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A NOTE ON JENSEN'S INEQUALITY INVOLVING MONETARY UTILITY FUNCTIONS

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Abstract. In this note, we extend the results in [8] and establish that Jensens inequality is valid for quasiconcave monetary utility functions regarding some convex, concave, quasiconvex and quasiconcave functions. In connection with quasiconvex and quasiconcave functions that are in linear fractional form, this paper establishes further that the Jensen's inequality is valid for the utility functions under study.

1. Introduction

Monetary utility functions are bounded non-linear functions that are asymmetric. In recent times, these functions attracted so much interest as a result of their applications and profound implications in the processes of decision-making especially in finance ([1], [3], [5], [7]).

In a business environment, the optimal performance for a given decision rests on the achievable results of all other decisions in the same period of time. So, outcomes of decisions influence utility. The interest at most times, is not only to measure the utility of an uncertain random variable, but also to estimate the utility of its function.

The inequality of Jensen demonstrates that it is a relevant tool for addressing this problem. It holds for classical expectation and thus is a form of monetary utility function. Monetary utility function U can be associated with the convex risk measure by $U(\xi) = -\rho(\xi)$ [7].

2. Preliminaries

Let (Ω, F, P) be a probability space with Ω describing the collection of all possible outcomes; F the collection of complex events used to characterize groups of outcomes and P the probability measure function. Assume that

$$L^\infty = \{\{x_n\}_{n=1}^\infty : |x_n| \leq M, \forall n \in N, M \in \Re^+\}$$

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is a space of bounded random variables.

Definition 1 Let $S, T \in L^{\infty}$ and $m \in \Re$. A function $U : L^{\infty} \longrightarrow \Re$ is called a monetary utility function if it is nondecreasing in relation to the order of L^{∞} and satisfies the following conditions:

- Concavity: $U(\lambda S + (1-\lambda)T) \geq \lambda U(S) + (1-\lambda)U(T)$ for all $\lambda \in [0,1]$.
- Normalization: $U(S) \geq 0$ if $S \geq 0$.
- Monotonicity: $U(S) \geq U(T)$ for $S \geq T$.
- Monetary or cash invariance property: $U(S+m) = U(S) + m$ (See also [5]).
- Fatou's property: If $\{\sup \|S_n\|\}_{n=1}^{\infty} < \infty$ and the sequence $S_n \to S$ in probability, then $U(S) \geq \limsup U(S_n)$.

Remark 1 The monotonicity and monetary property suggest that U is finite and Lipschitz-continuous on L^{∞} . Thus, the normalization $U(0) = 0$ does not put a restriction on the generality as it is obtaineable by adding a constraint [7].

Definition 2 A function $\varphi : X \to \Re$ defined on a convex subset X of a real vector space is called quasiconvex if

$$\varphi(\lambda x + (1-\lambda)y) \leq \max\{\varphi(x), \varphi(y)\},$$

for any $x, y \in X$ and $\lambda \in [0,1]$. Furthermore, if

$$\varphi(\lambda x + (1-\lambda)y) < \max\{\varphi(x), \varphi(y)\}$$

for any $x \neq y$ and $\lambda \in [0,1]$ then φ is strictly quasiconvex. A function φ is called quasiconcave if $-\varphi$ is quasiconvex and it is strictly quasiconcave if $-\varphi$ is strictly quasiconvex. That is, φ is quasiconcave if

$$\varphi(\lambda x + (1-\lambda)y) \geq \min\{\varphi(x), \varphi(y)\}$$

and strictly quasiconcave if

$$\varphi(\lambda x + (1-\lambda)y) > \min\{\varphi(x), \varphi(y)\}.$$

Some examples of Quasiconvex functions are $\sqrt{|x|}$ on $\Re$ and $\log x$ on $\Re^+$. Note also that $\log x$ on $\Re^+$ is quasilinear (both quasiconvex and quasiconcave). We also present the discrete form of the Jensen's inequality as follows:

Theorem 1 Let φ be a continuous and convex function on an interval I. If $u_1, u_2, \cdots, u_n$ are in I and $0 < \lambda_1, \lambda_2, \ldots, \lambda_n < 1$ with $\sum_{j=1}^{n} \lambda_j = 1$. Then

$$\varphi\left(\sum_{j=1}^{n} \lambda_j u_j\right) \leq \sum_{j=1}^{n} \lambda_j \varphi(u_j).$$

Proof See [6, p.73].

3. Results and Discussions

The discussion in this section is mainly on Jensen's Inequality involving Monetary utility functions. We begin with a Lemma which is of vital importance in establishing our main results. This Lemma is an extension of Proposition 2.1 in [8] to cover the notion of quasiconcavity on the monetary utility function.

Lemma 1 For any monotonic quasiconcave function U, the inequality

$$U(\lambda S + (1-\lambda)T) \geq \lambda U(S) + (1-\lambda)U(T) \geq \min\{U(S), U(T)\} \tag{1}$$

holds for all $S, T \in L^{\infty}$, $\lambda \in [0,1]$.

Proof The idea is to prove the second part of the double inequality (1) by geometrical notion. The first part is trivial since it defines concavity of U. The convex combination of the points $S, T \in L^{\infty}$ is the line segment connecting the two points expressed as $(1-\lambda)T + \lambda S$ for $\lambda \in [0,1]$. Similarly, the convex combination of the points $U(S), U(T) \in \Re$ is the line segment connecting the two points expressed as $(1-\lambda)U(T) + \lambda U(S)$ for $\lambda \in [0,1]$. Since U is nondecreasing, then $U(S) \geq U(T)$ for $S \geq T$. Hence

$$\lambda U(S) + (1-\lambda)U(T) \geq \min\{U(S), U(T)\}.$$

This concludes the proof.

Lemma 2 Let $U : L^{\infty} \longrightarrow \Re$ be a quasiconcave monetary utility function. For any $\lambda \in \Re$, $S \in L^{\infty}$, the inequalities

$$U(\lambda S) \geq \lambda U(S), \quad if \quad 0 \leq \lambda \leq 1 \tag{2}$$

or

$$U(\lambda S) \leq \lambda U(S), \quad if \quad \lambda \leq 0 \quad or \quad \lambda \geq 1 \tag{3}$$

hold.

Proof

Let $0 \leq \lambda \leq 1$. Since U is nondecreasing and $S \geq T$ then $U(T) = \min\{U(S), U(T)\}$. By inequality (1) we have

$$U(\lambda S + (1-\lambda)T) \geq \lambda U(S) + (1-\lambda)U(T) \geq U(T)$$

which yields

$$U(\lambda S + (1-\lambda)T) \geq \lambda U(S) + U(T) - \lambda U(T) - U(T) \geq 0.$$

This simplifies to

$$U(\lambda S + (1-\lambda)T) \geq \lambda U(S) - \lambda U(T).$$

Take $T = 0$, $U(0) = 0$ (Normalization condition). Thus,

$$U(\lambda S) \geq \lambda U(S)$$

which proves the inequality (2).

Now we prove (3) for $\lambda \geq 1$. That is $0 \leq \frac{1}{\lambda} \leq 1$. By applying (2) we obtain

$$U\left(\frac{1}{\lambda}(\lambda S)\right) \geq \frac{1}{\lambda}U(\lambda S)$$

$$\lambda U\left(\frac{1}{\lambda}(\lambda S)\right) \geq U(\lambda S)$$

$$\lambda U(S) \geq U(\lambda S)$$

$$U(\lambda S) \leq \lambda U(S)$$

Also, we prove (3) for $-1 \leq \lambda \leq 0$. That is $0 \leq -\lambda \leq 1$. By applying (2) we have

$$U(-\lambda S) \geq (-\lambda)U(S). \tag{4}$$

But
$0 = U(0) = U\{(\frac{1}{2}\lambda S) + \frac{1}{2}(-\lambda S)\} \geq \frac{1}{2}U(\lambda S) + \frac{1}{2}U((-\lambda)S)$

$$0 \geq \frac{1}{2}U(\lambda S) + \frac{1}{2}U((-\lambda)S)$$

$$-U(\lambda S) \geq U(-\lambda S)$$
$$U(-\lambda S) \leq -U(\lambda S) \tag{5}$$

Combining (4) and (5) yields

$$(-\lambda)U(S) \leq U(-\lambda S) \leq -U(\lambda S).$$

Thus

$$(-\lambda)U(S) \leq -U(\lambda S)$$

implies

$$U(\lambda S) \leq \lambda U(S).$$

as required.

The next considerations are some two theorems that characterize the inequality of Jensen involving monetary utility functions. In these cases, the notions of subdifferentiability, monotonicity and cash invariance of the monetary unitility function are used to establish the results.

Theorem 2 Let $U : L^{\infty} \longrightarrow \Re$ be a monetary utility function. If φ is any convex function on $\Re$ with a subdifferential at $U(S)$, then

$$\varphi(U(S)) \leq U(\varphi(S))$$

for all $S \in L^{\infty}$.

Proof Let $\lambda \in [0, 1]$ be a subdifferential of φ at $U(S)$. Following the discussion in [4, Remark 4.2], φ admits a support line on time scale at $U(S)$ and

$$\varphi(S) \geq \varphi(U(S)) + \lambda(S - U(S)). \tag{6}$$

Applying to (6), the properties of monotonicity and cash invariance of U together with inequality (2), we get

$$U(\varphi(S)) \geq U[\varphi(U(S)) + \lambda(S - U(S))]$$

$$U(\varphi(S)) \geq \varphi(U(S)) + \lambda U(S) - \lambda U(S)$$

$$U(\varphi(S)) \geq \varphi(U(S)).$$

Or

$$\varphi(U(S)) \leq U(\varphi(S)).$$

Theorem 3 Let $U : L^{\infty} \longrightarrow \Re$ be a monetary utility function and ϕ is any concave function on $\Re$ with a subdifferential at $U(S)$, then

$$\phi(U(S)) \geq U(\phi(S))$$

for all $S \in L^{\infty}$.

Proof Following the discussion in Theorem 2 with

$$\phi(S) \leq \phi(U(S)) + \lambda(S - U(S))$$

for $\lambda \in [0,1]$, we get

$$\phi(U(S)) \geq U(\phi(S)).$$

Thus, an extension has been established for Jensen's inequality involving quasiconcave monetary utility functions regarding convex and concave functions. Note that Jensen's inequality does not hold for all monetary utility functions in relation to concavity or convexity even when such cases are linear (see also [8]).

Now consider when φ and ϕ are quasiconvex and quasiconcave respectively.

Theorem 4 Let $U : L^{\infty} \longrightarrow \Re$ be a monetary utility function and φ is a quasiconvex function on $\Re$, then

$$\varphi(U(S)) \leq U(\varphi(S))$$

for all $S \in L^{\infty}$.

Proof Based on the subdifferential property for a quasiconvex function in [2], we have

$$\lambda(S - U(S)) \geq 0 \implies \varphi(S) \geq \varphi(U(S)).$$

Or

$$\lambda S - \lambda U(S) \geq 0 \implies \varphi(S) \geq \varphi(U(S)).$$

From the cash invariance property of U, we have

$$U(\lambda S) - \lambda U(S)) \geq 0 \implies U(\varphi(S)) \geq U(\varphi(U(S))).$$

Or

$$U(\lambda S) \geq \lambda U(S) \implies U(\varphi(S)) \geq \varphi(U(S)).$$

Thus

$$U(\lambda S) \geq \lambda U(S) \implies \varphi(U(S)) \leq U(\varphi(S)).$$

This proves that Jensen's inequality is valid for monetary utility functions in connection with quasiconcave functions based on the concept of subdifferential of a function.

Theorem 5 Let ϕ be a quasiconcave function on $\Re$ and $U : L^{\infty} \longrightarrow \Re$ be a monetary utility function. For all $S \in L^{\infty}$, $\phi(U(S)) \geq U(\phi(S))$.

Proof The proof is similar to those in Theorems 2 and 3.

The example below shows that Jensen's inequality is true for all quasiconcave monetary utility functions with respect to certain quasiconvex and quasiconcave functions.

Example 1 Let $\varphi(S) = \frac{\lambda S+c}{\Upsilon S+d}$, $\Upsilon S + d \neq 0, \lambda, \Upsilon \in \Re$ and

$\phi(S) = \frac{\alpha S+e}{\beta S+f}$, $\beta S + f \neq 0, \alpha, \beta \in \Re$ be quasiconvex and quasiconcave functions respectively on $\Re$. We see that Jensen's inequality holds for all quasiconcave monetary utility functions in respect of quasiconcave and quasiconvex functions that are linear-fractional functions.

Illustration

$$\varphi(S) = \frac{\lambda S + c}{\Upsilon S + d}$$

implies
$$\varphi(S) = \frac{(\lambda/\Upsilon)(\Upsilon S + d) + c - (\lambda d/\Upsilon)}{\Upsilon S + d}.$$
Then
$$\varphi(U(S)) = \frac{(\lambda/\Upsilon)(\Upsilon(U(S)) + d) + c - (\lambda d/\Upsilon)}{\Upsilon(U(S)) + d}$$
implies
$$\varphi(U(S)) \leq \frac{U[(\lambda/\Upsilon)(\Upsilon S + d) + c - (\lambda d/\Upsilon)]}{U(\Upsilon S + d)}$$
$$\varphi(U(S)) \leq U\left(\frac{(\lambda/\Upsilon)(\Upsilon S + d) + c - (\lambda d/\Upsilon)}{\Upsilon S + d}\right).$$
Thus,
$$\varphi(U(S)) \leq U(\varphi(S)).$$
Similarly,
$$\phi(S) = \frac{\alpha S + e}{\beta S + f}$$
implies
$$\phi(S) = \frac{(\alpha/\beta)(\beta S + f) + e - (\alpha f/\beta)}{\beta S + f}.$$
Then
$$\phi(U(S)) = \frac{(\alpha/\beta)(\beta(U(S)) + f) + e - (\alpha f/\beta)}{\beta(U(S)) + f}$$
implies
$$\phi(U(S)) \geq \frac{U[(\alpha/\beta)(\beta S + f) + e - (\alpha f/\beta)]}{U(\beta(S)) + f)}$$
$$\phi(U(S)) \geq U\left(\frac{(\alpha/\beta)(\beta S + f) + e - (\alpha f/\beta)}{\beta S + f}\right).$$
Therefore
$$\phi(U(S)) \geq U(\phi(S))$$
satisfying Jensen's inequality.

We also examine the entropic utility function [8] similar to [1] which is defined as
$$U(S) = -\ln E[\exp(-S)]$$
via quasiconvex and quasiconcave functions. The example below serves as an illustration.

Example 2 Let $\varphi(S) = \frac{S+1}{S+2}$ and $\phi(S) = -\frac{S+1}{S+2}$ be quasiconvex and quasiconcave functions respectively. Choose $S \in L^\infty$ such that $P(S = 0) = P(S = 1) = 0.5$.
For $S = 0$,

$\varphi(0) = \frac{1}{2}$ and $U(0) = -\ln E[\exp(-0)] = 0$

$\varphi(U(0)) = \varphi(0) = \frac{1}{2}$

$U(\varphi(0)) = U(\frac{1}{2}) = -\ln E[\exp(-\frac{1}{2})] = \frac{1}{2}$
Taking $S = 1$, we have
$\varphi(1) = \frac{2}{3}$ and $U(1) = -\ln E[\exp(-1)] = 1$.
Therefore
$\varphi(U(1)) = \varphi(1) = \frac{2}{3}$
$U(\varphi(1)) = U(\frac{2}{3}) = -\ln E[\exp(-\frac{2}{3})] = \frac{2}{3}$.

Similarly, for $S = 0$,
$\phi(0) = -\frac{1}{2}$ and $U(0) = -\ln E[\exp(-0)] = 0$
$\phi(U(0)) = \phi(0) = -\frac{1}{2}$
$U(\phi(0)) = U(-\frac{1}{2}) = -\ln E[\exp(\frac{1}{2})] = -\frac{1}{2}$
For, $S = 1$, we have
$\phi(1) = -\frac{2}{3}$ and $U(1) = -\ln E[\exp(-1)] = 1$
$\phi(U(1)) = \phi(1) = -\frac{2}{3}$
$U(\phi(0)) = U(-\frac{2}{3}) = -\ln E[\exp(\frac{2}{3})] = -\frac{2}{3}$.

Finally in the example below, Jensen's inequality is applied similar to the results in [8] for monetary utility functions.

Example 3 Using the entropic utility of the future outcome S, we can estimate the entropic utility of S^+ or S^-. The inequality of Jensen is a useful tool for this purpose. Suppose that $\varphi(S) = S^+$ and $\phi(S) = S^-$ are quasiconvex and quasiconcave functions satisfying Theorems 2 and 3 respectively, then
$-\ln E[\exp(-S^+)] \geq -\ln E[\exp(-(-S^+)]$ and
$-\ln E[\exp(-S^-)] \leq -\ln E[\exp(-(-S^-)]$.

4. Conclusion

Jensen's inequality for quasiconcave or quasiconvex utility functions of monetary nature were examined. The results established show that the inequality of Jensen holds for quasiconcave and quasiconvex functions on time scales.

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