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THIRD HANKEL DETERMINANT FOR A CLASS OF ANALYTIC UNIVALENT FUNCTIONS

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Abstract. Let $\mathcal{A}$ denote the class of all normalized analytic function f in the unit disc $\mathbb{U}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. The objective of this paper is to obtain an upper bound to the third Hankel determinant denoted by $H_3(1)$ for certain subclass of univalent functions, using Toeplitz determinants.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions defined on the unit disc $\mathbb{U} = \{z : |z| < 1\}$ with the normalization condition $f(0) = 0 = f'(0) - 1$. So $f \in \mathcal{A}$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let $\mathcal{S}$ be the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be starlike if it satisfies the condition $\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} > 0$, for $z \in \mathbb{U}$. Let $\mathcal{P}$ denote the class of functions $p(z)$, has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2}$$

which are regular in the open unit disc $\mathbb{U}$ and satisfy $\mathrm{Re}\, p(z) > 0$, for $z \in \mathbb{U}$. Here $p(z)$ is called the Caratheodory function [5].

Definition 1 ([4]) For $\alpha \geq 0$, a function $f \in \mathcal{A}$ with $\frac{f(z)f'(z)}{z} \neq 0$ said to be alpha-close-to-convex function if for a starlike function $\phi(z)$, satisfies the condition

$$\mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}\right\} > 0, \quad z \in \mathbb{U}.$$

We denote $\mathcal{C}_\alpha$ be the class of all alpha-close-to-convex functions. This class was introduced and studied by Chichra [4]. We denote the subclass of $\mathcal{C}_\alpha$ by $\tilde{\mathcal{R}}_\alpha$ for

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which $\phi(z) = z$.
Definition 2 ([4]) Let $\tilde{\mathcal{R}}_\alpha$ be the class of all functions $f \in \mathcal{A}$ which satisfy

$$\mathrm{Re}(f'(z) + \alpha z f''(z)) > 0, \quad \text{for all } z \in \mathbb{U}.$$

For $\alpha = 0$, $\tilde{\mathcal{R}}_\alpha \equiv \mathcal{R}_0$, the class of functions whose derivative has positive real part. These classes have been studied by many authors (see [4, 15, 16, 20]). It is well known that the n^{th} coefficient is bounded by n, for $f \in \mathcal{S}$. Also the bounds for the coefficients give information about the geometric properties of the univalent functions. For example, the growth and distortion properties of the normalized univalent function are determined by studying the bound of its second coefficient. In the recent years, several authors considered a more general coefficient problem of this type, which is the Hankel determinant problem.
Definition 3 The q-th Hankel determinant of $f(z)$ for $q \geq 1$ and $n \geq 1$ is defined by Pommerenke [22] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{3}$$

The Hankel determinants have been considered by several authors, to investigate its rate of growth as $n \to \infty$ and to determine the bounds of it for different specific values of q and n, (see [6, 13, 17, 18, 19]). It is interesting to note that, $H_2(1) = |a_3 - a_2^2|$, the Fekete-Szegö functional for $\mu = 1$ (see [9]). The Hankel determinant in the case of $q = 2$ and $n = 2$, is known as the second Hankel determinant (functional), given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \tag{4}$$

The bounds of $H_2(2)$ were obtained for various subclasses of univalent and multivalent analytic functions by many authors existed in the literature [2, 8, 14]. Similarly, the third Hankel determinant in the case of $q = 3$ and $n = 1$, denoted by $H_3(1)$, is defined by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}. \tag{5}$$

For $f \in \mathcal{A}$, $a_1 = 1$, we have

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$$

and by applying triangle inequality, we obtain

$$|H_3(1)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_4 - a_2 a_3| + |a_5||a3 - a_2^2|. \tag{6}$$

Recently, Babola [1], Bansal et.al [2], Prajapat et.al [23] and Vamshee Krishna et.al [10], have studied the third Hankel determinant $H_3(1)$ and obtained its bound for various subclasses of univalent and multivalent analytic functions. Motivated by the result obtained by Chichra [4], Babalola [1] and Vamshee Krishna et.al [10], we obtain an upper bound to the functional $|a_2 a_3 - a_4|$ and hence for $|H_3(1)|$, for the function $f(z)$ in the class $\tilde{\mathcal{R}}_\alpha$.

2. Preliminary Results

The following lemmas are required to prove our main results.

Lemma 1 ([21]). If $p(z) \in \mathcal{P}$, given by (2), then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $p_0(z) = \frac{1+z}{1-z}$.

Lemma 2 ([3]). If $p(z) \in \mathcal{P}$, given by (2). Then

$$\left|c_2 - \rho\frac{c_1^2}{2}\right| \leq \begin{cases} 2(1-\rho), & \rho \leq 0, \\ 2, & 0 \leq \rho \leq 2, \\ 2(\rho-1), & \rho \geq 2. \end{cases}$$

The inequality is sharp for the function

$$p(z) = \begin{cases} \frac{1+z^2}{1-z^2}, & 0 \leq \rho \leq 2, \\ \frac{1+z}{1-z}, & \rho \in (-\infty, 0] \cup [2, \infty). \end{cases}$$

Lemma 3 ([7]) The power series $1 + \sum_{n=1}^{\infty} c_n z^n$, converges in the open unit disc $\mathbb{U}$ to a function in $\mathcal{P}$ if and only if the Toeplitz determinants

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \dots & c_n \\ c_{-1} & 2 & c_1 & \dots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \dots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \\ c_{-n} & c_{-n+1} & c_{-n+2} & \dots & 2 \end{vmatrix}, \ n = 1, 2, 3 \dots$$

and $c_{-k} = \bar{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{\infty} \rho_k p_0(e^{it_k} z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n = 0$ for $n \geq m$.

This necessary and sufficient condition is due to Caratheodory and Toeplitz [7]. We may assume without restriction that $c_1 > 0$. Hence by using Lemma 3, for $n = 2$, we get

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} = [8 + 2\mathrm{Re}\{c_1^2 c_2\} - 2|c_2|^2 - 4|c_1|^2] \geq 0,$$

which is equivalent to

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{7}$$

for some x, $|x| \leq 1$. For $n = 3$, we get

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \bar{c}_1 & 2 & c_1 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 & c_1 \\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} \geq 0$$

and is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 + 2|(2c_2 - c_1^2)|^2. \tag{8}$$

By using (7) we get from (8) that

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{9}$$

for some x and z such that $|x| \leq 1$ and $|z| \leq 1$.

3. Main Results

To obtain our results, we refer to the classical method initiated by Libera and Zlotkiewicz [11, 12].

Theorem 1. For $0 \leq \alpha \leq \frac{1}{2}$, let $f \in \tilde{\mathcal{R}}_\alpha$. Then

$$|a_2 a_4 - a_3^2| \leq \frac{4}{9(1+2\alpha)^2}.$$

Proof. Let $f(z)$ given by (1), be in the class $\tilde{\mathcal{R}}_\alpha$. Then there exists an analytic function $p \in \mathcal{P}$ in the unit disc $\mathbb{U}$ with $p(0) = 1$ such that

$$f'(z) + \alpha z f''(z) = p(z). \tag{10}$$

By using the series representations for $f'(z)$, $f''(z)$ and $p(z)$ from (1) and (2) in (10), we get

$$1 + \sum_{n=1}^{\infty}(n+1)(1+n\alpha)a_{n+1}z^n = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{11}$$

Equating the coefficients of $z,\ z^2,\ z^3$ of both sides of (11), we have

$$a_2 = \frac{c_1}{2(1+\alpha)}; \quad a_3 = \frac{c_2}{3(1+2\alpha)}; \quad a_4 = \frac{c_3}{4(1+3\alpha)}. \tag{12}$$

On substituting the values of $a_2,\ a_3$ and a_3 from (12) in $|a_2a_4 - a_3^2|$ for the function $f \in \tilde{\mathcal{R}}_\alpha$, we have

$$|a_2a_4 - a_3^2| = \left|K(\alpha)\left[9(1+2\alpha)^2 c_1 c_3 - 8(1+\alpha)(1+3\alpha)c_2^2\right]\right|, \tag{13}$$

where

$$K(\alpha) = \frac{1}{72(1+\alpha)(1+2\alpha)^2(1+3\alpha)}. \tag{14}$$

By substituting the values of c_2 and c_3 from equations (7) and (9) in the equation (13), we get

$$\begin{aligned} |a_2a_4 - a_3^2| &= K(\alpha)\left[\frac{9}{4}(1+2\alpha)^2 c_1\{c_1^3 + 2(4-c_1^2)c_1 x - c_1(4-c_1^2)x^2\right. \\ &\left. +2(4-c_1^2)(1-|x|^2)z\} - \frac{8}{4}(1+\alpha)(1+3\alpha)\{c_1^2 + (4-c_1^2)x\}^2\right]. \end{aligned}$$

Then by using the triangle in equality and the fact $|z| < 1$, we get

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq \frac{K(\alpha)}{4}\left[(1+4\alpha+12\alpha^2)c_1^4 + 2(1+4\alpha+12\alpha^2)c_1^2(4-c_1^2)x\right. \\ &+\{(1+4\alpha+12\alpha^2)c_1^2 + 32(1+\alpha)(1+3\alpha)\}(4-c_1^2)x^2 \\ &\left.+18(1+2\alpha)^2(4-c_1^2)c_1(1-|x|^2)\right]. \end{aligned} \tag{15}$$

On replacing $c_1 = c$, $c \in [0,2]$ and $|x| = \mu$ in (15), we get

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq \frac{K(\alpha)}{4}\big[(1+4\alpha+12\alpha^2)c^4 + 2(1+4\alpha+12\alpha^2)c^2(4-c^2)\mu \\ &+\{(1+4\alpha+12\alpha^2)c^2 + 32(1+\alpha)(1+3\alpha)\}(4-c^2)\mu^2 \\ &+18(1+2\alpha)^2(4-c^2)c - 18(1+2\alpha)^2(4-c^2)c\mu^2\big] \\ &= \frac{K(\alpha)}{4}\big[(1+4\alpha+12\alpha^2)c^4 + 18(1+2\alpha)^2(4-c^2)c \\ &+2(1+4\alpha+12\alpha^2)c^2(4-c^2)\mu + (1+4\alpha+12\alpha^2)\times \\ &(4-c^2)(c-2)(c-\beta)\mu^2\big] = F(c,\mu)\ (say), \end{aligned} \tag{16}$$

where

$$\beta = \beta(\alpha) = \frac{16(1+\alpha)(1+3\alpha)}{1+4\alpha+12\alpha^2}, \quad 0 \leq c \leq 2 \quad \text{and} \quad 0 \leq \mu \leq 1,$$

and $K(\alpha)$ defined in (14). We next maximize the function $F(c,\mu)$ on the closed square $[0,2] \times [0,1]$. Since $c-2<0$ and $c-\beta<0$, so

$$\begin{aligned} \frac{\partial F}{\partial \mu} &= \frac{K(\alpha)}{4}[2(1+4\alpha+12\alpha^2)(4-c^2)\{c^2 + (c-2)(c-\beta)\mu\}] \\ &= \frac{(1+4\alpha+12\alpha^2)(4-c^2)\{c^2+(c-2)(c-\beta)\mu\}}{36(1+\alpha)(1+2\alpha)^2(1+3\alpha)} > 0. \end{aligned}$$

Thus for a fixed c, $F(c,\mu)$ is increasing function of μ and hence it can not have maximum in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for fixed $c \in [0,2]$, we have

$$\max_{0\leq\mu\leq1} F(c,\mu) = F(c,1) = G(c)\ (say). \tag{17}$$

Here

$$\begin{aligned} G(c) &= \frac{K(\alpha)}{4}\big[(1+4\alpha+12\alpha^2)c^4 + 18(1+2\alpha)^2(4-c^2)c + 2(1+4\alpha+12\alpha^2)\times \\ &c^2(4-c^2) + (1+4\alpha+12\alpha^2)(4-c^2)(c-2)(c-\beta)\big]. \end{aligned} \tag{18}$$

Next,

$$G'(c) = -\frac{1}{18(1+\alpha)(1+2\alpha)^2(1+3\alpha)}[(1+4\alpha+12\alpha^2)c^3 + (5+20\alpha-12\alpha^2)], \tag{19}$$

so that $G'(c) < 0$ for $0 \leq \alpha < \frac{1}{2}$ and $c \in [0,2]$. Thus $G(c)$ is a decreasing function in c and so

$$\max_{0\leq c\leq 2} G(c) = G(0) = \frac{4}{9(1+2\alpha)^2}. \tag{20}$$

Hence the desired result follows from (16), (18) and (20).

Theorem 2. For $0 \leq \alpha \leq \frac{1}{2}$, let $f \in \tilde{\mathcal{R}}_\alpha$. Then

$$|a_2a_3 - a_4| \leq \frac{1}{18\sqrt{3}(1+\alpha)(1+2\alpha)}\left[\frac{5(1+3\alpha)-6\alpha^2}{1+3\alpha}\right]^{3/2}.$$

Proof. Let $f(z)$ given by (1), be in the class $\tilde{\mathcal{R}}_\alpha$. Then substituting the values of $a_2,\ a_3$ and a_3 from (12) in $|a_2a_3 - a_4|$ for the function $f \in \tilde{\mathcal{R}}_\alpha$, we have

$$|a_2a_3 - a_4| = |M(\alpha)\,[2(1+3\alpha)c_1c_2 - 3(1+\alpha)(1+2\alpha)c_3]|, \tag{21}$$

where

$$M(\alpha)=\frac{1}{12(1+\alpha)(1+2\alpha)(1+3\alpha)}. \tag{22}$$

Substituting the values of c_2 and c_3 from equations (7) and (9) in equation (21), we get

$$\begin{aligned}|a_2a_3-a_4| &= M(\alpha)\Big|(1+3\alpha)c_1\{c_1^2+(4-c_1^2)x\}-\frac{3}{4}(1+\alpha)(1+2\alpha)\times\\ &\quad \{c_1^3+2(4-c_1^2)c_1x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)z\}\Big|.\end{aligned}$$

Using the triangle in equality and the fact $|z|\le 1$, after simplifying, we get

$$\begin{aligned}|a_2a_3-a_4| &\le \frac{M(\alpha)}{4}\left[(1+3\alpha-6\alpha^2)c_1^3+6(1+\alpha)(1+2\alpha)(4-c_1^2)\right.\\ &\quad +2(1+3\alpha-6\alpha^2)c_1(4-c_1^2)|x|\\ &\quad \left.+3(1+\alpha)(1+2\alpha)(c_1-2)(4-c_1^2)|x|^2\right].\end{aligned}$$

On replacing $c_1=c$, $c\in[0,2]$ and $|x|=\mu$, we get

$$\begin{aligned}|a_2a_3-a_4| &\le \frac{M(\alpha)}{4}\left[(1+3\alpha-6\alpha^2)c^3+6(1+\alpha)(1+2\alpha)(4-c^2)\right.\\ &\quad \left.+2(1+3\alpha+6\alpha^2)c(4-c^2)\mu+3(1+\alpha)(1+2\alpha)(c-2)(4-c^2)\mu^2\right]\\ &= F(c,\mu)(Say),\end{aligned} \tag{23}$$

where $M(\alpha)$ defined in (22). We next maximize the function $F(c,\mu)$ on the closed square $[0,2]\times[0,1]$. Since $c-2<0$ and $0\le\alpha\le\frac{1}{2}$, so

$$\frac{\partial F}{\partial\mu} = \frac{M(\alpha)}{2}\left[(4-c^2)(1+3\alpha-6\alpha^2)c+3(1+\alpha)(1+2\alpha)(c-2)\mu\right]>0.$$

Thus for a fixed c, $F(c,\mu)$ is increasing function of μ and hence it can not have maximum in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$, we have

$$\max_{0\le\mu\le1}F(c,\mu)=F(c,1)=G(c)\ (say). \tag{24}$$

Here

$$G(c)=-4(1+3\alpha)c^3+4(5+15\alpha-6\alpha^2)c. \tag{25}$$

Next,

$$G'(c)=-12(1+3\alpha)c^2+4[5(1+3\alpha)-6\alpha^2]=0,$$

which implies

$$c_0=\sqrt{\frac{5(1+3\alpha)-6\alpha^2}{3(1+3\alpha)}}. \tag{26}$$

So c_0 be the critical point of $G(c)$. Since $G''(c_0)=-24(1+3\alpha)c_0<0$, so $G(c)$ has maximum at c_0. Thus

$$\max_{0\le c\le2}G(c)=G(c_0)=8(1+3\alpha)\left(\frac{5(1+3\alpha)-6\alpha^2}{3(1+3\alpha)}\right)^{3/2}. \tag{27}$$

Hence the desired result follows from (23), (24) and (27).
Theorem 3 For $\mu \geq 0$, let $f \in \tilde{\mathcal{R}}_\alpha$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2}{3(1+2\alpha)}, & 0 \leq \mu \leq \frac{4(1+\alpha)^2}{3(1+2\alpha)}, \\ \\ \frac{\mu}{(1+\alpha)^2} - \frac{2}{3(1+2\alpha)}, & \mu \geq \frac{4(1+\alpha)^2}{3(1+2\alpha)}. \end{cases}$$

The inequality is sharp.
Proof. Let $f(z)$ given by (1), be in the class $\tilde{\mathcal{R}}_\alpha$. Then substituting the values of a_2 and a_3 from (12) in $|a_3 - \mu a_2^2|$ for the function $f \in \tilde{\mathcal{R}}_\alpha$, we have

$$|a_3 - \mu a_2^2| = \frac{1}{3(1+2\alpha)} \left| c_2 - \frac{3(1+2\alpha)\mu}{2(1+\alpha)^2} \frac{c_1^2}{2} \right|. \tag{28}$$

Let $\rho = \frac{3(1+2\alpha)\mu}{2(1+\alpha)^2} \geq 0$. Then by applying Lemma 2, we get the desired result.
The inequality derived in the above theorem is called as Fekete-Sezgö inequality.
Taking $\mu = 1$ in Theorem 3, we get the following theorem.
Theorem 4 For $\alpha \geq 0$, let $f \in \tilde{\mathcal{R}}_\alpha$. Then

$$|a_3 - a_2^2| \leq \frac{2}{3(1+2\alpha)}.$$

Theprem 5 If for $\alpha \geq 0$, $f \in \tilde{\mathcal{R}}_\alpha$, then

$$|a_k| \leq \frac{2}{k(1+(k-1)\alpha)}, \quad k \geq 2.$$

Proof Let $f \in \tilde{\mathcal{R}}_\alpha$. Then equating the coefficients of z^k of both sides of (11), we get

$$a_k = \frac{c_{k-1}}{k(1+(k-1)\alpha)}, \quad k \geq 2.$$

By using the Lemma 1, we get the desired result.
Theorem 6 If for $0 \leq \alpha \leq \frac{1}{2}$, $f \in \tilde{\mathcal{R}}_\alpha$, then

$$H_3(1) \leq \frac{1}{3(1+2\alpha)} \left[\frac{4}{5} + \frac{8}{9(1+2\alpha)} + \frac{1}{12\sqrt{3}(1+\alpha)(1+2\alpha)} \left\{ \frac{5(1+3\alpha) - 6\alpha^2}{1+3\alpha} \right\}^{3/2} \right].$$

Proof. Let for $0 \leq \alpha \leq \frac{1}{2}$, $f \in \tilde{\mathcal{R}}_\alpha$. Then from (6) we have,

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a3 - a_2^2|.$$

By using the bounds of $|a_2a_4 - a_3^2|$, $|a_2a_3 - a_4|$, $|a3 - a_2^2|$, $|a_3|$, $|a_4|$ and $|a_5|$ from Theorem 1, Theorem 2, Theorem 4 and Theorem 5, we get the desired result.

Taking $\alpha = 0$, in Theorem 6, we have the following result.
Corollary 1 If $f(z) \in \mathcal{R}_0$, then $|H_3(1)| \leq 0.742$
This inequality coincides with Babalola [1] and Vamashee Krishna et.al [10] for $\alpha = 0$.
Remark For $\alpha = 0$ the results in Theorem 1, Theorem 2, Theorem 4, coincides with the results due to Babalola [1], and also due to Vamashee Krishna et.al [10] for $\alpha = 0$.

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