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SUM OF ENTIRE FUNCTIONS OF BOUNDED INDEX IN JOINT VARIABLES

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Abstract. In the paper, we present sufficient conditions of index boundedness in joint variables for a sum of entire functions. Our propositions are generalizations of Pugh's result obtained for entire functions of one variable of bounded index. Also we give an example of two functions of bounded index in joint variables such that its sum are not bounded index in joint variables. Thus, a class of entire functions of bounded index in joint variables is not closed under the operation of addition of the functions.

1. Introduction

Let $l : \mathbb{C} \to \mathbb{R}_+$ be a fixed positive continuous function, where $\mathbb{R}_+ = (0, +\infty)$. An entire function f is said to be of bounded $l-$index [15] if there exists an integer m, independent of z, such that for all p and all $z \in \mathbb{C}$ $\frac{|f^{(p)}(z)|}{l^p(z)p!} \leq \max\{\frac{|f^{(s)}(z)|}{l^s(z)s!} : 0 \leq s \leq m\}$. The least such integer m is called the $l-$index of $f(z)$ and is denoted by $N(f, l)$. If $l(z) \equiv 1$ then we obtain a definition of function of *bounded index* [16] and in this case we denote $N(f) := N(f, 1)$.

In 1970, W. J. Pugh and S. M. Shah [21] posed some questions about properties of entire functions of bounded index. One of those questions is following: *II. Classes of functions of bounded index: is the sum (or product) of two functions of bounded index also of bounded index?*

Later W. J. Pugh [20] proved that sum of two functions of bounded index need not be of bounded index. He proposed an example of two entire functions of bounded index such that its sum is a function with unbounded multiplicities of zeros, i.e. it is of unbounded index. Also W. J. Pugh deduced sufficient conditions of index boundedness for a sum of entire functions.

Recently, there was generalized of Pugh's example and theorem for entire functions of bounded L-index in direction [1, 6]. But we also began [11, 7] to study properties of entire functions of bounded $\mathbf{L}$-index in joint variables (see definition below), where $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}^n_+$ is a continuous function.

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Clearly, the question of Shah and Pugh can be formulated for entire in $\mathbb{C}^n$ functions: *Is the sum of two functions of bounded* $\mathbf{L}$*-index in joint variables also of bounded* $\mathbf{L}$*-index in joint variables?*

In the paper, we give an answer to the question and generalize Pugh's example and theorem for this class of entire functions of several variables.

Nore that for every entire function F with bounded multiplicities of zero points [13, 5] there exists a positive continuous function $\mathbf{L}: \mathbb{C}^n \to \mathbb{R}^n_+$ such that F is of bounded $\mathbf{L}$-index in joint variables. Thus, the concept of bounded L-index allows to study properties of very wide class of entire functions.

The concepts of bounded L-index in a direction and bounded $\mathbf{L}$-index in joint variables help to investigate properties of entire solutions of partial differential equations. In particular, if an entire solution has bounded index [3, 12, 2] then it immediately yields its growth estimates, an uniform in a some sense distribution of its zeros, a certain regular behavior of the solution, etc. Also there is known such a result [24] that if entire functions f and f satisfy differential equations with appropriate conditions, then $f+g$ will be of bounded index.

2. Notations, definitions and auxiliary results

Let us introduce some standard notations. Let $\mathbb{R}^n$ and $\mathbb{C}^n$ be n-dimensional real and complex vector spaces, respectively, $n \in \mathbb{N}$. Denote $\mathbb{R}_+ = (0, +\infty)$, $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$. For $K = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$ let us to write $\|K\| = k_1 + \dots + k_n$, $K! = k_1! \cdot \ldots \cdot k_n!$. For $A = (a_1, \dots, a_n) \in \mathbb{C}^n$, $B = (b_1, \dots, b_n) \in \mathbb{C}^n$, we will use formal notations without violation of the existence of these expressions $A \pm B = (a_1 \pm b_1, \dots, a_n \pm b_n)$, $AB = (a_1 b_1, \cdots, a_n b_n)$, $A/B = (a_1/b_1, \dots, a_n/b_n)$, $A^B = a_1^{b_1} a_2^{b_2} \cdot \ldots \cdot a_n^{b_n}$. For $A, B \in \mathbb{R}^n$ $\max\{A, B\} = (\max\{a_1, b_1\}, \dots, \max\{a_n, b_n\})$, a notation $A < B$ means that $a_j < b_j$ for all $j \in \{1, \dots, n\}$; similarly, the relation $A \le B$ is defined.

For $R = (r_1, \dots, r_n)$ we denote by $\mathbb{D}^n(z^0, R) := \{z \in \mathbb{C}^n : \ |z_j - z_j^0| < r_j, \ j \in \{1, \dots, n\}\}$ the polydisc, by $\mathbb{T}^n(z^0, R) := \{z \in \mathbb{C}^n : \ |z_j - z_j^0| = r_j, \ j \in \{1, \dots, n\}\}$ its skeleton and by $\mathbb{D}^n[z^0, R] := \{z \in \mathbb{C}^n : \ |z_j - z_j^0| \le r_j, \ j \in \{1, \dots, n\}\}$ the closed polydisc.

For a partial derivative of entire function $F(z) = F(z_1, \dots, z_n)$ we will use the notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K} = \frac{\partial^{k_1 + \dots + k_n} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}, \text{ where } K = (k_1, \dots, k_n) \in \mathbb{Z}^n_+.$$

Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z)$ are positive continuous functions of variable $z \in \mathbb{C}^n$, $j \in \{1, 2, \dots, n\}$.

An entire function $F(z)$ is called *a function of bounded* $\mathbf{L}$*-index in joint variables* [11, 12], if there exists a number $m \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}^n$ and $J = (j_1, j_2, \dots, j_n) \in \mathbb{Z}^n_+$

$$\frac{|F^{(J)}(z)|}{J! \mathbf{L}^J(z)} \le \max\left\{ \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} : \ K \in \mathbb{Z}^n_+, \ \|K\| \le m \right\}. \tag{1}$$

The least integer m for which inequality (1) holds is called $\mathbf{L}$*-index in joint variables of the function* F and is denoted by $N(F, \mathbf{L})$. If $l_j(z_j) \equiv 1, \ j \in \{1, 2, \dots, n\}$, then the entire function is called a *function of bounded index in joint variables* [14, 17, 18, 19, 22] and $N(F) = N(F, \mathbf{1})$. Also there are papers about analytic functions in the unit ball [8, 10] and in the unit polydisc [9] of bounded L-index in joint variables.

Henceforth for simplicity we consider a case $\mathbf{L}(z) \equiv 1$, i.e. $l_j(z_j) \equiv 1$, $j \in \{1, 2, \ldots, n\}$. We need the following proposition. **Theorem 1** [[11]] An entire function F has bounded index in joint variables if and only if for any R', $R'' \in \mathbb{R}^n_+$ $\mathbf{0} < R' < R''$, there exists a number $p_1 = p_1(R', R'') \geq 1$ such that for every $z^0 \in \mathbb{C}^n$

$$\max\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, R''\right)\right\} \leq p_1 \max\left\{|F(z)|\colon z \in \mathbb{T}^n\left(z^0, R'\right)\right\}. \qquad (2)$$

Below we present an example of entire functions of bounded index in joint variables such that their sum is a function of unbounded index in joint variables. Our construction uses ideas with [20] (see also [23, c.36]).

Example. Let $F(z) = \prod_{j=1}^{n} f(z_j)$, where $f(t) = \cos t + \cosh t$. Since $f^{(4)}(t) \equiv f(t)$, for $\|I\| \geq 3n + 1$ we have

$$\frac{1}{I!}\left|\frac{\partial^{\|I\|} F(z)}{\partial z^I}\right| \leq \frac{G(z)}{4}, \qquad (3)$$

where $G(z) = \max\left\{\frac{1}{K!}\left|\frac{\partial^{\|K\|} F(z)}{\partial z^K}\right| : \|K\| \leq 3n\right\}$. Thus, $F(z)$ has bounded index in joint variables and $N(F) = 3n$.

By analogy to [20], it can be deduced

$$(\forall z \in \mathbb{C}^n)\colon\ G(z) \geq A \exp\left\{0,5 \sum_{j=1}^{n} |z_j|\right\},\ A = const > 0. \qquad (4)$$

Let us consider a function

$$\Phi(z) = \prod_{i=1}^{n} \prod_{j=1}^{\infty} \left(1 + z_i 2^{-j}\right)^j.$$

It is known that the function is of unbounded index in joint variables because multiplicities of its zero points are unbounded (see definition of bounded index in joint variables and [5]). The function Φ is a function with separable variables. Denote $\varphi(t) = \prod_{j=1}^{\infty}(1 + t2^{-j})^j$. This is a function of one complex variable $t \in \mathbb{C}$. In paper [20], for the function there are deduced the following properties

$$(\forall k \in \mathbb{Z}_+)\ \frac{|\varphi^{(k)}(t)|}{k!} \leq BM(2r, \varphi), \qquad (5)$$

$$M(2r, \varphi) \leq C \exp\{0,5|t|\} \leq C' \max\left\{\frac{|f^{(p)}(t)|}{p!} : 0 \leq p \leq 3\right\}, \qquad (6)$$

where $|t| = r$, $B = M(2, \varphi) \geq 1$, $M(r, \varphi) = \max\{|\varphi(t)| :\ |t| = r\}$, C, C' — positive constants.

Let $\varepsilon > 0$ be a sufficiently small number, $H(z) = F(z) + \varepsilon\Phi(z)$. Then in view of (3), (5) and (6) for $\|P\| \geq 3n + 1$

$$\frac{1}{P!}\left|\frac{\partial^P H(z)}{\partial z^P}\right| \leq \frac{G(z)}{4} + \varepsilon B^n \prod_{j=1}^{n} M(2r_j, \varphi) \leq \left(\frac{1}{4} + \varepsilon(BC')^n\right) G(z). \qquad (7)$$

where $r_j = |z_j|$. And for $\|P\| \le 3n$ inequalities (5) and (6) imply

$$\frac{1}{P!}\left|\frac{\partial^P H(z)}{\partial z^P}\right| \ge \frac{1}{P!}\left|\frac{\partial^P F(z)}{\partial z^P}\right| - \varepsilon B^n \prod_{j=1}^{n} M(2r_j,\varphi) \ge \frac{1}{P!}\left|\frac{\partial^P F(z)}{\partial z^P}\right| - \varepsilon (BC')^n G(z),$$

that is

$$\max\left\{\frac{1}{P!}\left|\frac{\partial^P H(z)}{\partial z^P}\right| : \ \|P\| \le 3n\right\} \ge G(z) - \varepsilon(BC')^n G(z) = \\ = (1-\varepsilon(BC')^n)G(z). \tag{8}$$

From (7) and (8) for $\varepsilon = \dfrac{3}{8(BC')^n}$ we deduce

$$\frac{1}{P!}\left|\frac{\partial^P H(z)}{\partial z^P}\right| \le \frac{1/4+\varepsilon(BC')^n}{1-\varepsilon(BC')^n} \max\left\{\frac{1}{K!}\left|\frac{\partial^K H(z)}{\partial z^K}\right| : \|K\| \le 3n\right\} = \\ = \max\left\{\frac{1}{K!}\left|\frac{\partial^K H(z)}{\partial z^K}\right| : \|K\| \le 3n\right\}.$$

Hence, H is a function of bounded index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(H) = 3$. Besides, F is a function of bounded index in the direction $\mathbf{b}$, but $\Phi(z) = \frac{H(z)-F(z)}{\varepsilon}$ is of unbounded index in the direction $\mathbf{b}$.

3. Sufficient conditions of index boundedness for a sum of entire functions

Theorem 2. Let F, G be the entire functions in $\mathbb{C}^n$ satisfying the following conditions

1) $G(z)$ has bounded index in joint variables with $N(G) = N < +\infty$;
2) there exists $\alpha \in (0,1)$ such that for all $z \in \mathbb{C}^n$ and for all $\|P\| \ge N+1$ $(P \in \mathbb{N}^n)$

$$\frac{|G^{(P)}(z)|}{P!} \le \alpha \max\left\{\frac{|G^{(K)}(z)|}{K!} : \|K\| \le N\right\}; \tag{9}$$

3) for some $z^0 \in \mathbb{C}^n$ $(F(z^0) \ne 0)$ and every $z \in \mathbb{C}^n$ the following inequality holds

$$\max\left\{|F(z')| : \ z' \in \mathbb{T}^n\left(z^0, 2R\right)\right\} \le \max\left\{\frac{|G^{(K)}(z)|}{K!} : \ \|K\| \le N\right\} \tag{10}$$

where $r_j = |z_j - z_j^0|$, $R = (r_1, \dots, r_n)$.
4) $(\exists c \ge 1)$ $(\forall z \in \mathbb{C}^n)$ such that $R^P \le 1$ for some $P \in Z_+^n$

$$\frac{\max\left\{|F(z')| : \ z' \in \mathbb{T}^n(z^0, 2R)\right\}}{\max\{\mathbf{1}, R\}^P |F(z^0)|} \le c < +\infty. \tag{11}$$

Then for each $\varepsilon \in \mathbb{C}$, $|\varepsilon| \le \frac{1-\alpha}{2c}$, the function

$$H(z) = G(z) + \varepsilon F(z) \tag{12}$$

has bounded index in joint variables and $N(H) \le N$.

Proof.

We write Cauchy's formula for the entire function $F(z)$

$$\frac{F^{(P)}(z)}{P!} = \frac{1}{(2\pi i)^n} \int_{z' \in \mathbb{T}^n(z,R)} \frac{F(z')}{(z'-z)^{P+\mathbf{1}}} dz'. \tag{13}$$

For the chosen $r_j = |z_j - z_j^0|$ the following inequality holds

$$r_j = |z_j' - z_j| \geq |z_j' - z_j^0| - |z_j - z_j^0| = |z_j' - z_j^0| - r_j.$$

Hence,

$$|z_j' - z_j^0| \leq 2r_j. \tag{14}$$

Equality (13) yields

$$\frac{|F^{(P)}(z)|}{P!} \leq \frac{1}{(2\pi)^n} \cdot \frac{1}{R^{P+\mathbf{1}}} \prod_{j=1}^{n} 2\pi r_j \cdot \max\{|F(z')| : z' \in \mathbb{T}^n(z,R)\} \leq$$
$$\leq \frac{1}{R^P} \max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R)\right\}. \tag{15}$$

If $R^P > 1$ then (15) implies

$$\frac{|F^{(P)}(z)|}{P!} \leq \max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R)\right\}. \tag{16}$$

Let $R^P \in (0;1]$. Then for some $j \in \{1, \dots, n\}$ $r_j = |z_j - z_j^0| \in (0;1]$. Setting $r_j = 1$ for these j in (13) and (14) and $R' = \max\{\mathbf{1}, R\}$ we similarly deduce

$$\frac{|F^{(P)}(z)|}{P!} \leq \frac{1}{R'^P} \max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R'/\mathbf{L}(z))\right\} =$$
$$= \frac{1}{R'^P} \frac{\max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R')\right\}}{\max\{|F(z')| : z' \in \mathbb{T}^n(z^0, 2R)\}} \max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R)\right\} \leq$$
$$\leq \frac{\max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R)\right\}}{R'^P |F(z^0)|} \max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R)\right\} \leq$$
$$\leq c \max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R)\right\}, \tag{17}$$

where

$$c = \sup_{\substack{z\in\mathbb{C}^n \ \exists P \in \mathbb{Z}_+^n \\ |(z-z^0)^P| \leq 1}} \frac{\max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R)\right\}}{\max\{\mathbf{1}, R\}^P |F(z^0)|}.$$

In view of (16) and (17) we have

$$\frac{|F^{(P)}(z)|}{P!} \leq c \max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R)\right\} \tag{18}$$

for all $P \in \mathbb{Z}_+^n$.

We differentiate (12) p times, $\|P\| = p \geq N+1$, and apply consequently (9), (18) and (10) to obtained equality

$$\frac{|H^{(P)}(z)|}{P!} \leq \frac{|G^{(P)}(z)|}{P!} + \frac{|\varepsilon||F^{(P)}(z)|}{P!} \leq \alpha \max\left\{\frac{|G^{(K)}(z)|}{K!} : \|K\| \leq N\right\} +$$
$$+ c|\varepsilon| \max\left\{|F(z')| :\ z' \in \mathbb{T}^n(z^0, 2R)\right\} \leq$$
$$\leq (\alpha + c|\varepsilon|) \max\left\{\frac{|G^{(K)}(z)|}{K!} : \|K\| \leq N\right\}. \tag{19}$$

If $\|S\| \leq N$, then (18) holds with $P = S$, but (9) is not valid. Therefore, the differentiation of (12) give us the lower estimate

$$\frac{|H^{(S)}(z)|}{S!} \geq \frac{|G^{(S)}(z)|}{S!} - \frac{|\varepsilon||F^{(S)}(z)|}{S!} \geq$$
$$\geq \frac{|G^{(S)}(z)|}{S!} - c|\varepsilon| \max\left\{|F(z')| : \ z' \in \mathbb{T}^n(z^0, 2R)\right\}, \tag{20}$$

where $\|S\| \leq N$. Hence, (10) and (20) imply that

$$\max\left\{\frac{|H^{(S)}(z)|}{S!} : \|S\| \leq N\right\} \geq (1 - c|\varepsilon|) \max\left\{\frac{|G^{(S)}(z)|}{S!} : \|S\| \leq N\right\}. \tag{21}$$

If $c|\varepsilon| < 1$, then (19) and (21) yield

$$\frac{|H^{(P)}(z)|}{P!} \leq \frac{\alpha + c|\varepsilon|}{1 - c|\varepsilon|} \max_{0 \leq s \leq N} \left\{\frac{|H^{(S)}(z)|}{S!} : \|S\| \leq N\right\} \tag{22}$$

for $\|P\| \geq N + 1$. We assume that $\frac{\alpha + c|\varepsilon|}{1 - c|\varepsilon|} \leq 1$. Hence, $|\varepsilon| \leq \frac{1-\alpha}{2c}$. For these ε the function H has bounded index in joint variables with $N(H) \leq N$. Theorem 2 is proved.

At the fact, condition 2) in Theorem 2 can be removed. The following theorem is true.

Theorem 3. Let $\alpha \in (0, 1)$ and F, G be the entire functions in $\mathbb{C}^n$ satisfying conditions:

1) $G(z)$ has bounded index in joint variables;
2) for some $z^0 \in \mathbb{C}^n$ $(F(z^0) \neq 0)$ and every $z \in \mathbb{C}^n$ the following inequality holds

$$\max\left\{|F(z')| : \ z' \in \mathbb{T}^n\left(z^0, 2R\right)\right\} \leq \max\left\{\frac{|G^{(K)}(z)|}{K!} : \ \|K\| \leq N(G_\alpha)\right\}$$

where $r_j = |z_j - z_j^0|$, $R = (r_1, \ldots, r_n)$.

If $|\varepsilon| \leq \frac{1-\alpha}{2c}$ then the function

$$H(z) = G(z) + \varepsilon F(z)$$

has bounded index in joint variables with $N(H) \leq N(G_\alpha)$, where $G_\alpha(z) = G(z/\alpha)$.

Proof. Condition 2) in Theorem 2 always holds with $N = N(G_\alpha)$ instead $N = N(G)$, where $G_\alpha(z) = G(z/\alpha)$, $\alpha \in (0, 1)$. Indeed, by Theorem 1 inequality (2) holds for the function G. Substituting $\frac{z^0}{\alpha}$, $\frac{z}{\alpha}$ instead of z^0, z in (2) we obtain

$$\max\left\{|G(z/\alpha)| \colon z \in \mathbb{T}^n\left(z^0, R''\alpha\right)\right\} \leq p_1 \max\left\{|G(z/\alpha)| \colon z \in \mathbb{T}^n\left(z^0, R'\alpha\right)\right\}. \tag{23}$$

By Theorem 1 inequality (23) implies that $G_\alpha(z) = G(z/\alpha)$ has bounded index in joint variables and vice versa. Then for $\|P\| \geq N(G_\alpha) + 1$ and $\alpha \in (0, 1)$ we have

$$\frac{|G_\alpha^{(P)}(z)|}{P!} = \frac{|G^{(P)}(z/\alpha)|}{\alpha^{\|P\|} P!} \leq \max\left\{\frac{|G_\alpha^{(S)}(z)|}{S!} \colon \ \|S\| \leq N(G_\alpha)\right\} =$$
$$= \max\left\{\frac{|G^{(S)}(z/\alpha)|}{\alpha^{\|S\|} S!} : \ \|S\| \leq N(G_\alpha)\right\}.$$

Hence, for all $z \in \mathbb{C}^n$

$$\frac{|G^{(P)}(z/\alpha)|}{P!} \leq \max\left\{\frac{\alpha^{\|P\|-\|S\|}|G(z/\alpha)|}{S!} : \|S\| \leq N(G_\alpha)\right\} \leq$$
$$\leq \alpha \max\left\{\frac{|G^{(S)}(z/\alpha)|}{S!} : \ \|S\| \leq N(G_\alpha)\right\}. \tag{24}$$

Thus, inequality (24) implies (9).

Remark. It is easy to prove that $N(G_\alpha) \leq N(G)$ for $\alpha \in (0,1)$. Therefore $N(G_\alpha)$ in Theorem 3 can be replaced by $N(G)$.

References

[1] A.I. Bandura, Product of two entire functions of bounded L-index in direction is a function with the same class, Bukovyn. Mat. Zh. Vol. 4, no. 1-2, 8-12, 2016.

[2] A.I. Bandura, O.B. Skaskiv, Directional logarithmic derivative and the distribution of zeros of an entire function of bounded L-index along the direction, Ukrain. Mat. J., Vol. 69, no. 3, 500-508, 2017, doi:10.1007/s11253-017-1377-8

[3] A. Bandura, O. Skaskiv, P. Filevych, Properties of entire solutions of some linear PDE's, J. Appl. Math. Comput. Mech., Vol. 16, no. 2, 17–28, 2017. doi:10.17512/jamcm.2017.2.02

[4] A.I. Bandura, O.B. Skaskiv, Asymptotic estimates of entire functions of bounded **L**-index in joint variables, https://arxiv.org/abs/1701.08276 (submitted to Novi Sad Journal of Mathematics).

[5] A.I. Bandura, O.B. Skaskiv, Iyer's metric space, existence theorem and entire functions of bounded **L**-index in joint variables, Bukovyn. Mat. Zh., Vol. 5, no. 3-4, 2017 (in print).

[6] A.I. Bandura, Sum of entire functions of bounded L-index in direction, Mat. Stud. Vol. 45, no. 2, 149–158, 2016, doi:10.15330/ms.45.2.149-158

[7] A. Bandura, New criteria of boundedness of **L**-index in joint variables for entire functions, Math. Bull. Shevchenko Sci. Soc., Vol. 13, 58-67, 2016.

[8] A. Bandura, O. Skaskiv, Functions analytic in a unit ball of bounded L-index in joint variables, J. Math. Sci. Vol. 227, no. 1, 1-12, 2017, doi:10.1007/s10958-017-3570-6

[9] A.I. Bandura, N.V. Petrechko, O.B. Skaskiv, Analytic functions in a polydisc of bounded **L**-index in joint variables, Mat. Stud., Vol. 46, no. 1, 72-80, 2016, doi:10.15330/ms.46.1.72-80

[10] A. Bandura, O. Skaskiv, Analytic functions in the unit ball. Bounded L-index in joint variables and solutions of systems of PDE's, Beau-Bassin: LAP Lambert Academic Publishing, 2017, 100 p. https://arxiv.org/abs/1705.09568

[11] A.I. Bandura, M.T. Bordulyak, O.B. Skaskiv, Sufficient conditions of boundedness of L-index in joint variables, Mat. Stud., Vol. 45, no. 1, 12–26, 2016.

[12] A. Bandura, O. Skaskiv, Entire functions of several variables of bounded index, Lviv: Chyslo, Publisher I. E. Chyzhykov, 2016, 128 p. http://chyslo.com.ua/

[13] M.T. Bordulyak, A proof of Sheremeta conjecture concerning entire function of bounded l-index, Mat. Stud., Vol. 11, no. 2, 108-110, 1999.

[14] G.J. Krishna, S.M. Shah, Functions of bounded indices in one and several complex variables, In: Mathematical essays dedicated to A. J. Macintyre, Ohio Univ. Press, Athens, Ohio, 1970, 223–235.

[15] A.D. Kuzyk, M.N. Sheremeta, Entire functions of bounded l-distribution of values, Math. Notes, **39** (1986), No 1, 3–8.

[16] B. Lepson, Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index, Proc. Sympos. Pure Math., Vol. 2, 298–307, 1968.

[17] F. Nuray, R.F. Patterson, Entire bivariate functions of exponential type, Bull. Math. Sci. **5**, no. 2, 171–177, 2015, doi:10.1007/s13373-015-0066-x

[18] F. Nuray, R.F. Patterson, Multivalence of bivariate functions of bounded index, Le Matematiche, 70, no. 2, 225–233, 2015, doi:10.4418/2015.70.2.14

[19] R.F. Patterson, F. Nuray, A characterization of holomorphic bivariate functions of bounded index, Mathematica Slovaca, Vol. 67, no.3, 731–736, 2017, doi:10.1515/ms-2017-0005

[20] W.J. Pugh, Sums of functions of bounded index, Proc. Amer. Math. Soc., Vol. 22, 319–323, 1969.

[21] W.J. Pugh, S.M. Shah, On the growth of entire functions of bounded index. Pacific J. Math. Vol. 33, no. 1, 191–201, 1970.
[22] M. Salmassi, Functions of bounded indices in several variables, Indian J. Math., Vol, 31, no. 3, 249–257, 1989.
[23] M. Sheremeta, Analytic functions of bounded index, Lviv: VNTL Publishers, 1999, 141 p.
[24] R. Roy, S.M. Shah, Sums of functions of bounded index and ordinary differential equations, Complex Var. Elliptic Equ. Vol. 12, 95–100, 1989.

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