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GEOMETRIC VIRTUES OF THIRD-ORDER DIFFERENTIAL EQUATION USING ADMISSIBLE FUNCTIONS IN A COMPLEX DOMAIN

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Abstract. We investigate geometric properties of solutions of the following third-order differential equation:

$$w'''(z) + a(z)w''(z) + b(z)w'(z) + c(z)w(z) = 0,$$

subject to the initial conditions $w(0) = 0$, $w'(0) = 1$ and $w''(0) = 0$, where $a(z), b(z)$ and $c(z)$ are analytic in the open unit disk (OUD). We indicate that the above differential equation has a univalent and starlike solution in the open unit disk. We employ the concept of admissible functions and we extend the study into some complex Banach spaces.

1. Introduction

Recently, the outcomes of the third order have been investigated by a large number of mathematicians, and they have obtained many results for some special cases. Sufficient conditions imposed for the boundedness of all solutions of the equation

$$w'''(z) = P(z, w, w', w''), \tag{1}$$

where P is a polynomial whose coefficients are functions in the independent variable z. Moreover, the existence and the uniqueness of the solutions of (1) established. This class of differential equations occurs in the flow of thin films of viscous fluid with a free surface [1]-[4]. Newly, in [5], the author studied the geometry of solutions of the Chazy$'$s equation (third order differential equation) in the complex plane, using semi-completeness of complex manifolds. In [6], the authors discussed the geometric properties (convexity) of solutions of the initial-value problem which contains the following third-order linear differential equations:

$$w'''(z) + Q(z)w'(z) = 0, \tag{2}$$

subject to the initial conditions $w(0) = 0$ and $w'(0) = 1$.

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In this paper, We propose some geometric properties of solutions of the following third-order differential equation:

$$w'''(z) + a(z)w''(z) + b(z)w'(z) + c(z)w(z) = 0, \tag{3}$$

subject to the initial conditions $w(0) = 0$, $w'(0) = 1$ and $w''(0) = 0$, where $a(z), b(z)$ and $c(z)$ are analytic in the open unit disk. We indicate that equation (3) has a univalent and starlike solution in OUD. We utilize the concept of admissible functions.

The class of admissible functions in a complex domain plays an important role in the theory of differential subordination. This function implies dominants of various differential subordinations and differential inequalities that would be difficult to find directly. There are diverse processes to utilize this type of functions; the first process concerns with the equation of the boundary of the complex domain Ω is known. The second process deals with the geometry of Ω. While the third process involves subordination chains. Moreover, this class of functions is extended to a complex Banach space to have extra applications [7]. Applications are illustrated to determine the Ulam stability of fractional differential equations in complex domain [8]-[10]. Recently, Antonino & Miller presented applications of results to third-order differential inequalities, third-order differential subordinations, univalent functions and Bessel functions [11].

2. Preliminaries

Let $\mathfrak{H}[a, n]$ be the class of all analytic functions in OUD $U := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ of the form $\phi(\zeta) = a + \varphi_n\zeta^n + \varphi_{n+1}\zeta^{n+1} + ...$ with $\mathfrak{H}[0, 1] \equiv \mathfrak{H}_0$. And let $\mathcal{A}$ be the class of functions $\phi(\zeta)$ formalized by

$$\phi(\zeta) = \zeta + \sum_{n=2}^{\infty} \varphi_n\zeta^n, \quad \zeta \in U. \tag{4}$$

In addition, consider $\mathcal{S}$ and $\mathcal{C}$ are subclasses of $\mathcal{A}$ including functions which are, respectively, univalent and convex in U. It is well known that; if the function $\phi(\zeta)$ given by (4) is in the class $\mathcal{S}$, then $|\varphi_n| \leq n, \quad n \in \mathbb{N}\backslash\{1\}$, where $\mathbb{N} := \{1, 2, 3, ...\}$. Equality holds for the Koebe function $\phi(\zeta) = \frac{\zeta}{(1-\zeta)^2}, \quad \zeta \in U$. Also, if the function $\phi(\zeta)$ given by (1) is in the class $\mathcal{C}$, then $|\varphi_n| \leq 1, \quad n \in \mathbb{N}$. Equality holds for the function $\phi(\zeta) = \frac{\zeta}{1-\zeta}, \quad \zeta \in U$.

A function $\phi \in \mathcal{A}$ is called starlike of order ν if it satisfies the following inequality

$$\Re\{\frac{\zeta\phi'(\zeta)}{\phi(\zeta)}\} > \nu, \ \ (\zeta \in U)$$

for some $0 \leq \nu < 1$. We symbolize this class by $\mathcal{S}^*(\nu)$.

A function $\phi \in \mathcal{A}$ is called convex of order ν if it satisfies the following inequality

$$\Re\{\frac{\zeta\phi''(\zeta)}{\phi'(\zeta)} + 1\} > \nu, \ \ (\zeta \in U)$$

for some $0 \leq \nu < 1$. We typify this class by $\mathcal{C}(\nu)$. Note that $\phi \in \mathcal{C}(\nu)$ if and only if $\zeta\phi' \in \mathcal{S}^*(\nu)$.

Definition 1 Let Ω, Σ be two sets in $\mathbb{C}$, let q be analytic in U such that $q(0) = a$ and let $\Phi(r, s, t; \zeta) : \mathbb{C}^3 \times U \to \mathbb{C}$. If a function q satisfies

$$\{\Phi(q(\zeta), \zeta q'(\zeta), \zeta^2 q''(\zeta); \zeta) \mid \zeta \in U\} \subset \Omega \Rightarrow q(U) \subset \Sigma \tag{5}$$

then Φ is called an admissible function. We symbolize this class by $\Phi_n[\Omega, q]$. As a special case, if $|a| < M$, $M > 0$, we put $\Phi_n[\Omega, q] \equiv \Phi_n[\Omega, M, a]$ and if $\Omega = \Sigma$, we signify $\Phi_n[\Omega, M, a] \equiv \Phi_n[M, a]$.

For example, let $q(z) = \frac{1+z}{1-z}$, $z \in U$, $\Omega = q(U)$ and $\phi(r, s, t; z) = r^2 s^2$. A simple calculation yields $\phi \in \Phi[q(U), q]$, where

$$\Re\{(p(z))^2.(zp'(z))^2\} > 0,$$

for some analytic function $p(z)$.

The following result can be located in [7].

Theorem 1 Let $p \in \mathfrak{H}[a, n]$. If $\Phi \in \Phi_n[M, a]$, then

$$|\Phi(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta)| < M \Rightarrow |p(z)| < M.$$

Definition 2 Let H_M be the set of complex functions $\Phi(u, v, w) \in \Phi_n[M, a]$ satisfying:

(i) $\Phi(u, v, w)$ is continuous in a domain $D \subset \mathbb{C} \times \mathbb{C} \times \mathbb{C}$,

(ii) $(0, \mathrm{O}) \in D$ and $|\Phi(0, \mathrm{O})| < M$,

(iii) $|\Phi(Me^{i\theta}, Ke^{i\theta}, Le^{i\theta})| \geq M$ when $(Me^{i\theta}, Ke^{i\theta}, Le^{i\theta}) \in D$, θ is real and $K, L \geq M$.

Example 3 Obviously that the following function $\Phi(u, v, w)$ is in H_M : $\Phi(u, v) = \rho u + v + w$ where $0 \leq \rho < 1$ and $D = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$.

Definition 4 Let $\Phi \in H_M$ with corresponding domain D . We indicate by $B_M(\Phi)$ those functions $w(z) = w_1 z + w_2 z^2 + \cdots$ which are analytic in U gratifying

(i) $(w(z), zw'(z), z^2 w''(z)) \in D$,

(ii) $|\Phi(w(z), zw'(z), z^2 w''(z))| < M$ $(z \in U)$.

Example 5 Clearly, that the set $B_M(\Phi)$ is not empty since for any $\Phi \in H_M$ it is true that $\omega(z) = \omega_1 z \in B_M(\Phi)$ for $|\omega_1|$ sufficiently small depending on Φ.

Definition 6 [7] Let X, Y be complex Banach spaces. The class of admissible functions $\mathcal{G}'(X, Y)$, involves of those functions $\varrho : X^2 \times U \to Y$ that achieve the admissibility conditions:

$$\|\varrho(r, ks, ; z)\| \geq 1, \text{ when } \|r\| = 1, \ \|s\| = 1, \ k \geq 1.$$

Definition 7 [8] Let X, Y be complex Banach spaces. The extend class of admissible functions $\mathcal{G}''(X, Y)$, encompasses of those functions $\varrho : X^3 \times U \to Y$ that fulfill the admissibility conditions:

$$\|\varrho(r, ks, lt; z)\| \geq 1, \text{ when } \|r\| = 1, \|s\| = 1, \|t\| = 1,$$

$$(\quad z \in U, k \geq 1, l \geq 1).$$

We need the following results:

Lemma 8[7] Let $\varrho \in \mathcal{G}'(X, Y)$. If $p : U \to X$ is the holomorphic vector-valued function defined in U with $p(0) = \Theta$, then

$$\left\|\varrho\Big(p(z), zp'(z)); z\Big)\right\| < 1 \Longrightarrow \|p(z)\| < 1.$$

Lemma 9 [8] Let $\varrho \in \mathcal{G}''(X, Y)$. If $p : U \to X$ is the holomorphic vector-valued function defined in the unit disk U with $p(0) = \Theta$, then

$$\left\|\varrho\Big(p(z), zp'(z), z^2p''(z); z\Big)\right\| < 1 \Longrightarrow \|p(z)\| < 1.$$

Lemmas 8 and 9, allow us to solve first and second order differential inequalities in Banach spaces respectively. Moreover, they employed to investigate Ulam stability for some classes of complex fractional differential equations in sense of the Srivastava-Owa operators [9, 10]. Note that the second differential equation was studied by Saitoh [11, 12] using different approach.

3. Geometric properties

Our first main result is the following theorem.

Theorem 10 Let $a(z), b(z)$ and $c(z)$ be analytic in U, with

$$b(z) = \frac{a^2(z)}{4} + \frac{3a'(z)}{2}$$

and

$$c(z) = \frac{a(z)b(z)}{2} - \frac{a^3(z)}{8} + \frac{a''(z)}{2} - \frac{a(z)a'(z)}{4}.$$

Let $w(z)(z \in U)$ be the solution of (3) with $w(0) = 0$, $w?(0) = 1, w''(0) = 0$. If $|a(z)| < 1$ then $w(z)$ is starlike in U.

Proof. To find a relation on $a(z), b(z)$ and $c(z)$ we define the transformation

$$\begin{aligned}
w(z) &= \exp(-\frac{1}{2}\int_0^z a(\xi)d\xi)v(z),\\
w'(z) &= \exp(-\frac{1}{2}\int_0^z a(\xi)d\xi)v'(z) - \exp(-\frac{1}{2}\int_0^z a(\xi)d\xi)v(z).\frac{a(z)}{2},\\
w''(z) &= \exp(-\frac{1}{2}\int_0^z a(\xi)d\xi)v''(z) - \exp(-\frac{1}{2}\int_0^z a(\xi)d\xi)v'(z).a(z)\\
&\quad + \exp(-\frac{1}{2}\int_0^z a(\xi)d\xi)v(z).(\frac{a^2(z)}{4} - \frac{a'(z)}{2}) \qquad (6)\\
w'''(z) &= \exp(-\frac{1}{2}\int_0^z a(\xi)d\xi)v'''(z) - \exp(-\frac{1}{2}\int_0^z a(\xi)d\xi)v''(z).\frac{3a(z)}{2}\\
&\quad + \exp(-\frac{1}{2}\int_0^z a(\xi)d\xi)v'(z).(\frac{3a^2(z)}{4} - \frac{3a'(z)}{2})\\
&\quad + \exp(-\frac{1}{2}\int_0^z a(\xi)d\xi)v(z)(\frac{3a(z)a'(z)}{4} - \frac{a''(z)}{2} - \frac{a^3(z)}{8}).
\end{aligned}$$

This reduces (3) into the form

$$v'''(z) - \frac{a(z)}{2}v''(z) = 0. \qquad (7)$$

Now by letting

$$u(z) = \frac{zv'(z)}{v(z)} - 1,$$

we obtain

$$|\Phi(u, zu', z^2u''; z)| < 1, \quad z \in U,$$

where $\Phi(u, zu', z^2u''; z) = z^2u''(z) + zu'(z) - u^2(z) - 3u(z)$. It is clear that $\Phi \in H_1$, where

(i) $\Phi(u, v, w)$ is continuous in a domain $D \subset \mathbb{C} \times \mathbb{C} \times \mathbb{C}$,

(ii) $(0, \mathrm{O}) \in D$ and $|\Phi(0, \mathrm{O})| < 1$,

(iii) $|\Phi(e^{i\theta}, Ke^{i\theta}, Le^{i\theta})| \geq 1$ when $(e^{i\theta}, Ke^{i\theta}, Le^{i\theta}) \in D$, θ is real and $K, L \geq 1$. By Theorem 2.1, we receive that

$$|u(z)| < 1, \quad z \in U.$$

Thus, we conclude that

$$|\frac{zv'(z)}{v(z)} - 1| < 1 \quad (z \in \Delta).$$

This yields that $\Re\{\frac{zv'(z)}{v(z)}\} > 0$. Since

$$v(z) = \exp(\frac{1}{2}\int_0^z a(\xi)d\xi)w(z),$$

then logarithmically differentiating of the last assertion, implies

$$\frac{zw'(z)}{w(z)} = \frac{zv'(z)}{v(z)} - \frac{z}{2}a(z), \quad |a(z)| < 1.$$

Hence $\Re\{\frac{zw'(z)}{w(z)}\} > 0$ and consequently, $w(z)$ is starlike in U. This completes the proof.

Example 11 Let $a(z) = z$ in (3), then the equation has a starlike solution $w(z)$ in U of the form
$$w(z) = exp(-1/4 * z^2) * z, \quad z \in U.$$

Next, we discuss the starlikeness of solution when $a(z)$ is a constant function. Then in view of Theorem 10, this leads that $b(z) = \frac{a^2}{4}$ and $c(z) = 0$. Therefore, we have
$$w'''(z) + aw''(z) + \frac{a^2}{4} w'(z) = 0, \tag{8}$$
subject to the initial conditions $w(0) = 0$, $w'(0) = 1$ and $w''(0) = 0$. We have the following result:

Theorem 12 Let $|a| < 1$ and $w(z)(z \in U)$ be the solution of (8) with $w(0) = 0$, $w?(0) = 1, w''(0) = 0$. Then $w(z)$ is starlike in U.

Proof. By putting
$$\omega(z) = \frac{zw'(z)}{w(z)} - 1,$$
we obtain
$$|\Psi(\omega, z\omega', z^2\omega''; z)| < \frac{1}{4}, \quad z \in U,$$
where $\Psi(\omega, z\omega', z^2\omega''; z) = -\frac{a^2}{4}$. It is clear that $\Psi \in H_{\frac{1}{4}}$, where
(i) $\Psi(.,.,.)$ is continuous in a domain $D \subset \mathbb{C} \times \mathbb{C} \times \mathbb{C}$,
(ii) $(0, \mathrm{O}) \in D$ and $|\Psi(0, \mathrm{O})| < \frac{1}{4}$,
(iii) $|\Psi(e^{i\theta}, Ke^{i\theta}, Le^{i\theta})| \geq \frac{1}{4}$ when $(e^{i\theta}, Ke^{i\theta}, Le^{i\theta}) \in D$, θ is real and $K, L \geq \frac{1}{4}$. By Theorem 1, we get
$$|\omega(z)| < 1, \quad z \in U.$$
Thus, we conclude that
$$|\frac{zw'(z)}{w(z)} - 1| < 1 \quad (z \in \Delta).$$
This implies that $\Re\{\frac{zw'(z)}{w(z)}\} > 0$ and completes the proof.

Example 13 Let $a = 1$ in [8], then the equation
$$w'''(z) + w''(z) + \frac{1}{4} w'(z) = 0$$
has a starlike solution in U which takes the form
$$w(z) = 4 - 4 * exp(-1/2 * z) - exp(-1/2 * z) * z, \quad z \in U.$$

4. Bounded solutions

In this section, we discuss the boundedness of solutions in complex Banach spaces using Lemma 8 and Lemma 9. Consider (3) in complex Banach spaces.

Theorem 14 Let $a, b : U \to \mathbb{C}$ be complex valued functions. And let $w : U \to X$, (X is a complex Banach space) be a holomorphic vector-valued function satisfying the equation

$$w'''(z) + a(z)w''(z) + b(z)w'(z) = 0 \tag{9}$$

and defined in U with $w(0) = \Theta$, $w' \neq 0$. If $|b(z)| < 1$ then $w(z)$ is bounded in U.

Proof. By utilizing the transform

$$f(z) = \frac{zw''(z)}{w'(z)},$$

(9) reduces to the form

$$zf'(z) + (za(z) - 1)f(z) + f^2(z) = -z^2 b(z), \quad z \in U.$$

Equivalently to

$$\Phi(f(z), zf'(z); z) = -z^2 b(z), \quad z \in U,$$

where Φ satisfies Definition 6 for some $k \geq 1$. Moreover, it fulfills that

$$\|\Phi(f(z), zf'(z); z)\| \leq |b(z)| < 1, \quad z \in U.$$

In view of Lemma 8 we have $\|f(z)\| < 1$, this yields that

$$\|\frac{z^2 w''(z)}{zw'(z)}\| < 1.$$

If we let $\varphi(w, zw'(z), z^2 w''(z); z) := \frac{z^2 w''(z)}{zw'(z)}$ then $\varphi \in \mathcal{G}''(X, Y)$ for sufficient k and ℓ. Thus according to Lemma 9, we have $\|w\| < 1$ in the open unit disk U and consequently it is bounded.

Theorem 15 Let $a, b, c : U \to \mathbb{C}$ be complex valued functions. And let $w : U \to X$, (X is a complex Banach space) be a holomorphic vector-valued function satisfied equation (3) and defined in U with $w(0) = \Theta$. If $|c(z)| < 1$ then $w(z)$ is bounded in U.

Proof. By employing the transform

$$f(z) = \frac{zw'(z)}{w(z)},$$

(3) implies that

$$z^2 f''(z) - zf'(z) + F(f, f^2, f^3) + za(z)(zf'(z) - f(z) + f^2) + z^2 b(z)f(z) = -z^2 c(z), \quad z \in U.$$

Equivalently to

$$\Psi(f(z), zf'(z), z^2 f''; z) = -z^2 c(z), \quad z \in U,$$

where Ψ satisfies Definition 7 for some $k, \ell \geq 1$. Furthermore, it satisfies

$$\|\Psi(f(z), zf'(z), z^2 f''; z)\| < 1, \quad z \in U.$$

Hence, in virtue of Lemma 9, we receive $\|f(z)\| < 1$, this implies that

$$\|\frac{zw'(z)}{w(z)}\| < 1.$$

If we put $\psi(w, zw'(z); z) := \frac{zw'(z)}{w(z)}$ then $\psi \in \mathcal{G}'(X, Y)$ for sufficient k and ℓ. Therefore, according to Lemma 8, we have $\|w\| < 1$ in the open unit disk U and consequently it is bounded.

Example 16 Let $a(z) = b(z) = \frac{1}{4}$ in Eq. (9), then the solution takes the form

$$w(z) = 1+7/15\sqrt(15)exp(-1/8z)sin(1/8\sqrt(15)z)-exp(-1/8z)cos(1/8\sqrt(15)z), \quad z \in U.$$

Example 17 Let $a(z) = b(z) = 0$, $c(z) = \frac{1}{4}$ in (3), then the solution takes the form

$$\begin{aligned} w(z) &= -1/3 * 2^{(2/3)} * exp(-1/2 * 2^{(1/3)} * z) \\ &+ 1/3 * \sqrt(3) * 2^{(2/3)} * exp(1/4 * 2^{(1/3)} * z) * sin(1/4 * \sqrt(3) \\ &* 2^{(1/3)} * z) + 1/3 * 2^{(2/3)} * exp(1/4 * 2^{(1/3)} * z) * cos(1/4 * \sqrt(3) * 2^{(1/3)} * z), \quad z \in U. \end{aligned}$$

5. Conclusion

We investigated geometric properties of solutions of third-order differential equation in the open unit disk. We showed that this differential equation has a univalent and starlike solution in the open unit disk. We applied some kinds of admissible functions and extended the study into some complex Banach spaces.

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