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SOLVABILITY OF URYSOHN-STIELTJES INTEGRAL EQUATION IN REFLEXIVE BANACH SPACE

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Abstract. In this paper we study the existence of weak solutions of the Urysohn-Stieltjes integral equation

$$x(t) = a(t) + \int_0^1 f(t, s, x(s))\ d_s g(t, s),\ \ t \in I = [0, 1]$$

in a reflexive Banach space E.

1. Introduction and Preliminaries

Volterra-Stieltjes integral operators and Volterra-Stieltjes integral equations have been studied (see [1]-[8]). The main tool utilized in our considerations is the technique associated with the Schauder fixed point theorem. This type of equations have been studied by Banaś (see [1]-[4]) and also by some other authors, for example (see [7], [8] and [14]-[16]).

Consider the nonlinear Urysohn-Stieltjes integral equation

$$x(t) = a(t) + \int_0^1 f(t, s, x(s))\ d_s g(t, s),\ \ t \in I = [0, 1]. \tag{1}$$

This equation have been studied by Banaś (see [3]), where proved the existence at least one solution $x \in C(I)$ to the equation (1).
In this paper, we generalize this results to study the existence of weak solutions $x \in C[I, E]$ of the Urysohn-Stieltjes integral equation (1). As an application, we study the existence of weak solutions $x \in C[I, E]$ of the Hammerstien-Stieltjes integral equation

$$x(t) = a(t) + \int_0^1 k(t, s)h(s, x(s))\ d_s g(t, s),\ \ t \in I = [0, 1]. \tag{2}$$

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In [10], the authors proved the existence of weak solutions $x \in C[I, E]$ of Volterra-Stieltjes integral equation

$$x(t) = p(t) + \int_0^t f(s, x(s))\ d_s g(t, s),\ t \in I = [0, T].$$

Let E be a reflexive Banach space with norm $\| \,.\, \|$ and dual E^*. Denote by $C[I, E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm.
Now, we shall present some auxiliary results that will be need in this work. Let E be a Banach space (need not be reflexive) and let $x : [a, b] \to E$, then

(1-) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in [a, b]$ if for every $\phi \in E^*$, $\phi(x(.))$ is continuous (measurable) at t_0.

(2-) A function $h : E \to E$ is said to be weakly sequentially continuous if h maps weakly convergent sequences in E to weakly convergent sequences in E.

If x is weakly continuous on I, then x is strongly measurable and hence weakly measurable (see [13] and [9]). It is evident that in reflexive Banach spaces, if x is weakly continuous function on $[a, b]$, then x is weakly Riemann integrable (see [13]). Since the space of all weakly Riemann-Stieltjes integrable functions is not complete, we will restrict our attention to the existence of weak solutions of equation (1) in the space $C[I, E]$.

Definition 1. Let $f : I \times E \to E$. Then $f(t, u)$ is said to be weakly-weakly continuous at (t_0, u_0) if given $\epsilon > 0$, $\phi \in E^*$ there exists $\delta > 0$ and a weakly open set U containing u_0 such that

$$| \phi(f(t, u) - f(t_0, u_0)) | < \epsilon$$

whenever $| t - t_0 | < \delta$ and $u \in U$.

Now, we have the following fixed point theorem, due to O'Regan, in the reflexive Banach space (see [17]) and some propositions which will be used in the sequel (see [11]).

Theorem 1. Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of $C[I, E]$ and let $F : Q \to Q$ be a weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in E for each $t \in I$. Then, F has a fixed point in the set Q.

Proposition 1. A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.

Proposition 2. Let E be a normed space with $y \in E$ and $y \neq 0$. Then there exists a $\phi \in E^*$ with $\| \phi \| = 1$ and $\| y \| = \phi(y)$.

Proposition 3. A convex subset of a normed space E is closed if and only if it is weakly closed.

2. Main results

In this section, we present our main result by proving the existence of weak solutions for equation (1) in the reflexive Banach space. Let us first state the following assumptions:

- (i) $a \in C[I, E]$.
- (ii) $f : I \times I \times D \subset E \to E$ satisfies the following conditions:
  - (1) $f(., s, x(s))$ is continuous function, $\forall s \in I, x \in D \subset E$.
  - (2) $f(t, ., .)$ is weakly-weakly continuous function, $\forall t \in I$.
  - (3) The weak closure of the range of $f(I \times I \times D)$ is weakly compact in E (or equivalently: there exists a constant M such that $\| f(t, s, x) \| \leq M$ for $t, s \in I$ and $x \in D$.
- (iii) The functions $t \to g(t, 1)$ and $t \to g(t, 0)$ are continuous on I, such that

$$\mu = \max\{\sup_t \mid g(t, 1) \mid + \sup_t \mid g(t, 0) \mid\}.$$

- (iv) For all $t_1, t_2 \in I$ such that $t_1 < t_2$ the function $s \to g(t_2, s) - g(t_1, s)$ is nondecreasing on I.
- (v) $g(0, s) = 0$ for any $s \in I$.

Remark 1. Observe that assumptions (iv) and (v) imply that the function $s \to g(t, s)$ is nondecreasing on the interval I, for any fixed $t \in I$ (Remark 1 in [4]). Indeed, putting $t_2 = t,\ t_1 = 0$ in (iv) and keeping in mind (v), we obtain the desired conclusion. From this observation, it follows immediately that, for every $t \in I$, the function $s \to g(t, s)$ is of bounded variation on I.

Theorem 2. Under the assumptions (i)-(v), the Urysohn-Stieltjes integral equation (1) has at least one weak solution $x \in C[I, E]$.

Proof. Define the operator A by

$$Ax(t) = a(t) + \int_0^1 f(t, s, x(s))\ d_s g(t, s),\ t \in I.$$

For every $x \in C[I, E]$, $f(., s, x(s))$ is continuous on I, and $f(t, ., .)$ is weakly-weakly continuous on I, then $\phi(f(t, s, x(s)))$ is continuous for every $\phi \in E^*$, g is of bounded variation. Hence $f(t, s, x(s))$ is weakly Riemann-Stieltjes integrable on I with respect to $s \to g(t, s)$. Thus A makes sense.

Now, define the set Q by

$$Q = \{x \in C[I, E] : \| x \|_0 \leq M_0\}$$

where

$$\| x(t) \| \leq \| a \|_0 + M\mu = M_0$$

and

$$\| x \|_0 = \sup_{t \in I} \| Ax(t) \| \leq M_0.$$

The remainder of the proof will be given in four steps.

Step 1 : The operator A maps $C[I, E]$ into $C[I, E]$.

Let $t_1,\ t_2 \in I,\ t_2 > t_1$ and $t_1 - t_2 < \epsilon$, without loss of generality, assume that $Ax(t_2) - Ax(t_1) \neq 0$. Then

$$\begin{aligned}
\| Ax(t_2) - Ax(t_1) \| &\leq | \phi(a(t_2) - a(t_1)) | + | \int_0^1 \phi(f(t_2, s, x(s)))\ d_s g(t_2, s) \\
&- \int_0^1 \phi(f(t_1, s, x(s)))\ d_s g(t_1, s) | \\
&\leq \| a(t_2) - a(t_1) \| + | \int_0^1 \phi(f(t_2, s, x(s)))\ d_s g(t_2, s) \\
&- \int_0^1 \phi(f(t_1, s, x(s)))\ d_s g(t_2, s) | \\
&+ | \int_0^1 \phi(f(t_1, s, x(s)))\ d_s g(t_2, s) \\
&- \int_0^1 \phi(f(t_1, s, x(s)))\ d_s g(t_1, s) | \\
&\leq \| a(t_2) - a(t_1) \| \\
&+ \int_0^1 | \phi(f(t_2, s, x(s)) - f(t_1, s, x(s))) |\ \ d_s(\bigvee_{z=0}^{s} g(t_2, z)) \\
&+ \int_0^1 | \phi(f(t_1, s, x(s))) |\ \ d_s(\bigvee_{z=0}^{s} [g(t_2, z) - g(t_1, z))]) \\
&\leq \| a(t_2) - a(t_1) \| \\
&+ \| f(t_2, s, x(s)) - f(t_1, s, x(s)) \| \int_0^1\ d_s g(t_2, s) \\
&+ \int_0^1 \| f(t_1, s, x(s)) \|\ \ d_s[g(t_2, s) - g(t_1, s))] \\
&\leq \| a(t_2) - a(t_1) \| \\
&+ \| f(t_2, s, x(s)) - f(t_1, s, x(s)) \| [g(t_2, 1) - g(t_2, 0)] \\
&+ M[(g(t_2, 1) - g(t_1, 1)) - (g(t_2, 0) - g(t_1, 0))] \\
&\leq \| a(t_2) - a(t_1) \| \\
&+ \| f(t_2, s, x(s)) - f(t_1, s, x(s)) \| [g(t_2, 1) - g(t_2, 0)] \\
&+ M[| g(t_2, 1) - g(t_1, 1) | + | g(t_2, 0) - g(t_1, 0) |]
\end{aligned}$$

Hence

$$\begin{aligned}
\| Ax(t_2) - Ax(t_1) \| &\leq \| a(t_2) - a(t_1) \| \\
&+ \| f(t_2, s, x(s)) - f(t_1, s, x(s)) \| [g(t_2, 1) - g(t_2, 0)] \\
&+ M[| g(t_2, 1) - g(t_1, 1) | + | g(t_2, 0) - g(t_1, 0) |,
\end{aligned}$$

then from the continuity of the function g assumption (iii) we deduce that A maps $C[I, E]$ into $C[I, E]$.

Step 2 : The operator A maps Q into Q.

Take $x \in Q$, we have

$$\begin{aligned}
\| Ax(t) \| &= \phi(Ax(t)) \\
&\leq | \phi(a(t)) | + | \phi(\int_0^1 f(t,s,x(s)) \ d_s g(t,s)) | \\
&\leq \| a \|_0 + \int_0^1 | \phi(f(t,s,x(s))) | \ \ d_s(\bigvee_{z=0}^{s} g(t,z)) \\
&\leq \| a \|_0 + \int_0^1 \| f(t,s,x(s)) \| \ \ d_s(\bigvee_{z=0}^{s} g(t,z)) \\
&\leq \| a \|_0 + M \int_0^1 \ d_s g(t,s) \\
&\leq \| a \|_0 + M[\ g(t,1) - g(t,0)\ ] \\
&\leq \| a \|_0 + M[\ \sup_{t\in I} | g(t,1) | + \sup_{t\in I} | g(t,0) |\ ] \\
&\leq \| a \|_0 + M\mu
\end{aligned}$$

Hence $Ax \in Q$, which prove that $A : Q \to Q$ and AQ is bounded in $C[I,E]$.

Step 3 : $AQ(t)$ is relatively weakly compact in E.
Note that Q is nonempty, uniformly bounded and strongly equi-continuous subset of $C[I,E]$, Also it can be shown that Q is convex and closed. According to Propositions 1 and 3, AQ is relatively weakly compact.

Step 4 : The operator A is weakly sequentially continuous.
Let $\{x_n(t)\}$ be sequence in Q weakly convergent to $x(t)$ in E, since Q is closed we have $x \in Q$. Fix $t, s \in I$, since f satisfies (1)-(2), then we have $f(t,s,x_n(s))$ converges weakly to $f(t,s,x(s))$. Furthermore, $(\forall\ \varphi \in E^*)$ $\varphi(f(t,s,x_n(s)))$ convergence strongly to $\varphi(f(t,s,x(s)))$.
Applying assumption (ii)-(3)and Lebesgue dominated convergence theorem

$$\begin{aligned}
\phi(\int_0^1 f(t,s,x_n(s)) \ d_s g(t,s)) &= \int_0^1 \phi(f(t,s,x_n(s))) \ d_s g(t,s) \\
&\to \int_0^1 \phi(f(t,s,x(s))) \ d_s g(t,s), \ \forall \phi \in E^*, \ t \in I.
\end{aligned}$$

i.e. $\phi(Ax_n(t)) \to \phi(Ax(t)), \ \forall\ t \in I, \ Ax_n(t)$ converging weakly to $Ax(t)$ in E.
Thus, A is weakly sequentially continuous on Q.

Since all conditions of Theorem 1 are satisfied, then the operator A has at least one fixed point $x \in Q$ and the Urysohn-Stieltjes integral equation (1) has at least one weak solution.■

3. Hammerstien-Stieltjes integral equation

Consider the following assumption:

$(ii)^*$ Let $h : I \times E \to E$ and $k : I \times I \to R_+$ assume that h, k satisfy the following assumptions:

(1)* $h(s, x(s))$ is weakly-weakly continuous function.
(2)* There exists a constant M such that $\parallel h(s, x) \parallel \leq M$ for $s \in I$ and $x \in E$.
(3)* $k(t, s)$ is continuous function on I and such that $K = \sup_t\{| k(t, s) |: t, s \in I\}$ is positive constant.

New for the existence of a weak solution of (2), we have the following theorem.
Theorem 3. Let the assumptions (i),(iii)-(v) and $(ii)^*$ be satisfied. Then the Hammerstien-Stieltjes integral equation (2) has at least one weak solution $x \in C[I, E]$.

Proof. Let

$$f(t, s, x(s)) = k(t, s)h(s, x(s)).$$

Then from the assumption $(ii)^*$, we find that the assumptions of Theorem 2 are satisfied and result follows.

In what follows, we provide some examples illustrating the above obtained results.
Example 1: Consider the function $g : I \times I \to R$ defined by the formula

$$g(t, s) = t^3 + ts,\ t \in I,$$

It can be easily seen that the function $g(t, s)$ satisfy assumptions (iii)-(v) given in Theorem 1. In this case, the Urysohn-Stieltjes integral equation (1) has the form

$$x(t) = a(t) + \int_0^1 tf(t, s, x(s)))\ ds,\ t \in I \tag{3}$$

Therefore, the equation (3) has at least one solution $x \in C[I, E]$, if the functions a and f satisfy the assumptions (i) and (ii).

Example 2: Consider the function $g : I \times I \to R$ defined by the formula

$$g(t, s) = t(t + s - 1),\ t \in I.$$

It can be easily seen that the function $g(t, s)$ satisfy assumptions (iii)-(v) given in Theorem 1. In this case, the Urysohn-Stieltjes integral equation (1) has the form (3).
Therefore, the equation (3) has at least one solution $x \in C[I, E]$, if the functions a and f satisfy the assumptions (i) and (ii).

Example 3: Consider the function $g : I \times I \to R$ defined by the formula

$$g(t, s) = \begin{cases} t \ln \frac{t+s}{t}, & \text{for } t \in (0, 1],\ s \in I, \\ 0, & \text{for } t = 0,\ s \in I. \end{cases}$$

Also, the function $g(t, s)$ satisfy assumptions (iii)-(v) given in Theorem 1. In this case, the Urysohn-Stieltjes integral equation (1) has the form

$$x(t) = a(t) + \int_0^1 \frac{t}{t + s} f(t, s, x(s)))\ ds,\ t \in I \tag{4}$$

Therefore, the equation (4) has at least one solution $x \in C[I, E]$, if the functions a and f satisfy the assumptions (i) and (ii).

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