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ASYMPTOTICALLY LACUNARY EQUIVALENT SEQUENCE SPACES DEFINED BY IDEAL CONVERGENCE AND AN ORLICZ FUNCTION

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Abstract. The purpose of this paper is to introduce certain new sequence spaces using ideal convergence, a lacunary sequence $\theta = (k_r)$,a strictly positive sequence $p = (p_k)$,and an Orlicz function and examine some of their properties.

1. Introduction

Let s, ℓ_∞, c denote the spaces of all real sequences, bounded, and convergent sequences,respectively. Any subspace of s is called a sequence space.

Following Freedman et al.[5], we call the sequence $\theta = (k_r)$ lacunary if it is an increasing sequence of integers such that $k_0 = 0, h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$ .The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = k_r/k_{r-1}$ .These notations will be used troughout the paper. The sequence space of lacunary strongly convergent sequences N_θ was defined by Freedman et al.[5], as follows:

$$N_\theta = \{x = (x_i) \in s : \lim_r h_r^{-1} \sum_{i \in I_r} |x_i - s| = 0 \text{ for some s}\}.$$

Orlicz [8] used the idea of Orlicz function to construct the space L^M. An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$,whic his continuous, nondecreasing and convex with $M(0) = 0, M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists constant $K > 0$, such that $M(2u) \leq KM(u)$ $(u \geq 0)$. It is also easy to see that always $K > 2$. The Δ_2-conditionis equivalent to the satisfactionofin equality $M(Lu) \leq KLM(u)$ for all values of u and $L > 1$.

Remark 1. An Orlicz function satisfies the inequality$M(\lambda x) < \lambda M(x)$ for all λ with $0 < \lambda < 1$

The following well known inequality will be used troughout the paper;

(1) $|a_i + b_i|^{p_i} \leq T(|a_i|^{p_i} + |b_i|^{p_i})$

where a_i and b_i are complex numbers, $T = max(1, 2^{H-1})$,and $H = supp_i < \infty$.

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Marouf presented definitions for asymptotically equivalent sequences and asymptotic regular matrices in [7]. Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices in [9].Subsequently,many

authors have shown their interest to solve different problems arising in this area (see [1],[3],and [10]).

Kostyrko et al. [6] introduced the notion of I-convergence with the help of an admissible ideal I,which denotes the ideal of subsets of N, which is a generalization of statistical convergence. Quite recently, Das et al. [4] unified these two approaches to introduce new concepts such as I- statistical convergence and I-lacunary statistical convergence and investigated some of their consequences. For more applications of ideals we refer to [2,6,11] where many important references can be found.

Recently, Karakuş and Bilgin[3] used an Orlicz function to define some notions of asymptotically equivalent sequences and studied some of their connections. This paper extended these concepts by presenting a non-trivial ideal I.We introduce some new notions,(M,p)-asymptotically equivalent of multiple L, strong (M,p)-asymptotically equivalent of multiple L, and strong (M,p)-asymptotically lacunary equivalent of multiple $L$with respect to the ideal I which is a natural comon-trivial ideal I, Lacunary sequence, a strictly positive sequence $p=(p_k)$,and Orlicz function. In addition to these definitions, we obtain some revelant connections between these notions.

2. Definitions and Notations

Now we recall some definitions of sequence spaces .

Definition 2.1. Two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically equivalent if $\lim_k \frac{x_k}{y_k}=1$,(denoted by $x\sim y$).

Definition 2.2. Two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon>0$,

$$\lim_n \frac{1}{n}\left|\left\{\ k\leq n:\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\right|=0,$$

(denoted by $x\overset{S}{\sim}y$) and simply asymptotically statistical equivalent, if $L=1$.

Definition 2.3.Two nonnegative sequences $[x]$ and $[y]$ are said to be strong asymptotically equivalent of multiple L provided that

$\lim_n \frac{1}{n}\sum\limits_{k=1}^{n}\left|\frac{x_k}{y_k}-L\right|=0$, (denoted by $x\overset{w}{\sim}y$) and simply strong asymptotically equivalent, if $L=1$.

Definition 2.4. Let θ be a lacunary sequence; the two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon>0$,

$\lim_r \frac{1}{h_r}\left|\left\{\ k\in I_r:\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\right|=0$,(denoted by $x\overset{S_\theta}{\sim}y$) and simply asymptotically lacunary statistical equivalent, if $L=1$.

Definition 2.5. Let θ be a lacunary sequence; the two nonnegative sequences $[x]$ and $[y]$ are said to be strong asymptotically lacunary equivalent of multiple L provided that $\lim_r \frac{1}{h_r}\sum\limits_{k\in I_r}\left|\frac{x_k}{y_k}-L\right|=0$ (denoted by $x\overset{N_\theta}{\sim}y$) and simply strong asymptotically lacunary equivalent, if $L=1$.

Definition 2.6. Let M be any Orlicz function; the two nonnegative sequences $[x]$ and $[y]$are said to be M-asymptotically equivalent of multiple L provided that, $lim_k M(\left|\frac{x_k}{y_k} - L\right| /\rho) = 0$,for some $\rho > 0$, (denoted by $x \overset{M}{\sim} y$) and simply strong $M-$asymptotically equivalent, if $L = 1$.

Definition 2.7. Let M be any Orlicz function; the two nonnegative sequences $[x]$ and $[y]$ are said to be strong M-asymptotically equivalent of multiple L provided that, $\lim_n \frac{1}{n} \sum_{k=1}^{n} M(\left|\frac{x_k}{y_k} - L\right| /\rho) = 0$,for some $\rho > 0$,(denoted by $x \overset{w^M}{\sim} y$) and simply strong $M-$asymptotically equivalent, if $L = 1$.

Definition 2.8. Let M be any Orlicz function and θ be a lacunary sequence; the two nonnegative sequences $[x]$ and $[y]$ are said to be strong M-asymptotically lacunary equivalent of multiple L provided that

$\lim_r \frac{1}{h_r} \sum_{k \in I_r} M(\left|\frac{x_k}{y_k} - L\right| /\rho) = 0$,for some $\rho > 0$, (denoted by $x \overset{N_\theta^M}{\sim} y$) and simply strong M-asymptotically lacunary equivalent, if $L = 1$.

For any non-empty set X, let $P(X)$ denote the power set of X.

Definition 2.9. A family $I \subseteq P(X)$is said to be an ideal in X if

(i) $\emptyset \in I$;

(ii) $A, B \in I$ imply $A \cup B \in I$ and

(iii) $A \in I, B \subset A$ imply $B \in I$.

Definition 2.10. A non-empty family $F \subseteq P(X)$is said to be a filter in X if

(i) $\emptyset \notin F$;

(ii) $A, B \in F$ imply $A \cap B \in F$ and

(iii) $A \in F, B \supset A$ imply $B \in F$.

An ideal I is said to be non-trivial if $I \neq \{\emptyset\}$ and $X \notin I$.A non-trivial ideal I is called admissible if it contains all the singleton sets. Moreover, if I is a non-trivial ideal on X, then $F = F(I) = \{X - A : A \in I\}$ is a filter on X and conversely. The filter$F(I)$ is called the filter associated with the ideal I.

Definition 2.11. Let $I \subset P(N)$ be a non-trivial ideal in N. A sequence $[x]$ in X is said to be I-convergent to ξ if for each $\varepsilon > 0$, the set $\{k \in N : |x_k - \xi| \geq \varepsilon\} \in I$.

In this case, we write $I-lim_{k\to\infty} x_k = \xi$.A sequence $[x]$ in X is said to be $I-null$ if L = 0. In this case we write $I - lim_{k\to\infty} x_k = 0$.

Definition 2.12. A sequence $[x]$ of numbers is said to be I-statistical convergent or $S(I)$-convergent to L, if for every $\varepsilon > 0$ and $\delta > 0$, we have

$$\left\{ n \in N; \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| \geq \delta \right\} \in I.$$

In this case, we write $x_k \to L(S(I))$ or$S(I)$ - $lim_{k\to\infty} x_k = L$.

Definition 2.13 Let $I \subset P(N)$ be a non-trivial ideal in N. The two non-negative sequences $[x]$ and $[y]$ are said to be strongly asymptotically equivalent of multiple $L$with respect to the ideal I provided that for each $\varepsilon > 0$

$\left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon \right\} \in I,$

(denoted by $x \overset{I(w)}{\sim} y$) and simply strongly asymptotically equivalent with respect to the ideal I, if $L = 1$.

Definition 2.14. Let $I \subset P(N)$ be a non-trivial ideal in N and $\theta = (k_r)$ be a lacunary sequence. The two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically lacunary statistical equivalent of multiple L with respect to the ideal I provided that for each $\varepsilon > 0$ and $\gamma > 0$,

$$\left\{r \in N; \frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\}\right| \geq \gamma\right\} \in I$$

(denoted by $x \overset{I(S_\theta)}{\sim} y$) and simply asymptotically lacunary statistical equivalent with respect to the ideal I, if $L = 1$.

Definition 2.15. Let $I \subset P(N)$ be a non-trivial ideal in N and $\theta = (k_r)$ be a lacunary sequence. The two non-negative sequences $[x]$ and $[y]$ are said to be strongly asymptotically lacunary equivalent of multiple L with respect to the ideal I provided that for $\varepsilon > 0$,

$$\left\{r \in N; \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\} \in I$$

(denoted by $x \overset{I(N_\theta)}{\sim} y$) and simply asymptotically lacunary equivalent with respect to the ideal I, if $L = 1$.

3. Main Results

We now consider our main results. We begin with the following definitions.

Definition 3.1. Let $I \subset P(N)$ be a non-trivial ideal in N and M be any Orlicz function. The two non-negative sequences $[x]$ and $[y]$ are said to be M-asymptotically equivalent of multiple L with respect to the ideal I provided that for each $\varepsilon > 0$

$$\left\{k \in N; M(\left|\frac{x_k}{y_k} - L\right|/\rho) \geq \varepsilon\right\} \in I, \text{for some } \rho > 0,$$

(denoted by $x \overset{I(M)}{\sim} y$) and simply M- asymptotically equivalent with respect to the ideal I, if $L = 1$.

Definition 3.2. Let $I \subset P(N)$ be a non-trivial ideal in N , M be any Orlicz function, and $p = (p_k)$ be a sequence of positive real numbers. Two number sequences $[x]$ and $[y]$ are said to be strongly (M,p)-asymptotically equivalent of multiple L with respect to the ideal I provided that for each $\varepsilon > 0$,

$$\left\{n \in N; \frac{1}{n}\sum_{k=1}^{n}\left[M(\left|\frac{x_k}{y_k} - L\right|/\rho)\right]^{p_k} \geq \varepsilon\right\} \in I, \text{for some } \rho > 0,$$

(denoted by $x \overset{I(w^{(M,p)})}{\sim} y$) and simply strongly (M,p) - asymptotically equivalent with respect to the ideal I, if $L = 1$.

If we take $M(x) = x$ for $x \geq 0$, we write $x \overset{I(w^p)}{\sim} y$ instead of $x \overset{I(w^{(M,p)})}{\sim} y$ and simply strongly p - asymptotically equivalent with respect to the ideal I, if $L = 1$.

If we take $p_k = p$ for all $k \in N$, we write $x \overset{I(w^{M_p})}{\sim} y$ instead of $x \overset{I(w^{(M,p)})}{\sim} y$.If we take $p = 1$,we write $x \overset{I(w^M)}{\sim} y$ instead of $x \overset{I(w^{M_p})}{\sim} y$ and simply strongly M-asymptotically equivalent with respect to the ideal I, if $L = 1$.

Definition 3.3. Let $I \subset P(N)$ be a non-trivial ideal in N , M be any Orlicz function , $\theta = (k_r)$ be a lacunary sequence, and $p = (p_k)$ be a sequence of positive real numbers. Two number sequences $[x]$ and $[y]$ are said to be (M,p)-asymptotically lacunary equivalent of multiple L with respect to the ideal I provided that for each $\varepsilon > 0$,

$\left\{r\in N; \frac{1}{h_r}\sum_{k\in I_r}\left[M(\left|\frac{x_k}{y_k}-L\right|/\rho)\right]^{p_k}\geq\varepsilon\right\}\in I$,for some $\rho > 0$, (denoted by $x \overset{I(N_\theta^{(M,p)})}{\sim} y$) and simply (M,p) -asymptotically lacunary equivalent with respect to the ideal I, if $L=1$.

If we take $p_k = p$ for all $k\in N$, we write $x \overset{I(N_\theta^{M_p})}{\sim} y$ instead of $x \overset{I(N_\theta^{(M,p)})}{\sim} y$

Note that,we put $p=1$, we write $x \overset{I(N_\theta^{M})}{\sim} y$ instead of $x \overset{I(N_\theta^{M_p})}{\sim} y$ and simply M-asymptotically lacunary equivalent with respect to the ideal I, if $L=1$.

Also if we put $M(x)=x$ for $x\geq 0$, we write $x \overset{I(N_\theta^{p})}{\sim} y$ instead of $x \overset{I(N_\theta^{(M,p)})}{\sim} y$.

Hence $x \overset{I(N_\theta^{p})}{\sim} y$ is the same as the $x \overset{N_\theta^{L(p)}(I)}{\sim} y$ of Savas and Gumus [11]

We start this section with the following Theorem to show that the relation between strongly M- asymptotically equivalence and strong asymptotically equivalence with respect to the ideal I

Theorem 3.1. Let $I\subset P(N)$ be a non-trivial ideal in N , M be any Orlicz function which satisfies the Δ_2-condition , $\theta=(k_r)$ be a lacunary sequence, then if $x \overset{I(w)}{\sim} y$ then $x \overset{I(w^M)}{\sim} y$

Proof. Let $x \overset{I(w)}{\sim} y$ and $\varepsilon > 0$. We choose $0<\delta<1$such that $M(u)<\varepsilon/2$ for every u with $0\leq u\leq\delta$. We can write

$$\frac{1}{n}\sum_{k=1}^{n}M(\left|\frac{x_k}{y_k}-L\right|/\rho)=\frac{1}{n}\sum_{1}M(\left|\frac{x_k}{y_k}-L\right|/\rho)+\frac{1}{n}\sum_{2}M(\left|\frac{x_k}{y_k}-L\right|/\rho)$$

where the first summation is over $(\left|\frac{x_k}{y_k}-L\right|/\rho)\leq\delta$ and the second summation over $(\left|\frac{x_k}{y_k}-L\right|/\rho)>\delta$.Since M is continuous

$\frac{1}{n}\sum_{1}M(\left|\frac{x_k}{y_k}-L\right|/\rho) < \varepsilon/2$ and for $(\left|\frac{x_k}{y_k}-L\right|/\rho) > \delta$ we use the fact that $(\left|\frac{x_k}{y_k}-L\right|/\rho)<(\left|\frac{x_k}{y_k}-L\right|/\rho)/\delta<1+(\left|\frac{x_k}{y_k}-L\right|/\rho)/\delta$.Since M is non-decreasing and convex,it follows that

$$\begin{aligned}M(\left|\frac{x_k}{y_k}-L\right|/\rho)&<M(1+(\left|\frac{x_k}{y_k}-L\right|/\rho)/\delta)\\&<\tfrac{1}{2}M(2)+\tfrac{1}{2}M(2(\left|\frac{x_k}{y_k}-L\right|/\rho)/\delta)\end{aligned}$$

Since M satisfies the Δ_2-condition, therefore

$$\begin{aligned}M(\left|\frac{x_k}{y_k}-L\right|/\rho)&<\tfrac{1}{2}K((\left|\frac{x_k}{y_k}-L\right|/\rho)/\delta)\ M(2)+\tfrac{1}{2}K((\left|\frac{x_k}{y_k}-L\right|/\rho)/\delta)\\&=K(\left|\frac{x_k}{y_k}-L\right|/\delta)M(2)\end{aligned}$$

Hence $\frac{1}{n}\sum_{2}M(\left|\frac{x_k}{y_k}-L\right|/\rho)\leq(KM(2)/\delta)\ \frac{1}{n}\sum_{k=1}^{n}(\left|\frac{x_k}{y_k}-L\right|/\rho)$,which together with

$$\frac{1}{n}\sum_{1}M(\left|\frac{x_k}{y_k}-L\right|/\rho)<\varepsilon$$

yields

$\frac{1}{n}\sum_{k=1}^{n}M(\left|\frac{x_k}{y_k}-L\right|/\rho)\leq\varepsilon/2+(KM(2)/\delta)\ \frac{1}{n}\sum_{k=1}^{n}(\left|\frac{x_k}{y_k}-L\right|/\rho)$.Thus,

$$\left\{n \in N; \frac{1}{n}\sum_{k=1}^{n} M(\left|\frac{x_k}{y_k} - L\right| /\rho) \geq \varepsilon\right\} \subset \left\{n \in N; \frac{1}{n}\sum_{k=1}^{n} \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\delta/2KM(2)\right\}.$$

Since $x \overset{I(w)}{\sim} y$ it follows the later set, and hence, the first set in above expression belongs to I. This proves that $x \overset{I(w^M)}{\sim} y$

Theorem 3.2. Let M_1, M_2 be Orlicz functions that satisfy the Δ_2-condition.

Then

(i) if $x \overset{I(M_2)}{\sim} y$ then $x \overset{I(M_1 o M_2)}{\sim} y$,

(ii) if $x \overset{I(M_1 \cap M_2)}{\sim} y$ then $x \overset{I(M_1 + M_2)}{\sim} y$

Proof. (i) Let $x \overset{I(M_2)}{\sim} y$ .Then there exists $\rho > 0$ such that

$$\left\{n \in N; \frac{1}{n}\sum_{k=1}^{n} M_2(\left|\frac{x_k}{y_k} - L\right| /\rho) \geq \varepsilon\right\} \in I$$

Let $\varepsilon > 0$ and choose $0 < \delta < 1$such that $M_1(u) < \varepsilon/2$ for every u with $0 \leq u \leq \delta$. Write $A_k = M_2(\left|\frac{x_k}{y_k} - L\right| /\rho)$ By the Remark, we have,for $A_k \leq \delta$

$M_1(M_2(\left|\frac{x_k}{y_k} - L\right| /\rho)) \leq M_1(2)M_2(\left|\frac{x_k}{y_k} - L\right| /\rho)\varepsilon/2$

For $A_k > \delta$, we have $A_k < A_k/\delta < 1 + A_k/\delta$. Since M is non-decreasing and convex,it follows that

$M_1(A_k) < M_1(1 + A_k/\delta)$

$< \frac{1}{2}M_1(2) + \frac{1}{2}M_1(2A_k/\delta)$

Since M satisfies the Δ_2-condition, therefore

$M_1(A_k) < \frac{1}{2}K(A_k/\delta)\ M_1(2) + \frac{1}{2}K(A_k/\delta)$

$= K(A_k/\delta)M_1(2)$

Hence $M_1(M_2(\left|\frac{x_k}{y_k} - L\right| /\rho)) \leq max(1, K\delta^{-1}M_1(2))M_2(\left|\frac{x_k}{y_k} - L\right| /\rho) + \varepsilon/2$

$\left\{n \in N; M_1(M_2(\left|\frac{x_k}{y_k} - L\right| /\rho)) \geq \varepsilon\right\}$

$\subset \left\{n \in N; M_2(\left|\frac{x_k}{y_k} - L\right| /\rho) \geq \varepsilon/2max(1, K\delta^{-1}M_1(2))\right\}$

we have$\left\{n \in N; M_1(M_2(\left|\frac{x_k}{y_k} - L\right| /\rho)) \geq \varepsilon\right\} \in I$ Hence $x \overset{I(M_1 o M_2)}{\sim} y$.

(ii) Let $x \overset{I(M_1 \cap M_2)}{\sim} y$.Then there exists $\rho > 0$ such that

$\left\{n \in N; \frac{1}{n}\sum_{k=1}^{n} M_1(\left|\frac{x_k}{y_k} - L\right| /\rho) \geq \varepsilon\right\} \in I$ and

$\left\{n \in N; \frac{1}{n}\sum_{k=1}^{n} M_2(\left|\frac{x_k}{y_k} - L\right| /\rho) \geq \varepsilon\right\} \in I.$

The rest of the proof follows from the following equality

$(M_1 + M_2)(\left|\frac{x_k}{y_k} - L\right| /\rho) = M_1(\left|\frac{x_k}{y_k} - L\right| /\rho) + M_2(\left|\frac{x_k}{y_k} - L\right| /\rho)$

The next theorem shows the relationship between the strongly $M-$asymptotically equivalence and the $M-$asymptotically lacunary equivalence with respect to the ideal I.

Theorem 3.3.Let $I \subset P(N)$ be a non-trivial ideal in N , M be an Orlicz function ,and $\theta = (k_r)$ be a lacunary sequence, then

(i) if $\limsup_r q_r < \infty$ then $x \overset{I(N_\theta^M)}{\sim} y$ implies $x \overset{I(w_M)}{\sim} y$

(ii) if $\liminf_r q_r > 1$ then $x \overset{I(w_M)}{\sim} y$ implies $x \overset{I(N_\theta^M)}{\sim} y$

(iii) if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$,then $x \overset{I(w_M)}{\sim} y \iff x \overset{I(N_\theta^M)}{\sim} y$.

Proof. Part (i): If $\limsup_r q_r < \infty$ then there exists $K > 0$ such that $q_r < K$ for every r.Now suppose that $x \overset{I(N_\theta^M)}{\sim} y$ and $\varepsilon > 0$.Let

$$A = \left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} M(\left| \frac{x_k}{y_k} - L \right| / \rho) < \varepsilon \right\}, \text{for some } \rho > 0$$

Hence,for all $j \in A$ and for some $\rho > 0$, we have $H_j = \frac{1}{h_j} \sum_{k \in I_j} M(\left| \frac{x_k}{y_k} - L \right| / \rho)$ $< \varepsilon$. Let n be any integer with $k_r \geq n > k_{r-1}$.Now write

$$\frac{1}{n} \sum_{k=1}^{n} M(\left| \frac{x_k}{y_k} - L \right| / \rho) \leq \frac{1}{k_{r-1}} \sum_{k=1}^{k_r} M(\left| \frac{x_k}{y_k} - L \right| / \rho)$$

$$= \frac{1}{k_{r-1}} \sum_{m=1}^{r} \sum_{k \in I_m} M(\left| \frac{x_k}{y_k} - L \right| / \rho)$$

$$= \frac{1}{k_{r-1}} \sum_{m=1}^{r} \frac{k_m - k_{m-1}}{h_m} \sum_{k \in I_m} M(\left| \frac{x_k}{y_k} - L \right| / \rho)$$

$$= \frac{1}{k_{r-1}} \sum_{m=1}^{r} (k_m - k_{m-1}) \sup_{j \in A} H_j$$

$$= \frac{k_r}{k_{r-1}} \sup_{j \in A} H_j$$

$$= q_r \sup_{j \in A} H_j$$

$$< K\varepsilon = \varepsilon'$$

it follows that for any $\varepsilon' > 0$, $\left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} M(\left| \frac{x_k}{y_k} - L \right| / \rho) < \varepsilon' \right\} \in F(I)$

which yields that $x \overset{I(w_M)}{\sim} y$.Because for any set $A \in F(I), \cup\{n : k_{r-1} < n < k_r, r \in A\} \in F(I)$.

Part (ii): Let $x \overset{I(w_M)}{\sim} y$ and $\liminf_r q_r > 1$ .There exist $\delta > 0$ such that $q_r = (k_r/k_{r-1}) \geq 1 + \delta$ for all $r \geq 1$. We have,for sufficiently large r, that $(k_r/h_r) \leq \frac{1+\delta}{\delta}$ and $(k_{r-1}/h_r) \leq \frac{1}{\delta}$. Let $\varepsilon > 0$ and define the set

$$A = \left\{ k_r \in N; \frac{1}{k_r} \sum_{k=1}^{k_r} M(\left| \frac{x_k}{y_k} - L \right| / \rho) < \varepsilon \right\}, \text{for some } \rho > 0.$$

We have $A \in F(I)$, which is the filter of the ideal I,For each $k_r \in A$, we have, for some $\rho > 0$,

$$\frac{1}{h_r} \sum_{k \in I_r} M(\left| \frac{x_k}{y_k} - L \right| / \rho) = \frac{1}{h_r} \sum_{k=1}^{k_r} M(\left| \frac{x_k}{y_k} - L \right| / \rho) - \frac{1}{h_r} \sum_{k=1}^{k_{r-1}} M(\left| \frac{x_k}{y_k} - L \right| / \rho)$$

$$= \frac{k_r}{k_r h_r} \sum_{k=1}^{k_r} M(\left| \frac{x_k}{y_k} - L \right| / \rho) - \frac{k_{r-1}}{h_r k_{r-1}} \sum_{k=1}^{k_{r-1}} M(\left| \frac{x_k}{y_k} - L \right| / \rho)$$

$$\leq \frac{k_r}{k_r h_r} \sum_{k=1}^{k_r} M(\left| \frac{x_k}{y_k} - L \right| / \rho)$$

$$< (\frac{1+\delta}{\delta})\varepsilon = \varepsilon'$$

it follows that for any $\varepsilon' > 0$,

$\left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} M(\left| \frac{x_k}{y_k} - L \right| / \rho) < \varepsilon' \right\} \in F(I)$ which yields that $x \overset{I(N_\theta^M)}{\sim} y$.

Part (iii): This immediately follows from (i) and (ii).

Now we give relation between asymptotically lacunary statistical equivalence and M-asymptotically lacunary equivalence with respect to the ideal I.

Theorem 3.4. Let $I \subset P(N)$ be a non-trivial ideal in N , M be an Orlicz function,and $\theta = (k_r)$ be a lacunary sequence,then

(i) if $x \overset{I(N_\theta^M)}{\sim} y$ then $x \overset{I(S_\theta)}{\sim} y$,

(ii) if M is bounded then $x \overset{I(N_\theta^M)}{\sim} y \iff x \overset{I(S_\theta)}{\sim} y$,

Proof. Part (i): Take $\varepsilon > 0$ and let $\sum_1$ denote the sum over $k \in I_r$ for some $\rho > 0$,with$\left| \frac{x_k}{y_k} - L \right| / \rho \geq \varepsilon$ .Then

$$\frac{1}{h_r} \sum_{k \in I_r} M(\left| \frac{x_k}{y_k} - L \right| / \rho) \geq \frac{1}{h_r} \sum_1 M(\left| \frac{x_k}{y_k} - L \right| / \rho)$$
$$\geq \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| / \rho \geq \varepsilon \right\} \right|,$$

and $\left\{ r \in N; \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| / \rho \geq \varepsilon \right\} \right| \geq \gamma \right\}$

$\subseteq \left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} M(\left| \frac{x_k}{y_k} - L \right| / \rho) \geq \gamma \right\} \in I$.But then, by definition of an ideal, later set belongs to I, and therefore $x \overset{I(S_\theta)}{\sim} y$

Part (ii): Suppose that M is bounded and $x \overset{I(S_\theta)}{\sim} y$. Since M is bounded, there exists an integer T such that $|M(x)| \leq T$ for all $x \geq 0$. We see that

$\frac{1}{h_r} \sum_{k \in I_r} M(\left| \frac{x_k}{y_k} - L \right| / \rho) \leq T \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| / \rho \geq \varepsilon \right\} \right| + M(\varepsilon)$ so we have

$\left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} M(\left| \frac{x_k}{y_k} - L \right| / \rho) \geq \varepsilon \right\}$

$\subseteq \left\{ r \in N; \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \frac{\varepsilon - M(\varepsilon)}{T} \right\} \in I$. Therefore we have $x \overset{I(N_\theta^M)}{\sim} y$

Let $p_k = p$ for all k , $t_k = t$ for all k and $0 < p \leq t$. Then it follows following Theorem.

Theorem 3.5. Let $I \subset P(N)$ be a non-trivial ideal in N ,M be an Orlicz function, ,and $\theta = (k_r)$ be a lacunary sequence, then

$x \overset{I(N_\theta^{M_t})}{\sim} y$ implies $x \overset{I(N_\theta^{M_p})}{\sim} y$,

Proof. Let $x \overset{I(N_\theta^{M_t})}{\sim} y$. It follows from Holder's inequality

$\frac{1}{h_r} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^p \leq (\frac{1}{h_r} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^t)^{p/t}$

and $\left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^p \geq \varepsilon \right\}$

$$\subseteq \left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^t \geq \varepsilon^{t/p} \right\} \in I.$$ Thus we have $x \overset{I(N_\theta^{M_p})}{\sim} y$

We now consider that (p_k) and (t_k) are not constant sequences.

Theorem 3.6. Let $I \subset P(N)$ be a non-trivial ideal in N , M be an Orlicz function, $\theta = (k_r)$ be a lacunary sequence, $0 < p_k \leq t_k$ for all k and (t_k/p_k) be bounded ,then $x \overset{I(N_\theta^{(M,t)})}{\sim} y$ implies $x \overset{I(N_\theta^{(M,p)})}{\sim} y$

Proof. Let $x \overset{I(N_\theta^{(M,t)})}{\sim} y$.$z_k = \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{t_k}$ and $\lambda_k = (p_k / \ t_k)$, so that

$0 < \lambda \leq \lambda_k \leq 1$:We define the

sequences (u_k) and (v_k) as follows: For $z_k \geq 1$;let $u_k = z_k$ and $v_k = 0$ and for $z_k < 1$; let $v_k = z_k$ and $u_k = 0$.Then we have $z_k = u_k + v_k$; $z_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$.Now it follows that $u_k{}^{\lambda_k} \leq u_k \leq z_k$ and $v_k^{\lambda_k} \leq v_k^{\lambda}$.Therefore

$$\frac{1}{h_r} \sum_{k \in I_r} z_k^{\lambda_k} = \frac{1}{h_r} \sum_{k \in I_r} (u_k{}^{\lambda_k} + v_k^{\lambda_k})$$
$$\leq \frac{1}{h_r} \sum_{k \in I_r} z_k + \frac{1}{h_r} \sum_{k \in I_r} v_k^{\lambda}$$

Now for each r;

$$\frac{1}{h_r} \sum_{k \in I_r} v_k^{\lambda} = \sum_{k \in I_r} (\tfrac{1}{h_r} v_k)^{\lambda} (\tfrac{1}{h_r})^{1-\lambda}$$
$$\leq (\sum_{k \in I_r} \left[(\tfrac{1}{h_r} v_k)^{\lambda} \right]^{1/\lambda})^{\lambda} (\sum_{k \in I_r} \left[(\tfrac{1}{h_r})^{1-\lambda} \right]^{1/1-\lambda})^{1-\lambda}$$
$$< (\tfrac{1}{h_r} \sum_{k \in I_r} v_k)^{\lambda} \text{ and so}$$

$$\frac{1}{h_r} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} = \frac{1}{h_r} \sum_{k \in I_r} z_k^{\lambda_k} \leq \frac{1}{h_r} \sum_{k \in I_r} z_k + (\frac{1}{h_r} \sum_{k \in I_r} v_k)^{\lambda}$$
$$= \left\{ \begin{array}{cc} \frac{1}{h_r} \sum_{k \in I_r} z_k & , z_k \geq 1 \\ \frac{1}{h_r} \sum_{k \in I_r} z_k + (\frac{1}{h_r} \sum_{k \in I_r} z_k)^{\lambda} & , z_k < 1 \end{array} \right\}$$
$$\leq \left\{ \begin{array}{cc} \frac{1}{h_r} \sum_{k \in I_r} z_k & , z_k \geq 1 \\ 2(\frac{1}{h_r} \sum_{k \in I_r} z_k)^{\lambda} & , z_k < 1 \end{array} \right\}$$

If $\frac{1}{h_r} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} \geq \varepsilon$ then

$$\left\{ \begin{array}{cc} \frac{1}{h_r} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{t_k} \geq \varepsilon & , z_k \geq 1 \\ \frac{1}{h_r} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{t_k} \geq \left(\frac{\varepsilon}{2} \right)^{1/\lambda} & , z_k < 1 \end{array} \right\}$$

Hence $\left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} \geq \varepsilon \right\}$

$$\subseteq \left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[M\left(\left| \frac{x_k}{y_k} - L \right| / \rho \right) \right]^{t_k} \geq \min \left\{ \varepsilon, \left(\frac{\varepsilon}{2} \right)^{1/\lambda} \right\} \right\} \in I.$$

Thus we have $x \overset{I(N_\theta^{(M,p)})}{\sim} y$.

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