONFLUENT HYPERGEOMETRIC FUNCTION OF THE SECOND KIND

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Abstract. In this paper, the authors establish some inequalities for the ratio of the Confluent hypergeometric function of the second kind using the method of analysis and theory of inequality and the integral representation. We also establish some completely monotonic functions involving the ratio of confluent hypergeometric function of the second kind.

1. Introduction

In this paper, we study the completely monotonic properties for the confluent hypergeometric functions of the second kind denoted by $\psi(a, c, x)$ also known as Tricomi confluent hypergeometric functions. This function is a particular solution of Kummers differential equation [1, p. 504].

$$xy''(x) + (c - x)y'(x) - ay(x) = 0,$$

and for $a > 0, c \in \mathbb{R}$, and $x > 0$ has the following integral representation [1, p. 505]

$$\psi(a, c, x) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1}(1 + t)^{c-a-1} e^{-xt} \mathrm{dt} \tag{1}$$

We recall that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I which alternate successively in sign, that is,

$$(-1)^n f^{(n)}(x) \geq 0 \tag{2}$$

for all $x \in I$ and for all $n \geq 0$. If inequality (2) is strict for all $x \in I$ and all $n \geq 0$, then f is said to be strictly completely monotonic. Completely monotonic functions play an eminent role in areas like mathematical analysis [5], probability theory [3], numerical analysis [4], physics [2], and the theory of special functions (see [6] - [22] and references therein). The celebrated Bernstein-Widder's Theorem [5] characterizes that a necessary and sufficient conditions that f should be completely monotonic for $0 < x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} \mathrm{d}\alpha(\mathrm{t}) \tag{3}$$

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where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. This expresses that a completely monotonic function on $(0, \infty)$ is a Laplace transform of the measure α. The Stieltjes-type transforms [19]

$$F(x) = \int_0^\infty \frac{f(t)\mathrm{dt}}{(x+t)^\rho}$$

are completely monotonic for any non-negative function f under the convergence conditions of the integral. In this paper, we prove our main results based on the integral representation of quotients of the confluent hypergeometric function of the second kind [25] and ideas from [23] and [24].

2. Inequalities involving ratio of confluent hypergeometric function of the second kind

Theorem 1 If $a > 0, 2 < c < a+1$ and $x > 0$ then the following inequality is valid

$$\psi(a, c-1, x) < \left(\frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1}\right) x\psi(a, c, x) \tag{4}$$

Proof. Consider the integral representation [24] for $|\arg z| < \pi$

$$\frac{\psi(a, c-1, x)}{\psi(a, c, x)} = \int_0^\infty \frac{x}{x+t} \Phi_{a,c}(t)\mathrm{dt} \tag{5}$$

where,

$$\Phi_{a,c}(t) = \frac{t^{-c}e^{-t}|\psi(a, c, te^{i\pi})|^{-2}}{\Gamma(a)\Gamma(a-c+2)}, a > 0, 1 < c < a+1$$

by equation (5), we have

$$\lim_{x\to 0} \frac{\psi(a, c-1, x)}{\psi(a, c, x)} = 0 \tag{6}$$

and

$$\left(\frac{\psi(a, c-1, x)}{\psi(a, c, x)}\right)' = \int_0^\infty \frac{t}{(x+t)^2} \Phi_{a,c}(t)\mathrm{dt} \tag{7}$$

Now,

$$\begin{aligned}
\left(\frac{\psi(a, c-1, x)}{\psi(a, c, x)}\right)' &= \frac{\psi(a, c, x)\psi'(a, c-1, x) - \psi(a, c-1, x)\psi'(a, c, x)}{\psi^2(a, c, x)} \\
&= a\left(\frac{\psi(a+1, c+1, x)\psi(a, c-1, x) - \psi(a+1, c, x)\psi(a, c, x)}{\psi^2(a, c, x)}\right)
\end{aligned}$$

and in view of the asymptotic expansion [1, p. 508]

$$\psi(a, c, x) \sim \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c}, c > 1, x \to 0. \tag{8}$$

we obtain that

$$\left(\frac{\psi(a, c-1, x)}{\psi(a, c, x)}\right)' \sim a\left(\frac{\Gamma(a)}{\Gamma(c-1)}\right)^2 \left(\frac{\Gamma(c)\Gamma(c-2)}{\Gamma(a)\Gamma(a+1)} - \frac{\Gamma^2(c-1)}{\Gamma(a)\Gamma(a+2)}\right)$$

where $a > 0$ and $2 < c < a+1$ are fixed and $x \to 0$. After some algebra we get

$$\left(\frac{\psi(a, c-1, x)}{\psi(a, c, x)}\right)' \sim \frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1} \tag{9}$$

where $a > 0$ and $2 < c < a + 1$ .Now, using (5) we obtain

$$\lim_{x\to 0}\left(\frac{\psi(a,c-1,x)}{\psi(a,c,x)}\right)' = \int_0^\infty \frac{\Phi_{a,c}(t)}{t}\mathrm{dt}$$

thus we have

$$\int_0^\infty \frac{\Phi_{a,c}(t)}{t}\mathrm{dt} = \frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1} \tag{10}$$

where $a > 0$ and $2 < c < a+1$. Now in view of (10) and the inequality $(x+t)^2 > t^2$ we have

$$\begin{aligned}\left(\frac{\psi(a,c-1,x)}{\psi(a,c,x)}\right)' &= \int_0^\infty \frac{x}{x+t}\Phi_{a,c}(t)\mathrm{dt} < \int_0^\infty \frac{\Phi_{a,c}(t)}{t}\mathrm{dt} = \frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1} \\ &< \left(\left(\frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1}\right)x\right)'\end{aligned}$$

thus the function

$$x \to \frac{\psi(a,c-1,x)}{\psi(a,c,x)} - \left(\frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1}\right)x$$

is strictly decreasing on $(0,\infty)$ for $a > 0$ and $2 < c < a+1$. Thus we have inequality (4). □

Theorem 2 If $a > 0, 2 < c < a + 1$ and $0 < x < 1$ then the following inequality is valid

$$\left(\frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1}\right) < \frac{\psi(a,c-1,x)}{\psi(a,c,x)} - \frac{1}{2}\ln(x) \tag{11}$$

and the inequality (11) reversed if $x \geq 1$.

Proof. For $a > 0, 1 < c < a + 1$ and $x > 0$ we have

$$\frac{\psi(a,c-1,x)}{\psi(a,c,x)} = \int_0^\infty \frac{x}{x+t}\Phi_{a,c}(t)\mathrm{dt} \tag{12}$$

For $a > 0, 2 < c < a + 1$, $x > 0$, (10) and the inequality $(1 + t) > t$ we have

$$\lim_{x\to 1}\frac{\psi(a,c-1,x)}{\psi(a,c,x)} < \int_0^\infty \frac{\Phi_{a,c}(t)}{1+t}\mathrm{dt} = \int_0^\infty \frac{\Phi_{a,c}(t)}{t}\mathrm{dt} = \frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1} \tag{13}$$

For $a > 0, 2 < c < a + 1$, $x > 0$,and the inequality $(x + t)^2 > 2xt$ we have

$$\left(\frac{\psi(a,c-1,x)}{\psi(a,c,x)}\right)' = \int_0^\infty \frac{t}{(x+t)^2}\Phi_{a,c}(t)\mathrm{dt} < \int_0^\infty \frac{t\Phi_{a,c}(t)}{2xt}\mathrm{dt} = \left(\frac{1}{2}\ln(x)\right)' \tag{14}$$

thus the function

$$x \to \frac{\psi(a,c-1,x)}{\psi(a,c,x)} - \frac{1}{2}\ln(x)$$

is strictly decreasing on $(0,\infty)$ for $a > 0$ and $2 < c < a + 1$.

$$\lim_{x\to 1}\left(\frac{\psi(a,c-1,x)}{\psi(a,c,x)} - \frac{1}{2}\ln(x)\right) < \frac{\psi(a,c-1,x)}{\psi(a,c,x)} - \frac{1}{2}\ln(x)$$

Thus for in view of (13) and for $a > 0$, $2 < c < a + 1$ and $0 < x < 1$ we have

$$\left(\frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1}\right) < \frac{\psi(a,c-1,x)}{\psi(a,c,x)} - \frac{1}{2}\ln(x)$$

which is same as the equation (11). □

3. Completely Monotonic functions involving ratio of confluent hypergeometric function of the second kind

Theorem 3 If $a > 0$ and $2 < c < a + 1$ then the following function is

$$\left(\frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1}\right)x - \frac{\psi(a, c-1, x)}{\psi(a, c, x)} \tag{15}$$

is strictly completely monotonic on $(0, \infty)$.

Proof. Let

$$f_{a,c}(x) = \left(\frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1}\right)x - \frac{\psi(a, c-1, x)}{\psi(a, c, x)}$$

using the integral representation (7) we have

$$(-1)^n f_{a,c}(x) = \int_0^\infty \frac{n!t\Phi_{a,c}(t)}{(x+t)^n}\mathrm{dt} > 0$$

for all $x > 0, a > 0$ and $2 < c < a + 1$. Thus the function

$$f_{a,c}(x) = \left(\frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1}\right)x - \frac{\psi(a, c-1, x)}{\psi(a, c, x)}$$

is strictly completely monotonic on $(0, \infty)$ for $a > 0$ and $2 < c < a + 1$. $\square$

If $a > 0, 2 < c < a + 1$ then the following function is

$$\frac{\psi(a, c-1, x)}{\psi(a, c, x)} - \frac{1}{2}\ln(x) - \left(\frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1}\right) \tag{16}$$

is strictly completely monotonic on $(0, 1)$.

Proof. Let

$$g_{a,c}(x) = \frac{\psi(a, c-1, x)}{\psi(a, c, x)} - \frac{1}{2}\ln(x) - \left(\frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1}\right)$$

then we have,

$$g'_{a,c}(x) = \left(\frac{\psi(a, c-1, x)}{\psi(a, c, x)}\right)' - \frac{1}{2x} < 0$$

implies that

$$(-1)^1 g'_{a,c}(x) > 0, \quad \text{for all } 0 < x < 1, a > 0 \text{ and } 2 < c < a + 1.$$

It is easy to see that the function $\frac{1}{2} - t\Phi_{a,c}(t)$ is positive on $(0, \infty)$. using the integral representation (7) and

$$\frac{1}{x} = \int_0^\infty \frac{1}{(x+t)^2}\mathrm{dt}, x > 0$$

we have

$$g'_{a,c}(x) = \int_0^\infty \frac{t}{(x+t)^2}\Phi_{a,c}(t)\mathrm{dt} - \frac{1}{2}\int_0^\infty \frac{1}{(x+t)^2}\mathrm{dt}$$

differentiating $n - 1$ times and for all $0 < x < 1, a > 0$ and $2 < c < a + 1$ we have

$$(-1)^n g_{a,c}^{(n)}(x) = \int_0^\infty \frac{n!(\frac{1}{2} - t\Phi_{a,c}(t))}{(x+t)^{n+1}}\mathrm{dt} > 0$$

thus the function

$$\frac{\psi(a, c-1, x)}{\psi(a, c, x)} - \frac{1}{2}\ln(x) - \left(\frac{\Gamma(c)\Gamma(c-2)}{\Gamma^2(c-1)} - \frac{1}{a+1}\right)$$

is strictly completely monotonic for all $0 < x < 1, a > 0$ and $2 < c < a + 1$. □

References

[1] M. Abramowitz, I.A. Stegun (eds.), Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Dover Publications, New York, 1965.

[2] M. E. H. Ismail, *Completely monotonic functions associated withthe gamma function and its q-analogues*, J. Math, Anl. Appl,**116** (1986),1-9.

[3] G. D. Anderson, S. L. Qiu, *A monotoneity property of gamma function* , Proc. Amer. Math. Soc, **125** (1997), 3355-3362.

[4] H. Alzer, *Some gamma function inequalities*, Math. Comp. **60** (1993), 337-346.

[5] D. V. Widder, *The Laplace Transform, Princeton Univ. Press, Princeton,* N.J., 1944.

[6] J. Wimp, *Sequence Transformations and Their Applications. Academic Press,* New York, 1981.

[7] S. Bochner, *Harmonic analysis and the theory of probability, Univ. of California Press*, Berkeley Los Angeles, 1955.

[8] L. Maligranda, J. E. Pečarić, and L. E. Persson, *Stolarsky's inequality with general weights,* Proc. Amer. Math. Soc. **123** (1995), 2113-2118.

[9] L. Gordon, *A stochastic approach to the gamma function*, Amer. Math. Monthly **101** (1994), 858-865.

[10] A. Z. Grinshpan and M. E. H. Ismail,*Completely monotonic functions involving the gamma and the q-gamma functions* , Proc. Amer. Math. Soc, **134** (2005), 1153-1160.

[11] Feng Qi, *Bounds for the ratio of two gamma functions*, Journal of Inequalities and Applications **2010**, Article ID 493058.

[12] Feng Qi, *Bounds for the ratio of two gamma functions:from Gautschi's and Kershaw's inequalities to completemonotonicity*, Turkish Journal of Analysis and Number Theory **2** (2014), no. 5, 152–164.

[13] Feng Qi and Qiu-Ming Luo, *Bounds for the ratio of two gamma functions: from Wendel's asymptotic relation to Elezović-Giordano-Pečarić's theorem*, Journal of Inequalities Applications **2013**: 542, 20 pages.

[14] Feng Qi and Qiu-Ming Luo, *Bounds for the ratio of two gamma functions:From Wendel's related inequalities to logarithmically completely monotonic functions*, Banach Journal Mathematical Analysis **6** (2012), no. 2, 132–158.

[15] Feng Qi and Wen-Hui Li, *A logarithmically completely monotonic function involving the of gamma functions*, Journal of Applied Analysis and Computation **5** (2015), no. 4, 626–634.

[16] Feng Qi, Mansour Mahmoud, Xiao-Ting Shi, and Fang-Fang Liu, *Some properties of the Catalan: Qi function related to the Catalan numbers*, SpringerPlus (2016), **5**: 1126, 20 pages.

[17] H. Alzer and C. Berg, *Some Classes of Completely Monotonic Functions*, Annales Academiae Scientiarum Fennicae, **27** (2002), 445-460.

[18] H. Alzer and C. Berg, *Some Classes of Completely Monotonic Functions, II*, The Ramanujan Journal, **11** (2006), No. 2, 225-248.

[19] K. S. Miller and S. G. Smako, *Completely Monotonic Functions*, Int. Transf. and Spec. Funct., **12** (2001), No. 4, 389-402.

[20] F. Qi and C. P. Chen , *A Complete Monotonicity Property of the Gamma Function*, J. Math Anal. Appl., **296** (2004), pp. 603-607.

[21] N Batir *Inequalities for the gamma function*, Archiv der Mathematik, 2008, 91: 554-563.

[22] G D Aderson, S L Qiu, *A monotonicity property of the gamma function*, Proc Amer Math Soc, 1997, **125** (11): 3355-3362.

[23] Á. Baricz, M. E. H. Ismail, *Turán type inequalities for Tricpmi confluent hypergeometric functions* , Constr. Approx. **37(2)** (2013) 195-221.

[24] Á. Baricz, S. Ponnusamy, S, Singh, *Turánn type inequalities for confluent hypergeometric function of second kind* , Studia Scientiarum Mathematicarum Hungarica **53(1)** (2016) 74-92.

[25] M. E. H. Ismail,D. H. Kelker, *Special functions, Stieltjes transforms ans infinity divisibility*, SAIM J. Math.Anal. **10(5)** (1979), 884-901.

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