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GLOBAL EXISTENCE AND ASYMPTOTIC STABILITY FOR A COUPLED VISCOELASTIC WAVE EQUATION WITH A TIME-VARYING DELAY TERM

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Abstract. In this paper, we consider the following viscoelastic coupled wave equation with a delay term

$$u_{tt} - L_1 u - \int_0^t g_1(t-s)L_1 u(s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau_2(t)) + f_1(u,\upsilon) = 0,$$
$$\upsilon_{tt} - L_2 \upsilon - \int_0^t g_2(t-s)L_2 \upsilon(s)ds + \alpha_1 \upsilon_t(x,t) + \alpha_2 \upsilon_t(x, t-\tau_2(t)) + f_2(u,\upsilon) = 0,$$

in a bounded domain. Under appropriate conditions on μ_1, μ_2, α_1 and α_2, we prove global existence of the solutions by combining the energy method with the Faedo-Galerkin's procedure. Furthermore, we study the asymptotic stability in using an appropriate Lyapunov functional. Finally, we show that the decay rates are the same as those obtained in [23].

1. Introduction

Our main interest lies in the following system of viscoelastic equations

$$\begin{cases} u_{tt} - L_1 u - \int_0^t g_1(t-s)L_1 u(s)ds + \mu_1 u_t(x,t)) \\ +\mu_2 u_t(x,t-\tau_2(t)) + f_1(u,\upsilon) = 0, \quad \text{in} \quad \Omega \times (0,\infty), \\ \upsilon_{tt} - L_2 \upsilon - \int_0^t g_2(t-s)L_2 \upsilon(s)ds + \alpha_1 \upsilon_t(x,t) \\ +\alpha_2 \upsilon_t(x,t-\tau_2(t)) + f_2(u,\upsilon) = 0, \quad \text{in} \quad \Omega \times (0,\infty), \\ u(x,t) = 0,\ \upsilon(x,t) = 0, \quad \text{on} \quad \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x),\ \upsilon(x,0) = \upsilon_0(x), \quad x \in \Omega, \\ u_t(x,0) = u_1(x),\ \upsilon_t(x,0) = \upsilon_1(x), \quad x \in \Omega, \\ u_t(x,t-\tau_2(t)) = \phi_0(x,t-\tau_2(t)), \quad x \in \Omega,\ t \geq 0, \\ \upsilon_t(x,t-\tau_2(t)) = \phi_1(x,t-\tau_2(t)),\ \tau_2(t) \neq 0, \quad x \in \Omega,\ t \geq 0. \end{cases} \tag{1}$$

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Where $L_1 u = -div(A_1 \nabla u) = -\sum_{i,j=1}^{N}\left(a_{1i,j}(x)\frac{\partial u}{\partial x_i}\right)$ and $\frac{\partial u}{\partial \nu_{L_1}} = \sum_{i,j=1}^{N}(a_{1i,j}(x)\frac{\partial u}{\partial x_i}\nu_i$.
$L_2 \upsilon = -div(A_2 \nabla \upsilon) = -\sum_{i,j=1}^{N}\left(a_{2i,j}(x)\frac{\partial \upsilon}{\partial x_i}\right)$ and $\frac{\partial \upsilon}{\partial \nu_{L_2}} = \sum_{i,j=1}^{N}(a_{2i,j}(x)\frac{\partial \upsilon}{\partial x_i}\nu_i$. Where Ω is a bounded domain in $\mathbb{R}^n, n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$ and $g_1, g_2 : \mathbb{R}^+ \to \mathbb{R}^+$, $\phi_i(.,.) : \mathbb{R}^2 \to \mathbb{R}$ $i = 1,2$, are given functions which will be specified later. Moreover, $\tau_2(t) > 0$ is the time delay and $\mu_1, \alpha_1, \alpha_2, \mu_2$ are positive real numbers. The initial data $(u_0, u_1, \phi_0), (\upsilon_0, \upsilon_1, \phi_1)$ belongs to a suitable space. Here u and υ denote the transverse displacements of waves. This problem arises in the theory of viscoelasticity and describes the interaction of two scalar fields (see [15]). To motivate our work, let us start with the wave equation proposed by [3], the authors considered the following coupled system of quasilinear viscoelastic equation in canonical form without delay terms

$$\begin{cases} |u_t|^{\rho} u_{tt} - \Delta u - \gamma_1 \Delta u_{tt} \\ + \int_0^t g_1(t-s)\Delta u(s)ds + f_1(x,u) = 0, \quad in\ \Omega \times (0,+\infty), \\ |\upsilon_t|^{\rho} \upsilon_{tt} - \Delta \upsilon - \gamma_2 \Delta \upsilon_{tt} \\ + \int_0^t g_2(t-s)\Delta \upsilon(s)ds + f_2(x,u) = 0, \quad in\ \Omega \times (0,+\infty), \end{cases} \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$with a smooth boundary $\partial\Omega, \gamma_1, \gamma_2 \geq 0$ are constants and ρ is a real number such that $0 < \rho < \frac{2n}{(n-2)}$ if $n \geq 3$ or $\rho > 0$ if $n = 1,2$. The functions u_0, u_1, υ_0 and υ_1 are given initial data. The relaxations functions g_1 and g_2 are continuous functions and $f_1(u,\upsilon), f_2(u,\upsilon)$ represent the nonlinear terms. The authors proved the energy decay result using the perturbed energy method.
Many authors considered the initial boundary value problem as follows

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + h_1(u_t) = f_1(x,u), \quad in\ \Omega \times (0,+\infty), \\ \upsilon_{tt} - \Delta \upsilon + \int_0^t g_2(t-s)\Delta \upsilon(s)ds + h_2(\upsilon_t) = f_2(x,u), \quad in\ \Omega \times (0,+\infty). \end{cases} \tag{3}$$

When the viscoelastic terms g_i $(i = 1,2.)$ are not taken into account in (3), Rammaha and Sakatusathian [4] obtained several results related to local and global existence of a weak solution. By using the same technique as in [5], they showed that any weak solution blows-up in finite time with negative initial energy. Later Said-Houari [6] extended this blow up result to positive initial energy. Conversely, in the presence of the memory term $(g_i \neq 0\ , i = 1,2.)$, there are numerous results related to the asymptotic behavior and blow up of solutions of viscoelastic systems. For example, Liang and Gao [7] studied problem (3) with $h_1(u_t) = -\Delta u_t$, $h_2(\upsilon_t) = -\Delta u_t$. They obtained that, under suitable conditions on the functions $g_i, f_i, i = 1,2$, and certain initial data in the stable set, the decay rate of the energy functions is exponential. On the contrary, for certain initial data in the unstable set, there are solutions with positive initial energy that blow-up in finite time. For $h_1(u_t) = |u_t|^{m-1}u_t$ and $h_2(\upsilon_t) = |\upsilon_t|^{r-1}\upsilon_t$, Hun and Wang [8] established several

results related to local existence, global existence and finite time blow-up (the initial energy $E(0) < 0$).
This latter has been improved by Messaoudi and Said-Houari [14] by considering a larger class of initial data for which the initial energy can take positive values. On the other hand, Muhammad I. Mustafa [26] considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + f_1(\upsilon, u) = 0, \quad in\ \Omega \times (0, +\infty), \\ \\ \upsilon_{tt} - \Delta \upsilon + \int_0^t g_2(t-s)\Delta \upsilon(s)ds + f_2(\upsilon, u) = 0, \quad in\ \Omega \times (0, +\infty), \end{cases} \tag{4}$$

and proved the well-posedness and energy decay result for wider class of relaxation functions.
The author in [24] have studied the following problem in $\Omega \times (0, +\infty)$

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + \left(|u|^k + |\upsilon|^q\right)|u_t|^{m-1}u_t = f_1(\upsilon, u), \\ \\ \upsilon_{tt} - \Delta \upsilon + \int_0^t h(t-s)\Delta \upsilon(s)ds + \left(|u|^\theta + |\upsilon|^\rho\right)|\upsilon_t|^{r-1}\upsilon_t = f_2(\upsilon, u), \end{cases} \tag{5}$$

with degenerate damping and source terms in a bounded domain. Under some assumptions on the relaxation functions, degenerate damping and source terms, he obtained the decay rate of the energy function for certain initial data.
It is widely known that delay effects, which arise in many practical problems, is a source of some instabilities. In this way, Datko and Nicaise ([11], [20], [21]) showed that a small delay in a boundary control turns a well-behaved hyperbolic system into a wild one which in turn, becomes a source of instability. They proved that the energy is exponentially stable under the condition

$$\mu_2 < \mu_1.$$

Motivated by the previous works, in the present paper, we analyze the influence of the viscoelastic damping and delay terms on the solutions to (1). Under suitable assumptions on the functions $g_i(.)$, $f_i(.,.)$ $(i = 1.2)$, the initial data and the parameters in the equations, to the best of our knowledge, there is no result concerning coupled system with the presence of delay term and elliptic operator. We establish several results concerning local and global existence, asymptotic stability and the boundedness of the solutions to (1).
Our work is organized as follows. In section 2, we present the preliminaries and some Lemmas. In section 3, the existence result is obtained. Finally in section 4, decay property is derived.

2. Preliminary Results

In this section, we present some material and assumptions for the proof of our results. We will use the embeddings $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 \le q \le \frac{2n}{n-2}$, if $n \ge 3$ and $q \ge 2$, if $n = 1, 2$; and $L^r(\Omega) \hookrightarrow L^q(\Omega)$, for $q < r$. We will use, in this case, the same embedding constant denoted by c_s

$$\|\nu\|_q \le c_s \|\nabla \nu\|_2, \quad \|\nu\|_q \le c_s \|\nu\|_r \quad \text{for} \quad \nu \in H_0^1(\Omega).$$

For studying the problem (1) we will need the following assumptions. For the relaxation functions g_i for $(i = 1, 2)$, we assume
$(A_0) : g_1(t),\ g_2(t)$: $[0, \infty) \to [0, \infty)$ are of class C^2 and satisfying, for $s \geq 0$

$$g_1(0) = g_{10} > 0,\ 1 - \int_0^\infty g_1(s)ds = l_1 > 0, \quad g_2(0) = g_{20} > 0,\ 1 - \int_0^\infty g_2(s)ds = l_2 > 0,$$

and there exist a nonincreasing functions $\zeta_1(t)$ and $\zeta_1(t)$ such that

$$g_1'(t) \leq -\zeta_1(t)g_1(t), \quad g_2'(t) \leq -\zeta_2(t)g_2(t),\ \forall t \geq 0. \tag{6}$$

(A_1) : The matrix $A_1 = (a_{1i,j}(x))$, $A_2 = (a_{2i,j}(x))$, where $a_{1i,j}, a_{2i,j} \in C^1(\overline{\Omega})$, are symmetric and there exists a constants $a_{01}, a_{02} > 0$ such that for all $x \in \overline{\Omega}$ and $\eta = (\eta_1, ...\eta_N) \in \mathbb{R}^N$ we have

$$\sum_{i,j=1}^{N} a_{1i,j}(x)\eta_j\eta_i \geq a_{01}|\eta|^2, \quad \sum_{i,j=1}^{N} a_{2i,j}(x)\eta_j\eta_i \geq a_{02}|\eta|^2. \tag{7}$$

(A_2) : We take $f_1(u, \upsilon)$ and $f_2(u, \upsilon)$ as in [23], namely

$$f_1(u, \upsilon) = a|u + \upsilon|^{p-1}(u + \upsilon) + b|u|^{\frac{p-3}{2}}|\upsilon|^{\frac{p+1}{2}}u, \tag{8}$$

$$f_2(u, \upsilon) = a|u + \upsilon|^{p-1}(u + \upsilon) + b|\upsilon|^{\frac{p-3}{2}}|u|^{\frac{p+1}{2}}\upsilon. \tag{9}$$

With $a, b > 0$. Further, one can easily verify that

$$uf_1(u, \upsilon) + \upsilon f_2(u, \upsilon) = (p + 1)F(u, \upsilon), \forall (u, \upsilon) \in \mathbb{R}^2.$$

Where

$$F(u, \upsilon) = \frac{1}{(p+1)}\left(a|u + \upsilon|^{p+1} + 2b|u\upsilon|^{\frac{p+1}{2}}\right), \quad f_1(u, \upsilon) = \frac{\partial F}{\partial u}, \quad f_2(u, \upsilon) = \frac{\partial F}{\partial \upsilon}.$$

And there exists C, such that

$$\left|\frac{\partial f_i}{\partial u}(u, \upsilon)\right| + \left|\frac{\partial f_i}{\partial \upsilon}(u, \upsilon)\right| \leq C\left(|u|^{p-1} + |\upsilon|^{p-1}\right), \quad i = 1, 2 \quad where \quad 1 \leq p < 6.$$

(A_3) : We assume that

$$if\ n = 1, 2; \quad p \geq 3 \quad if\ n = 3; \quad p = 3. \tag{10}$$

(A_4) : τ_i is a function such that

$$\tau_i \in W^{2.\infty}([0, T]), \quad \forall\ T > 0, \quad i = 1, 2, \tag{11}$$

$$0 < \tau_0 \leq \tau_2(t) \leq \tau_1,\ \forall t > 0, \tag{12}$$

$$\tau_2'(t) \leq d < 1,\ \tau_2'(t) \leq d' < 1, \tag{13}$$

where τ_0 and τ_1 are two positive constants.
(A_5) : We suppose further that

$$\mu_2 < \sqrt{1 - d}\mu_1, \tag{14}$$

$$\alpha_2 < \sqrt{1 - d'}\alpha_1. \tag{15}$$

As in [28] we choose ξ_1 and ξ_2 such that

$$\frac{\mu_2}{\sqrt{1-d}} < \xi_1 < 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}, \tag{16}$$

$$\frac{\alpha_2}{\sqrt{1-d'}} < \xi_2 < 2\alpha_1 - \frac{\alpha_2}{\sqrt{1-d'}}. \tag{17}$$

Lemma 1([23]) Suppose that (10) holds. Then there exists $\rho > 0$ such that for any $(u, \upsilon) \in (H_0^1(\Omega))^2$, we have

$$\|u+\upsilon\|_{p+1}^{p+1} + 2\|u\upsilon\|_{\frac{p+1}{2}}^{\frac{p+1}{2}} \le \rho\Big(l_1\|\nabla u\|_2^2 + l_2\|\nabla \upsilon\|_2^2\Big)^{\frac{p+1}{2}}.$$

Lemma 2([23]) There exist two positive constants c_0 and c_1 such that

$$c_0\left(|u|^{p+1} + |\upsilon|^{p+1}\right) \le F(u,\upsilon) \le c_1\left(|u|^{p+1} + |\upsilon|^{p+1}\right), \quad \forall (u,\upsilon) \in \mathbb{R}^2.$$

Remark 1 For seeking of simplicity, we take $a = b = 1$ in $(8) - (9)$.

3. Global existence

In order to prove the existence of solutions of problem (1), we introduce the new variables z_1, z_2 as in [28]

$$\begin{aligned} z_1(x,k_1,t) &= u_t(x, t-\tau_2(t)k_1), && x \in \Omega,\ k_1 \in (0,1),\\ z_2(x,k_2,t) &= \upsilon_t(x, t-\tau_2(t)k_2), && x \in \Omega,\ k_2 \in (0,1), \end{aligned}$$

which implies that

$$\begin{aligned} \tau_2(t)z_{1t}(x,k_1,t) + (1-\tau_2'(t))z_{k_1}(x,k_1,t) &= 0, \quad \text{in} \quad \Omega\times(0,1)\times(0,\infty),\\ \tau_2(t)z_{2t}(x,k_2,t) + (1-\tau_2'(t))z_{k_2}(x,k_2,t) &= 0, \quad \text{in} \quad \Omega\times(0,1)\times(0,\infty). \end{aligned}$$

Therefore, problem (1) is equivalent to

$$\begin{cases} u_{tt} - L_1u + \displaystyle\int_0^t g_1(t-s)L_1u(s)ds & \\ +\mu_1u_t(x,t) + \mu_2 z_1(x,1,t) + f_1(u,\upsilon) = 0, & in\ \Omega\times(0,\infty),\\ \upsilon_{tt} - L_2\upsilon + \displaystyle\int_0^t g_2(t-s)L_2\upsilon(s)ds & \\ +\alpha_1\upsilon_t(x,t) + \alpha_2 z_2(x,1,t) + f_2(u,\upsilon) = 0, & in\ \Omega\times(0,\infty),\\ \tau_2(t)z_{1t}(x,k_1,t) + (1-\tau_2'(t))z_{k_1}(x,k_1,t) = 0, & in\ \Omega\times(0,1)\times(0,\infty),\\ \tau_2(t)z_{2t}(x,k_2,t) + (1-\tau_2'(t))z_{k_2}(x,k_2,t) = 0, & in\ \Omega\times(0,1)\times(0,\infty),\\ z_1(x,0,t) = u_t(x,t), x\in\Omega, & t>0,\\ z_2(x,0,t) = \upsilon_t(x,t), x\in\Omega, & t>0,\\ z_1(x,k_1,0) = \phi_0(x,-\tau_2(0)k_1), & x\in\Omega,\\ z_2(x,k_2,0) = \phi_1(x,-\tau_2(0)k_2), & x\in\Omega,\\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x\in\Omega,\\ \upsilon(x,0) = \upsilon_0(x), \upsilon_t(x,0) = \upsilon_1(x), & x\in\Omega,\\ u(x,t) = 0, \upsilon(x,t) = 0,\ x\in\partial\Omega, & t\ge 0. \end{cases} \tag{18}$$

In the following, we will give sufficient conditions for the well-posedness of problem (18) by using the Fadeo-Galerkin's method.

Theorem 1 Let $(u_0,\upsilon_0) \in \left(H_0^1(\Omega)\cap H^2(\Omega)\right)^2, (u_1,\upsilon_1) \in (H_0^1(\Omega))^2$ and $(\phi_0,\phi_1) \in \left(L^2(\Omega\times(0,1))\right)^2$ satisfying the compatibility conditions

$$\phi_0 = (.,0) = u_1, \quad \phi_1 = (.,0) = \upsilon_1.$$

Assume that the hypotheses $(A_0) - (A_5)$ hold. Then there exists a unique weak solution $((u,z_1),(\upsilon,z_2))$ of (18) such for all $T>0$, we have

$$u(t),\upsilon(t) \in C\left([-\tau_2(0),T]; H_0^1(\Omega)\right) \cap C^1\left([-\tau_2(0),T]; L^2(\Omega)\right),$$

$$u_t(t), \upsilon_t(t) \in L^2\left([-\tau_2(0), T]; H_0^1(\Omega)\right) \cap L^2\left([-\tau_2(0), T] \times \Omega\right),$$

Proof. We use Faedo-Galerkin's method to construct approximate solution. Let $T > 0$ be fixed and denote by V_n the space generated by the set $\{w_n, n \in \mathbb{N}\}$ is a basis of $H^2(\Omega) \cap H_0^1(\Omega)$. We define also for $1 \leq j \leq n$, the sequence $\varphi_j(x, n)$ as follows $\varphi_j(x, 0) = w_j(x)$. Then we may extend $\varphi_j(x, 0)$ by $\varphi_j(x, n)$ over $L^2(\Omega \times [0, 1])$ and denote Z_n to be the space generated by $\{\varphi_1, \ldots.\ \varphi_n\}$, $(n = 1, 2, 3\ldots)$. We construct approximate solutions $(u^n(t), z_1^n(t), \upsilon^n(t), z_2^n(t))$ $(n = 1, 2, 3...)$ in the form

$$u^n(t) = \sum_{j=1}^n u_{n,j}(t) w_j(x), \quad z_1^n(t) = \sum_{j=1}^n z_{1n,j}(t)\varphi_j(x, k_1),$$
$$\upsilon^n(t) = \sum_{j=1}^n \upsilon_{n,j}(t) w_j(x), \quad z_2^n(t) = \sum_{j=1}^n z_{2n,j}(t)\varphi_j(x, k_2),$$

where $((u^n(t), z_1^n(t)), (\upsilon^n(t), z_2^n(t)))$ are the solutions of the following approximate problem corresponding to (18) then $((u^n(t), z_1^n(t)), (\upsilon^n(t), z_2^n(t)))$ verify the following system of ODEs:

$$\begin{aligned} &\langle u_{tt}^n(t), w_j\rangle_\Omega + a_1(u^n(t), w_j) + \left\langle \int_0^t g_1(t-s) A_1 \nabla u^n(s) ds, \nabla w_j \right\rangle_\Omega \\ &+ \langle \mu_1 u_t^n(x,t), w_j\rangle_\Omega + \langle \mu_2 z_1^n(x,1,t), w_j\rangle_\Omega + \langle f_1(u^n(t), \upsilon^n(t)), w_j\rangle_\Omega = 0, \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\langle \upsilon_{tt}^n(t), w_j\rangle_\Omega + a_2(\upsilon^n(t), w_j) + \left\langle \int_0^t g_2(t-s) A_2 \nabla \upsilon^n(s) ds, \nabla w_j \right\rangle_\Omega \\ &+ \langle \alpha_1 \upsilon_t^n(x,t), w_j\rangle_\Omega + \langle \alpha_2 z_2^n(x,1,t), w_j\rangle_\Omega + \langle f_2(u^n(t), \upsilon^n(t)), w_j\rangle_\Omega = 0, \end{aligned} \tag{20}$$

for $j = 1.....n$. More specifically

$$\begin{aligned} u^n(0) &= \sum_{j=1}^n u_{n,j}(0) w_j, \upsilon^n(0) = \sum_{j=1}^n \upsilon_{n,j}(0) w_j, \\ u_t^n(0) &= \sum_{j=1}^n u'_{n,j}(0) w_j, \upsilon_t^n(0) = \sum_{j=1}^n \upsilon'_{n,j}(0) w_j, \end{aligned} \tag{21}$$

where

$$u^{n,j}(0) = \langle u^0, w_j\rangle, \upsilon^{n,j}(0) = \langle \upsilon^0, w_j\rangle, \upsilon_t^{n,j}(0) = \langle \upsilon^1, w_j\rangle, \upsilon_t^{n,j}(0) = \langle \upsilon^1, w_j\rangle,$$

$j = 1, ..., n$. Obviously, $u^n(0) \to u^0$, $\upsilon^n(0) \to \upsilon^0$ strongly in $H_0^1(\Omega)$, $u_t^n(0) \to u^1$, $\upsilon_t^n(0) \to \upsilon^1$ strongly in $L^2(\Omega)$ as $n \to \infty$. Also

$$\left\langle \tau_2(t) z_{1t}^n(x, k_1, t) + (1 - \tau_2'(t)) z_{1k_1}^n(x, k_1, t), \varphi_j \right\rangle_\Omega = 0, \tag{22}$$
$$\left\langle \tau_2(t) z_{2t}^n(x, k_2, t) + (1 - \tau_2'(t)) z_{2k_2}^n(x, k_2, t), \varphi_j \right\rangle_\Omega = 0, \tag{23}$$
$$z_1^n(0) = z_1^n \to \phi_0,\ z_2^n(0) = z_2^n \to \phi_1\ in\ L^2\left(\Omega \times (0,1)\right). \tag{24}$$

For $j = 1.....n$. Where

$$a_1(\psi(t), \phi(t)) = \sum_{i,j=1}^N \int_\Omega a_{1i,j}(x) \frac{\partial \psi(t)}{\partial x_j} \frac{\partial \phi(t)}{\partial x_i} dx = \int_\Omega A_1 \nabla \psi(t) \phi(t) dx,$$

$$a_2(\psi(t),\phi(t)) = \sum_{i,j=1}^{N} \int_\Omega a_{2i,j}(x) \frac{\partial \psi(t)}{\partial x_j} \frac{\partial \phi(t)}{\partial x_i} dx = \int_\Omega A_2 \nabla \psi(t) \phi(t) dx.$$

By using hypothesis (A_1), we verify that the bilinear forms $a_1(.,.), a_2(.,.) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ are symmetric and continuous. On the other hand, from (7) for $\zeta = \nabla\psi$, we get

$$a_1(\psi(t),\psi(t)) \geq a_{01} \int_\Omega \sum_{i,j=1}^{N} \left| \frac{\partial \psi}{\partial x_i} \right|^2 dx = a_{01} \|\nabla\psi(t)\|_2^2, \tag{25}$$

$$a_2(\psi(t),\psi(t)) \geq a_{02} \int_\Omega \sum_{i,j=1}^{N} \left| \frac{\partial \psi}{\partial x_i} \right|^2 dx = a_{02} \|\nabla\psi(t)\|_2^2. \tag{26}$$

Which implies that $a_1(.,.)$, $a_2(.,.)$ are coercive. We will utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for $(u^n(t), z_1^n(t), \upsilon^n(t), z_2^n(t))$ such that $n \in \mathbb{N}$.

(1) **Estimate 1.** Multiplying equation (19) by $u'_{n,j}(t)$ and the equation (20) by $\upsilon'_{n,j}(t)$ and summing with respect to j, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|u_t^n(t)\|_2^2 + \|\upsilon_t^n(t)\|_2^2 + a_1(u^n(t),u_t^n(t)) + a_2(\upsilon^n(t),\upsilon_t^n(t))\right] \\ &+ \frac{1}{2}\frac{d}{dt}\int_\Omega F(u^n(t),\upsilon^n(t))dx + \frac{\mu_1}{2}\|u_t^n(t)\|_2^2 + \frac{\alpha_1}{2}\|\upsilon_t^n(t)\|_2^2 \\ &+ \mu_2 \int_\Omega z_1(x,1,t) u_t^n(x,t) dx + \alpha_2 \int_\Omega z_2(x,1,t)\upsilon_t^n(x,t)dx \\ &- \int_0^t g_1(t-s)\int_\Omega A_1 \nabla u^n(s) \nabla u_t^n(t) dx ds \\ &- \int_0^t g_2(t-s)\int_\Omega A_2 \nabla \upsilon^n(s) \nabla \upsilon_t^n(t) dx ds = 0. \end{aligned} \tag{27}$$

Note that

$$a_1(u^n(t),u_t^n(t)) = \frac{1}{2}\frac{d}{dt} a_1(u^n(t),u^n(t)), \tag{28}$$

and

$$a_2(\upsilon^n(t),\upsilon_t^n(t)) = \frac{1}{2}\frac{d}{dt} a_2(\upsilon^n(t),\upsilon^n(t)). \tag{29}$$

Following the same technique as in [31], we obtain

$$
\begin{aligned}
&\int_0^t g_1(t-s)\int_\Omega A_1\nabla u^n(s)\nabla u_t^n(t)dxds\\
&=\sum_{i,j=1}^N\int_0^t\int_\Omega g_1(t-s)a_{1i,j}(x)\frac{\partial u^n(s)}{\partial x_j}\frac{\partial u_t^n(t)}{\partial x_i}dxds\\
&=\sum_{i,j=1}^N\int_0^t\int_\Omega g_1(t-s)a_{1i,j}(x)\frac{\partial u^n(t)}{\partial x_i}\frac{\partial u_t^n(t)}{\partial x_i}dxds\\
&-\sum_{i,j=1}^N\int_0^t\int_\Omega g_1(t-s)a_{1i,j}(x)\left(\frac{\partial u^n(t)}{\partial x_i}-\frac{\partial u_t^n(s)}{\partial x_j}\right)\frac{\partial u_t^n(t)}{\partial x_i}dxds\\
&=\frac{1}{2}\int_0^t g_1(t-s)\left(\frac{d}{dt}a_1(u^n(t),u^n(t))ds\right)\\
&-\frac{1}{2}\int_0^t g_1(t-s)\left(\frac{d}{dt}a_1(u^n(t)-u^n(s),u^n(t)-u^n(s))ds\right)\\
&=\frac{1}{2}\frac{d}{dt}\left(\int_0^t g_1(t-s)a_1(u^n(t),u^n(t)\right)ds-\frac{1}{2}g_1(t)a_1(u^n(t),u^n(t))\\
&-\frac{1}{2}\frac{d}{dt}\left(\int_0^t g_1(t-s)a_1(u^n(t)-u^n(s),u^n(t)-u^n(s))\right)ds\\
&\frac{1}{2}\int_0^t g'(t-s)a_1(u^n(t)-u^n(s),u^n(t)-u^n(s))ds\\
&=-\frac{1}{2}\frac{d}{dt}(g_1\circ u^n)(t)+\frac{1}{2}(g_1'\circ u^n)(t)+\frac{1}{2}\frac{d}{dt}\left[a_1(u^n(t),u^n(t))\int_0^t g_1(s)ds\right]\\
&-\frac{1}{2}g_1(t)a_1(u^n(t),u^n(t)),
\end{aligned}
\tag{30}
$$

where

$$
(g_1\circ u^n)(t)=\int_0^t g_1(t-s)a_1(u^n(t)-u^n(s),u^n(t)-u^n(s))ds, \tag{31}
$$

in the same way

$$
\begin{aligned}
&\int_0^t g_2(t-s)\int_\Omega A_2\nabla \upsilon^n(s)\nabla \upsilon_t^n(t)dxds=-\frac{1}{2}\frac{d}{dt}(g_2\circ \upsilon^n)(t)+\frac{1}{2}(g_2'\circ \upsilon^n)(t)\\
&+\frac{1}{2}\frac{d}{dt}\left[a_2(\upsilon^n(t),\upsilon^n(t))\int_0^t g_2(s)ds\right]-\frac{1}{2}g_2(t)a_2(\upsilon^n(t),\upsilon^n(t)).
\end{aligned}
\tag{32}
$$

Inserting (28)- (32) in (27) and integrating over $(0,t)$, we get

$$\begin{aligned}
&\frac{1}{2}\|u_t^n(t)\|_2^2+\frac{1}{2}\|\upsilon_t^n(t)\|_2^2+\int_\Omega F(u^n(t),\upsilon^n(t))dx\\
&+\frac{1}{2}\left(1-\int_0^t g_1(s)ds\right)a_1(u^n(t),u^n(t))-\frac{1}{2}\int_0^t(g_2' o\upsilon^n)(s)ds=0\\
&+\frac{1}{2}\left(1-\int_0^t g_2(s)ds\right)a_2(\upsilon^n(t),\upsilon^n(t))+\frac{1}{2}(g_1ou^n)(t)+\frac{1}{2}(g_2o\upsilon^n)(t)\\
&+\mu_1\int_0^t\|u_s^n(s)\|_2^2ds+\alpha_1\int_0^t\|\upsilon_s^n(s)\|_2^2ds+\mu_2\int_0^t\int_\Omega z_1^n(x,1,s)u_s^n(s)dxds\\
&+\alpha_2\int_0^t\int_\Omega z_2^n(x,1,s)\upsilon_s^n(s)dxds+\frac{1}{2}\int_0^t g_1(s)a_1(u_t^n(t),u_t^n(t))ds\\
&+\frac{1}{2}\int_0^t g_2(s)a_2(\upsilon_t^n(t),\upsilon_t^n(t))ds-\frac{1}{2}\int_0^t(g_1' ou^n)(s)ds.
\end{aligned}\tag{33}$$

Now, we multiply (22) by $\xi_1(t)e^{-k_1\tau_2(t)}z_{1n,j}(t)$, summing with respect to j and integrating over $\Omega\times(0,1)$ to obtain

$$\begin{aligned}
&\xi_1(t)e^{-k_1\tau_2(t)}\int_\Omega\int_0^1 z_{1t}^n z_1^n(x,k_1,t)dk_1dx\\
&=-\frac{\xi_1(t)e^{-k_1\tau_2(t)}}{2\tau_0}\int_\Omega\int_0^1(1-\tau_2'(t)k_1)\frac{\partial}{\partial k_1}\left(z_1^n(x,k_1,t)\right)^2dk_1dx.
\end{aligned}\tag{34}$$

Consequently,

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\xi_1(t)e^{-k_1\tau_2(t)}}{2}\int_\Omega\int_0^1(z_1^n(x,k_1,t))^2dk_1dx\right)\\
&=-\frac{\xi_1(t)}{2}\int_0^1\int_\Omega\frac{\partial}{\partial k_1}((1-\tau_2'(t)k_1)e^{-k_1\tau_2(t)}(z_1^n(x,k_1,t))^2dk_1dx\\
&+\frac{\xi_1'(t)e^{-k_1\tau_2(t)}}{2}\int_0^1\int_\Omega(z_1^n(x,k_1,t))^2dk_1dx.\\
&=\frac{\xi_1(t)}{2}\int_\Omega\left[(z_1^n(x,0,t))^2-(z_1^n(x,1,t))^2\right]e^{-\tau_2(t)}dx\\
&+\frac{\xi_1(t)\tau_2'(t)e^{-\tau_2(t)}}{2}\int_\Omega(z_1^n(x,1,t))^2dx\\
&+\frac{\xi_1'(t)e^{-k_1\tau_2(t)}}{2}\int_0^1\int_\Omega(z_1^n(x,k_1,t))^2dk_1dx.
\end{aligned}\tag{35}$$

In the same way for (23), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\xi_2(t)e^{-k_2\tau_2(t)}}{2}\int_\Omega\int_0^1(z_2^n(x,k_2,t))^2dk_2dx\right)\\
&=\frac{\xi_2(t)}{2}\int_\Omega\left[(z_2^n(x,0,t))^2-(z_2^n(x,1,t))^2\right]e^{-\tau_2(t)}dx\\
&+\frac{\xi_2(t)\tau_2'(t)e^{-\tau_2(t)}}{2}\int_\Omega(z_2^n(x,1,t))^2dx\\
&+\frac{\xi_2'(t)e^{-k_2\tau_2(t)}}{2}\int_0^1\int_\Omega(z_2^n(x,k_2,t))^2dk_2dx.
\end{aligned}\tag{36}$$

Due to Young's inequality, we have

$$\mu_2 \int_\Omega z_1^n(x,1,t)u_t^n(x,t)dx \le \frac{\mu_2}{2\sqrt{1-d}}\|u_t^n(t)\|_2^2 + \frac{\mu_2\sqrt{1-d}}{2}\|z_1^n(x,1,t)\|_2^2, \quad (37)$$

$$\alpha_2 \int_\Omega z_2^n(x,1,t)\upsilon_t^n(x,t)dx \le \frac{\alpha_2}{2\sqrt{1-d'}}\|\upsilon_t(t)\|_2^2 + \frac{\alpha_2\sqrt{1-d'}}{2}\|z_2^n(x,1,t)\|_2^2. \quad (38)$$

Taking into account the fact that

$$\int_0^t (g_1' \circ u_s^n)(s)ds \le 0, \qquad \int_0^t (g_2' \circ \upsilon_s^n)(s)ds \le 0,$$

we arrive at

$$(g_1 \circ u_t^n)(t) - \int_0^t (g_1' \circ u_s^n)(s)ds + \int_0^t g_1(s)a_1(u_t^n(t), u_t^n(t))ds \ge 0, \quad (39)$$

$$(g_2 \circ \upsilon_t^n)(t) - \int_0^t (g_2' \circ \upsilon_s^n)(s)ds + \int_0^t g_2(s)a_2(\upsilon_t^n(t), \upsilon_t^n(t))ds \ge 0. \quad (40)$$

Summing (33), (35), (36), (37), (38), (39)and (40), we get

$$\begin{aligned}
&E^n(t) + \sigma_1 \int_0^t \|u_s^n(s)\|_2^2 ds + \sigma_2 \int_0^t \|\upsilon_s^n(s)\|_2^2 ds \\
&+ \sigma_3 \int_0^t \|z_1^n(x,1,s)\|_2^2 ds + \sigma_4 \int_0^t \|z_2^n(x,1,s)\|_2^2 ds \\
&- \frac{1}{2}\int_0^t (g_1' o u)(s)ds + \frac{1}{2}\int_0^t g_1(s)a_1(u(s),u(s))ds \\
&- \frac{1}{2}\int_0^t (g_2' o \upsilon)(s)ds + \frac{1}{2}\int_0^t g_2(s)a_2(\upsilon(s),\upsilon(s))ds \\
&\le \frac{1}{2}\left[\|u^{1n}\|_2^2 + \|u^{1n}\|_2^2 + \int_\Omega F(u^n(0),\upsilon^n(0))dx\right] = E^n(0).
\end{aligned} \quad (41)$$

Such that

$$\sigma_1 = \left(\mu_1 - \frac{\xi_1}{2} - \frac{\mu_2}{2\sqrt{1-d}}\right), \quad \sigma_3 = \left(\frac{\xi_1(1-\tau_2'(t))}{2} - \frac{\mu_2\sqrt{1-d}}{2}\right),$$

$$\sigma_2 = \left(\alpha_1 - \frac{\xi_2}{2} - \frac{\alpha_2}{2\sqrt{1-d'}}\right), \quad \sigma_4 = \left(\frac{\xi_2(1-\tau_2'(t))}{2} - \frac{\alpha_2\sqrt{1-d'}}{2}\right).$$

Where $E(t)$ is the energy of the solution defined by the following formula

$$\begin{aligned}
E^n(t) =& \frac{1}{2}\|u_t^n(t)\|_2^2 + \frac{1}{2}\|\upsilon_t^n(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right) a_1(u^n(t),u^n(t)) \\
&+ \frac{1}{2}\left(1 - \int_0^t g_2(s)ds\right) a_2(\upsilon^n(t),\upsilon^n(t)) + \frac{1}{2}(g_1 o u^n)(t) \\
&+ \frac{e^{-k_1\tau_2(t)}\xi_1(t)}{2}\int_\Omega\int_0^1 (z_1^n(x,k_1,t))^2 dk_1 dx + \frac{1}{2}(g_2 o \upsilon^n)(t) \\
&+ \frac{e^{-k_2\tau_2(t)}\xi_2(t)}{2}\int_\Omega\int_0^1 (z_2^n(x,k_2,t))^2 dk_2 dx + \int_\Omega F(u^n(t),\upsilon^n(t))dx.
\end{aligned} \quad (42)$$

We shall prove that the problem (19)-(24) admits a local solution in $[0, t_m)$, $0 < t_m < T$, for an arbitrary $T > 0$. The extension of the solution to the whole interval $[0,T]$ is a consequence of the estimates below.

(2) **Estimate 2.** As in [26] replacing w_j by $-\Delta w_j$ in (19)-(24), multiplying the equation (19) by $u'_{n,j}(t)$ and equation (20) by $\upsilon'_{n,j}(t)$, summing over j from 1 to n and using (30), (32), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|\nabla u_t^n(t)\|_2^2 + a_{01}\left(1-\int_0^t g_1(s)ds\right)\|\Delta u^n(t)\|_2^2 + (g_1 o\Delta u^n)(t)\right]\\&+\frac{1}{2}g_1(t)\|\Delta u^n(t)\|_2^2 - \frac{1}{2}(g_1' o\Delta u^n)(t) + \frac{\mu_1}{2}\|\nabla u_t^n(t)\|_2^2\\&+\mu_2\int_\Omega z_1^n(x,1,t)\Delta u_t^n(t)dx + \int_\Omega f_1(u^n(t),\upsilon^n(t))\Delta u^n(t)dx = 0,\end{aligned}\tag{43}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|\nabla \upsilon_t^n(t)\|_2^2 + a_{02}\left(1-\int_0^t g_2(s)ds\right)\|\Delta \upsilon^n(t)\|_2^2 + (g_2 o\Delta \upsilon^n)(t)\right]\\&+\frac{1}{2}g_2(t)\|\Delta \upsilon^n(t)\|_2^2 - \frac{1}{2}(g_2' o\Delta \upsilon^n)(t) + \frac{\alpha_1}{2}\|\nabla \upsilon_t^n(t)\|_2^2\\&+\alpha_2\int_\Omega z_2^n(x,1,t)\Delta \upsilon_t^n(t)dx + \int_\Omega f_2(u^n(t),\upsilon^n(t))\Delta \upsilon^n(t)dx = 0.\end{aligned}\tag{44}$$

Replacing φ_j by $-\Delta\varphi_j$ in (22)-(23), multiplying (22) by $z_{1n,j}(t)$, summing over j, it follows that

$$\left(\frac{\tau_2(t)}{2(1-\tau_2'(t)k_1)}\right)\frac{d}{dt}\|\nabla z_1^n(t)\|_2^2 + \frac{1}{2}\frac{d}{dk_1}\|\nabla z_1^n(t)\|_2^2 = 0.\tag{45}$$

Then

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\|\nabla z_1^n(t)\|_2^2\right) - \frac{1}{2}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)'\|\nabla z_1^n(t)\|_2^2\\&+\frac{1}{2}\frac{d}{dk_1}\|\nabla z_1^n(t)\|_2^2 = 0.\end{aligned}\tag{46}$$

In the same way for (23)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\|\nabla z_2^n(t)\|_2^2\right) - \frac{1}{2}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)'\|\nabla z_2^n(t)\|_2^2\\&+\frac{1}{2}\frac{d}{dk_2}\|\nabla z_2^n(t)\|_2^2 = 0.\end{aligned}\tag{47}$$

Using Young's inequality, summing (43), (46) and (47), integrating over $(0,t)$, we get

$$
\begin{aligned}
&\frac{1}{2}\left[\|\nabla u_t^n(t)\|_2^2+\|\nabla \upsilon_t^n(t)\|_2^2+a_{01}\left(1-\int_0^t g_1(s)ds\right)\|\Delta u^n(t)\|_2^2\right]\\
&+\int_0^1 \frac{\tau_2(t)}{1-\tau_2'(t)k_1}\|\nabla z_{1t}^n(x,k_1,t)\|_{L^2(\Omega)}^2 dk_1-\frac{1}{2}\int_0^t (g_1' o\Delta \upsilon^n)(s)ds\\
&+\int_0^1 \frac{\tau_2(t)}{1-\tau_2'(t)k_2}\|\nabla z_{2t}^n(x,k_2,t)\|_{L^2(\Omega)}^2 dk_2+\frac{1}{2}(g_2 o\Delta \upsilon^n)(t)\\
&+\frac{a_{02}}{2}\left(1-\int_0^t g_2(s)ds\right)\|\Delta \upsilon^n(t)\|_2^2+\frac{1}{2}(g_1 o\Delta u^n)(t)\\
&+\frac{1}{2}\int_0^t g_1(s)\|\Delta u^n(s)\|_2^2 ds+\frac{1}{2}\int_0^t g_2(s)\|\Delta \upsilon^n(s)\|_2^2 ds\\
&-\frac{1}{2}\int_0^t (g_2' o\Delta \upsilon^n)(s)ds+\frac{\mu_1}{2}\int_0^t \|\nabla u_s^n(s)\|_2^2 ds+\frac{\alpha_1}{2}\int_0^t \|\nabla \upsilon_s^n(s)\|_2^2 ds\\
&+\mu_2\int_0^t\int_\Omega |\nabla z_1^n(x,1,s)|^2 dsdx+\mu_2\int_0^t \|\nabla u_s^n(s)\|_2^2 ds\\
&+\alpha_2\int_0^t\int_\Omega |\nabla z_2^n(x,1,s)|^2 dsdx+\alpha_2\int_0^t \|\nabla \upsilon_s^n(s)\|_2^2 ds\\
&\le \int_\Omega \begin{pmatrix} f_1(u^n,\upsilon^n)\Delta u^n-f_1(u^0,\upsilon^0)\Delta u^0\\ \\ f_2(u^n,\upsilon^n)\Delta \upsilon^n-f_2(u^0,\upsilon^0)\Delta \upsilon^0\end{pmatrix}dx\\
&-\int_0^t\int_\Omega \begin{pmatrix} \frac{\partial}{\partial u}f_1(u^n,\upsilon^n)u_s^n\Delta u^n+\frac{\partial}{\partial \upsilon}f_1(u^n,\upsilon^n)\upsilon_s^n\Delta u^n\\ \\ +\frac{\partial}{\partial \upsilon}f_2(u^n,\upsilon^n)u_s^n\Delta \upsilon^n+\frac{\partial}{\partial \upsilon}f_2(u^n,\upsilon^n)\upsilon_s^n\Delta \upsilon^n\end{pmatrix}dxds\\
&+\int_0^t\int_0^1\left(\frac{\tau_2(s)}{1-\tau_2'(s)k_1}\right)\|\nabla z_{1s}^n(x,k_1,s)\|_{L^2(\Omega)}^2 dk_1+\|\Delta u^{1n}\|_2^2\\
&+\int_0^t\int_0^1\left(\frac{\tau_2(s)}{1-\tau_2'(s)k_2}\right)\|\nabla z_{2s}^n(x,k_2,s)\|_{L^2(\Omega)}^2 dk_2+\|\Delta \upsilon^{1n}\|_2^2\\
&+\int_0^t \|\nabla u_s^n(s)\|_2^2 ds+\int_0^t \|\nabla \upsilon_s^n(s)\|_2^2 ds+\frac{1}{2}\|\nabla u^{1n}\|_2^2+\frac{1}{2}\|\nabla \upsilon^{1n}\|_2^2.
\end{aligned}
\tag{48}
$$

Where $c_0=\frac{1}{2}\|\nabla u^{1n}\|_2^2+\frac{1}{2}\|\nabla \upsilon^{1n}\|_2^2+\|\Delta u^{1n}\|_2^2+\|\Delta \upsilon^{1n}\|_2^2$, is a positive constant. We just need to estimate the right hand terms of (48). Applying Holder's inequality and Sobolev's embedding theorem inequality, we infer

$$
\begin{aligned}
&\left|\int_\Omega f_1(u^n(t),\upsilon^n(t))\Delta u^n(t)dx\right|\\
&\le \int_\Omega \left(|u^n|^p+|\upsilon^n|^p+|u^n|^{\frac{p-1}{2}}|\upsilon^n|^{\frac{p+1}{2}}\right)|\Delta u^n|dx\\
&\le C\left(\|u^n\|_{2p}^p+\|\upsilon^n\|_{2p}^p+\|u^n\|_{\frac{p-1}{2}}^{3(p-1)}\|u^n\|_{\frac{p+1}{2}}^{\frac{3(p+1)}{2}}\right)\|\Delta u^n\|_2\\
&\le C\left(\|\nabla u^n\|_2^p+\|\nabla \upsilon^n\|_2^p+\|\nabla u^n\|_2^{\frac{p-1}{2}}\|\nabla \upsilon^n\|_2^{\frac{p+1}{2}}\right)\|\Delta u^n\|_2\\
&\le C\left(\|\Delta u^n\|_2^2+\|\nabla u^n\|_2^{2p}+\|\nabla \upsilon^n\|_2^{2p}+\|\nabla u^n\|_2^{p-1}\|\nabla u^n\|_2^{p+1}\right)\\
&\le C\|\Delta u^n\|_2^2+c.
\end{aligned}
\tag{49}
$$

Likewise, we obtain

$$\left|\int_\Omega f_2(u^n(t),\upsilon^n(t))\Delta\upsilon^n(t)dx\right| \le C\|\Delta\upsilon^n\|_2^2 + c. \tag{50}$$

Now we estimate $I = \int_\Omega \frac{\partial}{\partial u} f_1(u^n(t),\upsilon^n(t))u_t^n(t)\Delta u^n(t)dx$, then, by (A_2) and Young's inequality, we get

$$\begin{aligned} |I| &\le c\int_\Omega \left(|u^n|^{p-1} + |\upsilon^n|^{p-1}\right)|u_t^n||\Delta u^n|dx, \\ &\le c\|u_t^n\|_2\|u^n\|_{2p}^{p-1}\|\Delta u^n\|_{2p} + \|\upsilon^n\|_{2p}^{p-1}\|u_t^n\|_{2p}\|\Delta u^n\|_2. \end{aligned} \tag{51}$$

Hence

$$\begin{aligned} |I| &\le c\left(\|\nabla u^n\|_2^{p-1} + \|\nabla\upsilon^n\|_2^{p-1}\right)\|\nabla u_t^n\|_2\|\Delta u^n\|_2, \\ &\le c\|\nabla u_t^n\|_2\|\Delta u^n\|_2, \\ &\le c\|\Delta u^n\|_2^2 + c\|\nabla u_t^n\|_2^2. \end{aligned} \tag{52}$$

Then, we infer from (49) – (52), using Gronwall's Lemma, we deduce that

$$\begin{aligned} &\|\nabla u_t^n(t)\|_2^2 + \|\nabla\upsilon_t^n(t)\|_2^2 + a_{01}\left(1 - \int_0^t g_1(s)ds\right)\|\Delta u^n(t)\|_2^2 \\ &+ a_{02}\left(1 - \int_0^t g_2(s)ds\right)\|\Delta\upsilon^n(t)\|_2^2 \\ &+ \int_0^1 \left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|\nabla z_{1t}^n(x,k_1,t)\|_{L^2(\Omega)}^2 dk_1 \\ &+ \int_0^1 \left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|\nabla z_{2t}^n(x,k_2,t)\|_{L^2(\Omega)}^2 dk_2 \\ &\le e^{cT}\left(\|\nabla u_t^n(0)\|_2^2 + \|\nabla\upsilon_t^n(0)\|_2^2 + \|\Delta u^n(0)\|_2^2 + \|\Delta\upsilon^n(0)\|_2^2\right) \\ &+ e^{cT}\left(\int_0^1 \left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|\nabla z_{1t}^n(x,k_1,0)\|_{L^2(\Omega)}^2 dk_1\right) \\ &+ e^{cT}\left(\int_0^1 \left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|\nabla z_{2t}^n(x,k_2,0)\|_{L^2(\Omega)}^2 dk_2\right), \end{aligned} \tag{53}$$

we have also from (41)

$$\begin{aligned} &\|u_t^n(t)\|_2^2 + \|\upsilon_t^n(t)\|_2^2 + \|\nabla u^n(t)\|_2^2 + \|\nabla\upsilon^n(t)\|_2^2 + (g_1 o u^n)(t) \\ &+ \int_0^1\int_\Omega z_1^n(x,1,s)dxds + \int_0^1\int_\Omega z_2^n(x,1,s)dxds + (g_2 o\upsilon^n)(t) \\ &+ \int_0^1\int_\Omega z_1^n(x,k_1,s)dxds + \int_0^1\int_\Omega z_2^n(x,k_2,s)dxds + \int_\Omega F(u,\upsilon)dx \le C_1, \end{aligned} \tag{54}$$

where C_1 is a positive constant depending on the parameter $E(0)$.

(3) **Estimate 3.** First, we estimate $(u_n^{''}(0))$ and $(\upsilon_n^{''}(0))$ in (19)-(20) and taking $t=0$, we obtain

$$\begin{aligned} \|u_{tt}^n(0)\|_2^2+\|\upsilon_{tt}^n(0)\|_2^2 &\le \|a_{01}\ u^{0n}\|_2^2+\mu_1\|u^{1n}\|_2^2+\mu_2\|z_1^{0n}\|_2^2 \\ &+a_{02}\|\upsilon^{0n}\|_2^2+\alpha_1\|\upsilon^{1n}\|_2^2+\alpha_2\|z_2^{0n}\|_2^2 \\ &\le a_{01}\|u^0\|_2^2+\mu_1\|u^1\|_2^2+\mu_2\|z_1^0\|_2^2 \\ &+a_{02}\|\upsilon^0\|_2^2+\alpha_1\|\upsilon^1\|_2^2+\alpha_2\|z_2^0\|_2^2\le C. \end{aligned} \tag{55}$$

Where C is a positive constant. Now, differentiating (19) and (20) with respect to t, we have

$$\begin{aligned} &\left\langle \sum_{i,j=1}^N a_{1ij}(x)\left(\int_0^t g_1(t-s)\nabla u^n(t)ds\right)^{'},\nabla w_j\right\rangle_\Omega \\ &+\left\langle \mu_1\left(u_t^n(x,t)\right)^{'},w_j\right\rangle_\Omega+\langle u_{ttt}^n(t),w_j\rangle_\Omega+a_1(u_t^n(t),w_j) \\ &+\left\langle \mu_2\left(z_1^n(x,1,t)\right)^{'},w_j\right\rangle_\Omega+\langle (Df_1(u^n(t),\upsilon^n(t)),w_j\rangle_\Omega=0, \end{aligned} \tag{56}$$

and

$$\begin{aligned} &\left\langle \sum_{i,j=1}^N a_{2ij}(x)\left(\int_0^t g_2(t-s)\nabla \upsilon^n(t)ds\right)^{'},\nabla w_j\right\rangle_\Omega \\ &+\langle \upsilon_{ttt}^n(t),w_j\rangle_\Omega+a_2(\upsilon_t^n(t),w_j)+\left\langle \alpha_1\left(\upsilon_t^n(x,t)\right)^{'},w_j\right\rangle_\Omega \\ &+\left\langle \alpha_2\left(z_2^n(x,1,t)\right)^{'},w_j\right\rangle_\Omega+\langle (Df_2(u^n(t),\upsilon^n(t)),w_j\rangle_\Omega=0. \end{aligned} \tag{57}$$

Multiplying (56) by $u_{n,j}''(t)$ and (57) by $\upsilon_{n,j}''(t)$, summing over j from 1 to n, it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_{tt}^n(t)\|_2^2+a_1(u_t^n(t),u_t^n(t))+\mu_1\|u_{tt}^n(t)\|_2^2+\mu_2\left\langle u_{tt}^n(t),z_{1t}^n(x,1,t)\right\rangle_\Omega \\ &-g_1(0)\frac{d}{dt}A_1\left\langle \nabla u^n(t),\nabla u_t^n(t)\right\rangle_\Omega+g_1(0)a_1(u_t^n(t),u_t^n(t)) \\ &-\frac{d}{dt}\int_0^t A_1g_1^{'}(t-s)\left\langle \nabla u_s^n(s),\nabla u_t^n(t)\right\rangle_\Omega ds+\left\langle Df_1(u^n(t),\upsilon^n(t)),u_{tt}^n(t)\right\rangle_\Omega \\ &-\int_0^t A_1g_1^{''}(t-s)\left\langle \nabla u_s^n(s),\nabla u_t^n(t)\right\rangle_\Omega ds+g_1'(0)\left\langle A_1\nabla u_t^n(t),\nabla u_t^n(t)\right\rangle_\Omega, \end{aligned} \tag{58}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\upsilon_{tt}^n(t)\|_2^2+a_2(\upsilon_t^n(t),\upsilon_t^n(t))+\alpha_1\|\upsilon_{tt}^n(t)\|_2^2+\alpha_2\left\langle \upsilon_{tt}^n(t),z_{2t}^n(x,1,t)\right\rangle_\Omega \\ &-g_2(0)\frac{d}{dt}\left\langle A_2\nabla \upsilon^n(t),\nabla \upsilon_t^n(t)\right\rangle_\Omega+g_2(0)a_2(\upsilon_t^n(t),\upsilon_t^n(t)) \\ &-\frac{d}{dt}\int_0^t A_2g_2^{'}(t-s)\left\langle \nabla \upsilon_s^n(s),\nabla \upsilon_t^n(t)\right\rangle_\Omega ds+g_2'(0)\left\langle A_2\nabla \upsilon_t^n(t),\nabla \upsilon_t^n(t)\right\rangle_\Omega \\ &+\int_0^t A_2g_2^{''}(t-s)\left\langle \nabla \upsilon_s^n(s),\nabla \upsilon_t^n(t)\right\rangle_\Omega ds+\left\langle Df_2(u^n(t),\upsilon^n(t)),\upsilon_{tt}^n(t)\right\rangle_\Omega ds. \end{aligned} \tag{59}$$

Differentiating (22) with respect to t, we get

$$\left(\frac{\tau_2(t)}{(1-\tau_2'(t)k_1}\right) z_{1t}^n(t) + \frac{\tau_2(t)}{(1-\tau_2'(t)k_1)} z_{1tt}^n(t) + \frac{\partial}{\partial k_1}\|z_{1t}^n(t)\|_2^2 = 0. \tag{60}$$

Multiplying (60) by $z'_{1n,j}(t)$, summing over j from 1 to n, it follows that

$$\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)' \|z_{1t}^n(t)\|_2^2 + \frac{1}{2}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\frac{d}{dt}\|z_{1t}^n(t)\|_2^2 + \frac{1}{2}\frac{d}{dk_1}\|z_{1t}^n(t)\|_2^2 = 0. \tag{61}$$

Then, we have

$$\begin{aligned} &\frac{1}{2}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)' \|z_{1t}^n(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\|z_{1t}^n(t)\|_2^2\right) \\ &+ \frac{1}{2}\frac{d}{dk_1}\|z_{1t}^n(t)\|_2^2 = 0. \end{aligned} \tag{62}$$

In the same way for (23), we get

$$\begin{aligned} &\frac{1}{2}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)' \|z_{2t}^n(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\|z_{2t}^n(t)\|_2^2\right) \\ &+ \frac{1}{2}\frac{d}{dk_2}\|z_{2t}^n(t)\|_2^2 = 0. \end{aligned} \tag{63}$$

Taking the sum of (58), (59), (61) and (63), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|u_{tt}^n(t)\|_2^2 + \|\upsilon_{tt}^n(t)\|_2^2 + a_1(u_t^n(t), u_t^n(t)) + a_2(\upsilon_t^n(t), \upsilon_t^n(t))\right] \\ &+ \frac{1}{2}\frac{d}{dt}\left(\int_0^1 \frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|z_{1t}^n(t)\|_2^2 dk_1 + \mu_1\|u_{tt}^n(t)\|_2^2 + \alpha_1\|\upsilon_{tt}^n(t)\|_2^2 \\ &+ \frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|z_{2t}^n(t)\|_2^2 dk_2 + g_1(0)a_1(u_t^n(t), u_t^n(t)) \\ &+ g_2(0)a_2(\upsilon_t^n(t), \upsilon_t^n(t)) + \frac{1}{2}\|z_{1t}^n(x,1,t)\|_2^2 + \frac{1}{2}\|z_{2t}^n(x,1,t)\|_2^2 \\ &= -\frac{1}{2}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)'\|z_{1t}^n(t)\|_2^2 dk_1 + \frac{1}{2}\|u_{tt}^n(t)\|_2^2 \\ &- \frac{1}{2}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)'\|z_{2t}^n(t)\|_2^2 dk_2 + \frac{1}{2}\|\upsilon_{tt}^n(t)\|_2^2 \\ &+ \langle Df_1(u^n(t), \upsilon^n(t)), u_{tt}^n(t)\rangle_\Omega + \langle Df_2(u^n(t), \upsilon^n(t)), \upsilon_{tt}^n(t)\rangle_\Omega \\ &+ g_1(0)\frac{d}{dt}\langle A_1\nabla u^n(t), \nabla u_t^n(t)\rangle_\Omega + g_2(0)\frac{d}{dt}\langle A_2\nabla\upsilon^n(t), \nabla\upsilon_t^n(t)\rangle_\Omega \\ &- g_1'(0)\langle A_1\nabla u_t^n(t), \nabla u_t^n(t)\rangle_\Omega - \int_0^t A_1 g_1''(t-s)\langle\nabla u^n(s), \nabla u_t^n(t)\rangle_\Omega ds \\ &- \frac{d}{dt}\int_0^t A_2 g_2'(t-s)\langle\nabla\upsilon_s^n(s), \nabla\upsilon_t^n(t)\rangle_\Omega ds - g_2'(0)\langle A_2\nabla\upsilon_t^n(t), \nabla\upsilon_t^n(t)\rangle_\Omega \\ &- \int_0^t A_2 g_2''(t-s)\langle\nabla\upsilon^n(s), \nabla\upsilon_t^n(t)\rangle_\Omega ds - \mu_2\langle u_{tt}^n(t), z_{1t}^n(x,1,t)\rangle_\Omega \\ &- \frac{d}{dt}\int_0^t A_1 g_1'(t-s)\langle\nabla u_s^n(s), \nabla u_t^n(t)\rangle_\Omega ds - \alpha_2\langle\upsilon_{tt}^n(t), z_{2t}^n(x,1,t)\rangle_\Omega. \end{aligned} \tag{64}$$

Using Holder, Young's inequalities and the same technique as in [29], we conclude the following estimates

$$
\begin{aligned}
g_1'(0)A_1\left\langle \nabla u^n(t), \nabla u_t^n(t)\right\rangle_\Omega &= \sum_{i,j=1}^{N} g_1'(0)a_{1ij}(x)\left\langle \frac{\partial u^n(t)}{\partial x_j}, \frac{\partial u_t^n(t)}{\partial x_i}\right\rangle_\Omega \\
&\le \sum_{i,j=1}^{N} \frac{(g_1'(0))^2}{2\mu}\int_\Omega \left|\frac{\partial u^n(t)}{\partial x_j}\right|^2 dx + 2\mu \sum_{i,j=1}^{N}\int_\Omega \left| a_{1ij}(x)\frac{\partial u_t^n(t)}{\partial x_i}\right|^2 dx \\
&\le \frac{(g_1'(0))^2}{2\mu}\|\nabla u^n(t)\|_2^2 + 2\mu \max_{1\le i\le N}\left(\sum_{j=1}^{N}\|a_{1ij}\|_\infty^2\right)\|\nabla u_t^n(t)\|_2^2 \\
&\le \frac{(g_1'(0))^2}{2\mu}\|\nabla u^n(t)\|_2^2 + 2a_{11}\mu\|\nabla u_t^n(t)\|_2^2,
\end{aligned}
\tag{65}
$$

where

$$
a_{11} = \max_{1\le i\le N}\left(\sum_{j=1}^{N}\|a_{1ij}\|_\infty^2\right), \quad a_{22} = \max_{1\le i\le N}\left(\sum_{j=1}^{N}\|a_{2ij}\|_\infty^2\right).
$$

And

$$
g_2'(0)A_2\left\langle \nabla \upsilon^n(t), \nabla \upsilon_t^n(t)\right\rangle_\Omega \le \frac{(g_2'(0))^2}{2\mu}\|\nabla \upsilon^n(t)\|_2^2 + 2a_{22}\mu\|\nabla \upsilon_t^n(t)\|_2^2, \tag{66}
$$

$$
\begin{aligned}
&\int_0^t A_1 g_1''(t-s)\left\langle \nabla u^n(s), \nabla u_t^n(t)\right\rangle_\Omega ds \\
&= \int_0^t \sum_{j=1}^{N} a_{1ij}(x) g_1''(t-s)\left\langle \nabla u^n(s), \nabla u_t^n(t)\right\rangle_\Omega ds \\
&\le \sum_{j=1}^{N}\frac{a_{11}}{\epsilon}\int_\Omega\left|\frac{\partial u_t^n(t)}{\partial x_i}\right|^2 dx + 2\epsilon\|g_1''\|_{L^1}\sum_{j=1}^{N}\int_0^t\int_\Omega\left|\frac{\partial u^n(s)}{\partial x_j}\right|^2 dxds \\
&\le \frac{a_{11}}{\epsilon}\|\nabla u_t^n(t)\|_2^2 + 2\epsilon\|g_1''\|_{L^1}\int_0^t\|\nabla u^n(s)\|_2^2 ds,
\end{aligned}
\tag{67}
$$

also

$$
\begin{aligned}
\left|\int_0^t A_2 g_2''(t-s)\left\langle \nabla \upsilon^n(s), \nabla \upsilon_t^n(t)\right\rangle_\Omega ds\right| &\le \frac{a_{22}}{\epsilon}\|\nabla \upsilon_t^n(t)\|_2^2 \\
&\quad + 2\epsilon\|g_2''\|_{L^1}\int_0^t\|\nabla \upsilon^n(s)\|_2^2 ds.
\end{aligned}
\tag{68}
$$

Using (A_2) and the Sobolev's embedding, gives us

$$
\begin{aligned}
&|\langle Df_1 u^n(t), \upsilon^n(t)), u_{tt}^n(t)\rangle_\Omega| \le C\left[(\|u^n\|^{p-1} + \|\upsilon^n\|^{p-1})\|u_t^n\|_2\right]\|u_{tt}^n\|_2 \\
&+ C\left[(\|u^n\|^{p-1} + \|\upsilon^n\|^{p-1})\|\upsilon_t^n\|_2\right]\|u_{tt}^n\|_2 \\
&\le C\left[\|u^n\|_2^{2(p-1)} + \|\upsilon^n\|_2^{2(p-1)} + \|u_t^n\|_2^2 + \|\upsilon_t^n\|_2^2\right]\|u_{tt}^n\|_2 \le \|u_{tt}^n\|_2^2 + c,
\end{aligned}
\tag{69}
$$

where c is a positive constant. In the same way, we obtain

$$
|\langle Df_2(u^n(t), \upsilon^n(t))\upsilon_{tt}^n\rangle_\Omega| \le \|\upsilon_{tt}^n\|_2^2 + c. \tag{70}
$$

Replacing (65)-(70) in (64), we get

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\Big[\|u^n_{tt}(t)\|_2^2+\|\upsilon^n_{tt}(t)\|_2^2+a_1(u^n_t(t),u^n_t(t))+a_2(\upsilon^n_t(t),\upsilon^n_t(t))\Big]\\
&+\tfrac{1}{2}\tfrac{d}{dt}\int_0^1\left(\tfrac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|z^n_{1t}(t)\|_2^2dk_1+\tfrac{1}{2}\|z^n_{1t}(x,1,t)\|_2^2\\
&+\tfrac{1}{2}\tfrac{d}{dt}\int_0^1\left(\tfrac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|z^n_{2t}(t)\|_2^2dk_2+\tfrac{1}{2}\|z^n_{2t}(x,1,t)\|_2^2\\
&+g_1(0)a_{11}(u^n_t(t),u^n_t(t))+g_2(0)a_{22}(\upsilon^n_t(t),\upsilon^n_t(t))\\
&\le c\|u^n_{tt}(t)\|_2^2+c\|\upsilon^n_{tt}(t)\|_2^2+c\left(\int_0^1\left(\tfrac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|z^n_{1t}(t)\|_2^2dk_1\right)\\
&+c\left(\int_0^1\left(\tfrac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|z^n_{2t}(t)\|_2^2dk_2\right)+2\mu a_{22}\|\nabla\upsilon^n_t(t)\|_2^2\\
&+\tfrac{(g_1'(0))^2}{2\mu}\|\nabla u^n(t)\|_2^2+2\mu a_{11}\|\nabla u^n_t(t)\|_2^2+\tfrac{(g_2'(0))^2}{2\mu}\|\nabla\upsilon^n(t)\|_2^2\\
&+\epsilon\|g_1''\|_{L^1}\int_0^t A_1g_1''(t-s)\|\nabla u^n(s)\|_2^2ds+\frac{a_{11}}{\epsilon}\|\nabla u^n_t(t)\|_2^2\\
&+\epsilon\|g_2''\|_{L^1}\int_0^t A_2g_2''(t-s)\|\nabla\upsilon^n(s)\|_2^2ds+\frac{a_{22}}{\epsilon}\|\nabla\upsilon^n_t(t)\|_2^2\\
&-\tfrac{d}{dt}\int_0^t A_1g_1'(t-s)\left\langle\nabla u^n_s(s),\nabla u^n_t(t)\right\rangle_\Omega ds+g_1(0)\tfrac{d}{dt}A_1\left\langle\nabla u^n(t),\nabla u^n_t(t)\right\rangle_\Omega\\
&-\tfrac{d}{dt}\int_0^t A_2g_2'(t-s)\left\langle\nabla\upsilon^n_s(s)\nabla\upsilon^n_t(t)\right\rangle_\Omega ds+g_2(0)\tfrac{d}{dt}A_2\left\langle\nabla\upsilon^n(t),\nabla\upsilon^n_t(t)\right\rangle_\Omega.
\end{aligned}
\tag{71}
$$

Integrating the last inequality over $(0,t)$ and using Gronwall's Lemma, we obtain

$$
\begin{aligned}
&\|u^n_{tt}(t)\|_2^2+\|\upsilon^n_{tt}(t)\|_2^2+a_1(u^n_t(t),u^n_t(t))+a_2(\upsilon^n_t(t),\upsilon^n_t(t))\\
&+\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|z^n_{1t}(t)\|_2^2dk_1+\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|z^n_{2t}(t)\|_2^2dk_2\\
&\le\|u^n_{tt}(0)\|_2^2+\|\upsilon^n_{tt}(0)\|_2^2+a_1(u^n_t(0),u^n_t(0))+a_2(\upsilon^n_t(0),\upsilon^n_t(0))\\
&+\int_0^1\left(\frac{\tau_2(0)}{1-\tau_2'(0)k_1}\right)\|z^n_{1t}(x,k_1,0)\|_2^2dk_1+g_1(0)a_1(u^n(t),u^n_t(t))\\
&+\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|z^n_{2t}(x,k_2,0)\|_2^2dk_2+g_2(0)a_2(\upsilon^n(t),\upsilon^n_t(t))\\
&-\int_0^t g_1'(t-s)A_1\left\langle\nabla u^n_s(s),\nabla u^n_t(s)\right\rangle_\Omega ds-g_1(0)a_1(u^n_t(0),u^n_t(0))\\
&-\int_0^t g_2'(t-s)A_2\left\langle\nabla\upsilon^n_s(s),\nabla\upsilon^n_t(t)\right\rangle_\Omega ds-g_2(0)a_2(\upsilon^n_t(0),\upsilon^n_t(0))\\
&+\left(\frac{1}{4\epsilon}+\frac{g_1'(0)^2}{4\epsilon}\right)\int_0^t\|\nabla u^n_t(s)\|_2^2ds+\left(\frac{1}{4\epsilon}+\frac{g_2'(0)^2}{4\epsilon}\right)\int_0^t\|\nabla\upsilon^n_t(s)\|_2^2ds\\
&+(\epsilon+\epsilon\|g_1\|_{L^1}^2)\int_0^t\|\nabla u^n(s)\|_2^2ds+(\epsilon+\epsilon\|g_2\|_{L^1}^2)\int_0^t\|\nabla\upsilon^n(s)\|_2^2ds\\
&+\int_0^t\left(\|u^n_{ss}(s)\|_2^2+\|\upsilon^n_{ss}(s)\|_2^2+a_1(u^n_s(s),u^n_s(s))+a_2(\upsilon^n_s(s),\upsilon^n_s(s))\right)ds\\
&+\int_0^t\left(\int_0^1\left(\frac{\tau_2(s)}{1-\tau_2'(s)k_1}\right)\|z^n_{1t}(x,k_1,s)\|_2^2dk_1\right)ds\\
&+\int_0^t\left(\int_0^1\left(\frac{\tau_2(s)}{1-\tau_2'(s)k_2}\right)\|z^n_{2t}(x,k_2,s)\|_2^2dk_2\right)ds.
\end{aligned}
\tag{72}
$$

We have to estimate the right hand side of (72). We get easily

$$\begin{aligned}
g_1(0)A_1\left\langle \nabla u^n(t),\nabla u_t^n(t)\right\rangle_\Omega &= \sum_{i,j=1}^{N} g_1(0)a_{1ij}(x)\left\langle \frac{\partial u^n(t)}{\partial x_j},\frac{\partial u_t^n(t)}{\partial x_i}\right\rangle_\Omega \\
&\le \sum_{i,j=1}^{N}\frac{(g_1(0))^2}{2\mu}\int_\Omega\left|\frac{\partial u^n(t)}{\partial x_j}\right|^2dx+2\mu\sum_{i,j=1}^{N}\int_\Omega\left|a_{1ij}(x)\frac{\partial u_t^n(t)}{\partial x_i}\right|^2dx \\
&\le \frac{(g_1(0))^2}{2\mu}\|\nabla u^n(t)\|_2^2+2\mu\max_{1\le i\le N}\left(\sum_{j=1}^{N}\|a_{1ij}\|_\infty^2\right)\|\nabla u_t^n(t)\|_2^2 \\
&\le \frac{(g_1(0))^2}{2\mu}\|\nabla u^n(t)\|_2^2+2a_{11}\mu\|\nabla u_t^n(t)\|_2^2.
\end{aligned} \tag{73}$$

In the same way

$$g_2(0)\left\langle \nabla \upsilon^n(t),\nabla \upsilon_t^n(t)\right\rangle_\Omega \le \frac{(g_2(0))^2}{2\mu}\|\nabla \upsilon^n(t)\|_2^2+2a_{22}\mu\|\nabla \upsilon_t^n(t)\|_2^2. \tag{74}$$

As previously, we can obtain

$$\begin{aligned}
&\left|\int_0^t g_1'(t-s)A_1\left\langle \nabla u^n(s),\nabla u_t^n(t)\right\rangle_\Omega ds\right| \le \epsilon\|\nabla u_t^n(t)\|_2^2 \\
&+\frac{a_{11}\|g_1\|_{L^1}\|g_1\|_{L^\infty}}{4\epsilon}\int_0^t\|\nabla u^n(s)\|_2^2ds,
\end{aligned} \tag{75}$$

and

$$\begin{aligned}
&\left|\int_0^t g_2'(t-s)A_2\left\langle \nabla \upsilon^n(s),\nabla \upsilon_t^n(t)\right\rangle_\Omega ds\right| \le \epsilon\|\nabla \upsilon_t^n(t)\|_2^2 \\
&+\frac{a_{22}\|g_2\|_{L^1}\|g_2\|_{L^\infty}}{4\epsilon}\int_0^t\|\nabla \upsilon^n(s)\|_2^2ds.
\end{aligned} \tag{76}$$

Replacing (73)-(76) in (72), after choosing ϵ small enough and using Gronwall's Lemma, we obtain

$$\begin{aligned}
&\|u_{tt}^n(t)\|_2^2+\|\upsilon_{tt}^n(t)\|_2^2+a_{01}\|\nabla u_t^n(t)\|_2^2+a_{02}\|\nabla \upsilon_t^n(t)\|_2^2 \\
&+\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|z_{1t}^n(x,k_1,t)\|_2^2dk_1 \\
&+\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|z_{2t}^n(x,k_2,t)\|_2^2dk_2\le M.
\end{aligned} \tag{77}$$

Where M is some positive constant. Therefore, from (41), (53) and (77), we conclude that

$$u^n \textit{ is bounded in } L^\infty(0,T;H_0^1(\Omega)), \tag{78}$$

$$\upsilon^n \textit{ is bounded in } L^\infty(0,T;H_0^1(\Omega)), \tag{79}$$

$$u_t^n \textit{ is bounded in } L^\infty(0,T;H_0^1(\Omega)), \tag{80}$$

$$\upsilon_t^n \textit{ is bounded in } L^\infty(0,T;H_0^1(\Omega)), \tag{81}$$

$$u_{tt}^n \textit{ is bounded in } L^\infty(0,T;L^2(\Omega)), \tag{82}$$

$$\upsilon_{tt}^n \textit{ is bounded in } L^\infty(0,T;L^2(\Omega)), \tag{83}$$

$$z_1^n(x,1,t) \textit{ is bounded in } L^2(\Omega\times(0,T)), \tag{84}$$

$$z_2^n(x,1,t) \textit{ is bounded in } L^2(\Omega\times(0,T)), \tag{85}$$

$$\tau_2(t)z_{1t}^n(x,k_1,t) \textit{ is bounded in } L^\infty(0,T;L^1(\Omega\times(0,1)), \tag{86}$$

$$\tau_2(t)z_{2t}^n(x,k_2,t) \textit{ is bounded in } L^\infty(0,T;L^1(\Omega\times(0,1)), \tag{87}$$

$$z_{1t}^n(x,1,t) \textit{ is bounded in } L^\infty(0,T;L^2(\Omega\times(0,T))), \tag{88}$$

$$z_{2t}^n(x,,t) \textit{ is bounded in } L^\infty(0,T;L^2(\Omega\times(0,T))). \tag{89}$$

Applying Dunford-Pettis theorem, we deduce from (78)-(89) that there exists a subsequence $(u^n, z_1^n), (\upsilon^n, z_2^n)$ such that

$$u^n \to u \textit{ weakly star in } L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \tag{90}$$

$$\upsilon^n \to \upsilon \textit{ weakly star in } L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \tag{91}$$

$$u_t^n \to u_t \textit{ weakly star in } L^\infty(0,T;H_0^1(\Omega)), \tag{92}$$

$$\upsilon_t^n \to \upsilon_t \textit{ weakly star in } L^\infty(0,T;H_0^1(\Omega)), \tag{93}$$

$$u_t^n \to \chi_1 \textit{ weakly star in } L^2(\Omega\times(0,T)), \tag{94}$$

$$\upsilon_t^n \to \chi_2 \textit{ weakly star in } L^2(\Omega\times(0,T)), \tag{95}$$

$$u_{tt}^n \to u_{tt} \textit{ weakly star in } L^\infty(0,T;L^2(\Omega)), \tag{96}$$

$$\upsilon_{tt}^n \to \upsilon_{tt} \textit{ weakly star in } L^\infty(0,T;L^2(\Omega)), \tag{97}$$

$$z_1^n(x,k_1,t) \to z_1(x,k_1,t) \textit{ weakly star in } L^\infty(0,T;H_0^1(\Omega;L^2(0,1)), \tag{98}$$

$$z_2^n(x,k_2,t) \to z_2(x,k_2,t) \textit{ weakly star in } L^\infty(0,T;H_0^1(\Omega;L^2(0,1)), \tag{99}$$

$$z_{1t}^n(x,k_1,t) \to z_{1t}^n(x,k_1,t) \textit{ weakly star in } L^\infty(0,T;L^2(\Omega\times(0,T))), \tag{100}$$

$$z_{2t}^n(x,k_2,t) \to z_{2t}^n(x,k_2,t) \textit{ weakly star in } L^\infty(0,T;L^2(\Omega\times(0,T))), \tag{101}$$

$$z_{1t}^n(x,1,t) \to \psi_1 \textit{ weakly star in } L^2(\Omega\times(0,1)), \tag{102}$$

$$z_{2t}^n(x,1,t) \to \psi_2 \textit{ weakly star in} \mathrm{L}^2(\Omega\times(0,1)). \tag{103}$$

Further, by Aubin's Lemma [30], it follows from (90)-(103) that there exists a subsequence (u^n, υ^n) still represented by the same notation, such that

$$u^n \to u \textit{ strongly in } L^2(0,T;L^2(\Omega)), \tag{104}$$

$$\upsilon^n \to \upsilon \textit{ strongly in } L^2(0,T;L^2(\Omega)), \tag{105}$$

hence

$$u^n \to u \textit{ and } \upsilon^n \to \upsilon \textit{ a.e in } (0,T)\times\Omega, \tag{106}$$

and
$$u_t^n \to u_t \text{ and } \upsilon_t^n \to \upsilon_t \text{ a.e in } (0,T)\times\Omega. \tag{107}$$

(4) **Analysis of nonlinear term.**

$$\begin{aligned}
&\|f_1(u^n,\upsilon^n)\|_{L^2(\Omega\times(0,T))} \le \int_0^T\int_\Omega (|u^n(s)|^p + |\upsilon^n(s)|^p)dsdx \\
&+ \int_0^T\int_\Omega (|u^n(s)|^{\frac{p-1}{2}}|\upsilon^n(s)|^{\frac{p+1}{2}})dsdx \\
&\le c_s^p\int_0^T \|\nabla u^n(s)\|^p ds + c_s^p\int_0^T \|\nabla \upsilon^n(s)\|^p ds, \\
&+ c_s^{\frac{p-1}{2}}\int_0^T \|\nabla u^n(s)\|^{\frac{p-1}{2}} ds + c_s^{\frac{p+1}{2}}\int_0^T \|\nabla \upsilon^n(s)\|^{\frac{p+1}{2}} ds \\
&\le 2c_s^pTC_1^P + c_s^{\frac{p-1}{2}}TC_1^{\frac{p-1}{2}}TC_1^{\frac{p-1}{2}} + c_s^{\frac{p+1}{2}}TC_1^{\frac{p+1}{2}}TC_1^{\frac{p+1}{2}} = C.
\end{aligned} \tag{108}$$

Where C is a positive constant. In the same way for $f_2(u^n,\upsilon^n)$
$$\|f_2(u^n,\upsilon^n)\|_{L^2(\Omega\times(0,T))} \le C. \tag{109}$$

From the (108) and (109), we deduce that
$$\begin{aligned}
f_1(u^n,\upsilon^n) &\to f_1(u,\upsilon) \text{ weakly in } L^2(0,T;L^2(\Omega)), \\
f_1(u^n,\upsilon^n) &\to f_1(u,\upsilon) \text{ weakly in } L^2(0,T;L^2(\Omega)).
\end{aligned} \tag{110}$$

For suitable functions $(u,\upsilon) \in (L^\infty(0,T;H_0^1(\Omega)))^2$, $(z_1,z_2) \in (L^\infty(0,T;L^2(\Omega\times(0.1))))^2$, $\psi_1,\psi_2 \in L^2(\Omega\times(0,T))$, $(\chi_1,\chi_2) \in L^2(\Omega\times(0,T))^2$, $\xi \in L^\infty((0,T);L^2(\Omega))$. We have to show that $((u,z_1),(\upsilon,z_2))$ is a solution of $(19)-(24)$. Using the embedding
$$L^\infty(0,T;H_0^1(\Omega)) \hookrightarrow L^2(0,T;H_0^1(\Omega)),$$
$$H^1((0,T)\times\Omega) \hookrightarrow L^2((0,T)\times\Omega).$$

From (80)-(81) we have that u_t^n, υ_t^n are bounded in
$$L^\infty((0,T);H_0^1(\Omega)) \hookrightarrow L^2((0,T);H_0^1(\Omega)),$$
then u_{tt}^n and υ_{tt}^n are bounded in
$$L^\infty((0,T);L^2(\Omega)) \hookrightarrow L^2((0,T);L^2(\Omega)).$$

Consequently, u_t^n, υ_t^n are bounded in $H^1((\Omega)\times(0,T))$. Using Aubin-Lions theorem [30], we can extract a subsequence (u^ξ) of (u^n) and (υ^ξ) of (υ^n) such that
$$u_t^\xi \to u_t \text{ strongly in } L^2(\Omega\times(0,T)), \tag{111}$$
$$\upsilon_t^\xi \to \upsilon_t \text{ strongly in } L^2(\Omega\times(0,T)), \tag{112}$$
therefore
$$u_t^\xi \to u_t \text{ strongly and a.e. in } (\Omega\times(0,T)), \tag{113}$$
$$\upsilon_t^\xi \to \upsilon_t \text{ strongly and a.e. in } (\Omega\times(0,T)). \tag{114}$$

Similarly
$$z_1^\xi \to z_1 \text{ strongly in } L^2(0,T;L^2(\Omega\times(0,1))), \tag{115}$$
$$z_2^\xi \to z_2 \text{ strongly in } L^2(0,T;L^2(\Omega\times(0,1))). \tag{116}$$

It follows at once from the convergence (90), (91),(96), (97), (100), (101), (111) and (112) for each fixed $\vartheta \in L^2(0,T,L^2(\Omega))$, $\sigma \in L^2(0,T,L^2(\Omega)\times(0,1))$ as $\xi \to \infty$ permits us to deduce that

$$\begin{aligned}
&\int_0^T \left[\int_\Omega u_{tt}^\xi(t)\vartheta dx + \int_\Omega A_1\nabla u^\xi(t)\nabla\vartheta dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \int_0^t g_1(t-s)A_1\nabla u^\xi(t)\nabla\vartheta ds dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \mu_1 u_t^\xi(t)\vartheta dx + \int_\Omega \mu_2 z_1^\xi(x,1,t)\vartheta dx = \int_\Omega f_1(u^\xi(t)\upsilon^\xi(t))\vartheta dx\right] dt \\
&\to \int_0^T \left[\int_\Omega u_{tt}(t)\vartheta dx + \int_\Omega A_1\nabla u(t)\nabla\vartheta dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \int_0^t g_1(t-s)A_1\nabla u(t)\nabla\vartheta ds dx dt\right] dt \\
&+ \int_0^T \left[\int_\Omega \mu_1 u_t(t)\vartheta dx + \int_\Omega \mu_2 z_1(x,1,t)\vartheta dx = \int_\Omega f_1(u(t),\upsilon(t))\vartheta dx\right] dt,
\end{aligned} \tag{117}$$

and

$$\begin{aligned}
&\int_0^T \left[\int_\Omega \upsilon_{tt}^\xi(t)\vartheta dx dt + \int_\Omega A_2\nabla \upsilon^\xi(t)\nabla\vartheta dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \int_0^t A_2 g_2(t-s)\nabla \upsilon^\xi(t)\nabla\vartheta ds dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \mu_1 \upsilon_t^\xi(t)\vartheta dx + \int_\Omega \mu_2 z_2^\xi(x,1,t)\vartheta dx = \int_\Omega f_2(u^\xi(t)\upsilon^\xi(t))\vartheta dx\right] dt \\
&\to \int_0^T \left[\int_\Omega \upsilon_{tt}(t)\vartheta dx + \int_\Omega A_2\nabla \upsilon(t)\nabla\vartheta dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \int_0^t g_2(t-s)A_2\nabla \upsilon(t) ds \nabla\vartheta dx\right] dt \\
&+ \int_0^T \left[\int_\Omega \alpha_1 \upsilon_t(t)\vartheta dx + \int_\Omega \alpha_2 z_2(x,1,t)\vartheta dx = \int_\Omega f_2(u(t),\upsilon(t))\vartheta dx\right] dt.
\end{aligned} \tag{118}$$

Exploiting the convergence (115) and (116) we deduce

$$\begin{aligned}
&\int_0^T \int_0^1 \int_\Omega \left(\tau_2(t)\frac{\partial}{\partial t} z_1^\xi + (1-\tau_2'(t)k_1)\frac{\partial}{\partial k_1} z_1^\xi\right)\sigma dx dk_1 dt \\
&\to \int_0^T \int_0^1 \int_\Omega \left(\tau_2(t)\frac{\partial}{\partial t} z_1 + (1-\tau_2'(t)k_1)\frac{\partial}{\partial k_1} z_1\right)\sigma dx dk_1 dt,
\end{aligned} \tag{119}$$

and

$$\begin{aligned}
&\int_0^T \int_0^1 \int_\Omega \left(\tau_2(t)\frac{\partial}{\partial t} z_2^\xi + (1-\tau_2'(t)k_2)\frac{\partial}{\partial k_2} z_2^\xi\right)\sigma dx dk_2 dt \\
&\to \int_0^T \int_0^1 \int_\Omega \left(\tau_2(t)\frac{\partial}{\partial t} z_2 + (1-\tau_2'(t)k_2)\frac{\partial}{\partial k_2} z_2\right)\sigma dx dk_2 dt.
\end{aligned} \tag{120}$$

Uniqueness. Let (u_1, υ_1) and (u_2, υ_2) be two solutions of problem 1. Then $(w, q) = (u_1, \upsilon_1) - (u_2, \upsilon_2)$ and we put also $\tilde{w} = u_1'(x, t-k_1\tau_2(t)) - u_2'(x, t-k_1\tau_2(t))$, $\tilde{q} = \upsilon_1'(x, t-k_2\tau_2(t)) - \upsilon_2'(x, t-k_2\tau_2(t))$.

Multiplying the first equation in (18) by w', integrating over Ω and using integration by parts, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|w'(t)\|_2^2+\left(1-\int_0^t g_1(s)ds\right)a_1(w(t),w(t))+(g_1ow)(t)\right)\\
&+\mu_1\|w'(t)\|_2^2+\frac{1}{2}\|\tilde{w}(x,1,t)\|_2^2+g_1(t)a_1(w(t),w(t))-(g_1'\circ w)(t)\\
&=-\mu_2\int_\Omega \tilde{w}(x,1,t)w'(t)dx+\frac{1}{2}\|w'(t)\|_2^2\\
&+\int_\Omega\left[f_1(u_1,\upsilon_1)-f_1(u_2,\upsilon_2)\right]w'(t)dx.
\end{aligned} \tag{121}$$

In the same way for second equation in (18). Multiplying the second equation in (18) by q', integrating over Ω and using integration by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|q'(t)\|_2^2+\left(1-\int_0^t g_2(s)ds\right)a_2(q(t),q(t))+(g_2oq)(t)\right)\\
&+\alpha_1\|q'(t)\|_2^2+\frac{1}{2}\|\tilde{q}(x,1,t)\|_2^2+g_2(t)a_2(q(t),q(t))-(g_2'\circ q)(t)\\
&=-\alpha_2\int_\Omega \tilde{q}(x,1,t)q'dx+\frac{1}{2}\|q'(t)\|_2^2+\int_\Omega\left[f_2(u_1,\upsilon_1)-f_2(u_2,\upsilon_2)\right]q'(t)dx.
\end{aligned} \tag{122}$$

Multiplying the third equation in (18) by $\tilde{w}$, integrating over $\Omega\times(0,1)$, we arrive at

$$\frac{1}{2}\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\frac{d}{dt}\|\tilde{w}(t)\|_2^2+\frac{1}{2}\frac{d}{dk_1}\|\tilde{w}(t)\|_2^2=0. \tag{123}$$

Then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\|\tilde{w}(t)\|_2^2\right)dk_1-\frac{1}{2}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)'\|\tilde{w}(t)\|_2^2\,dk_1\\
&+\frac{1}{2}\left(\|\tilde{w}(x,1,t)\|_2^2-\|w'(t)\|_2^2\right)=0.
\end{aligned} \tag{124}$$

We use the same technique for the forth equation in (18), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\|\tilde{q}(t)\|_2^2\right)dk_2-\frac{1}{2}\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)'\|\tilde{q}(t)\|_2^2\,dk_2\\
&+\frac{1}{2}\left(\|\tilde{q}(x,1,t)\|_2^2-\|q'(t)\|_2^2\right)=0.
\end{aligned} \tag{125}$$

Combining (121)-(122) and (123)-(125), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\begin{array}{c} \|w'(t)\|_2^2+\|q'(t)\|_2^2+\left(1-\int_0^t g_1(s)ds\right)a_2(w(t),w(t)) \\ +\left(1-\int_0^t g_2(s)ds\right)a_1(q(t),q(t))+(g_1ow)(t)+(g_2oq)(t)\end{array}\right)dx \\
&+\frac{1}{2}\frac{d}{dt}\left(\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\|\tilde{w}(t)\|_2^2\right)dk_1+\int_0^1\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\|\tilde{q}(t)\|_2^2\right)dk_2\right) \\
&+\mu_1\|\omega'(t)\|_2^2+\alpha_1\|q'(t)\|_2^2+\frac{1}{2}\|\tilde{q}(x,1,t)\|_2^2+\frac{1}{2}\|\tilde{w}(x,1,t)\|_2^2 \\
&=-\mu_2\int_\Omega \tilde{w}(x,1,t)w'(t)dx-\alpha_2\int_\Omega \tilde{q}(x,1,t)q'(t)dx \\
&+\int_\Omega\left[f_1(u_1,\upsilon_1)-f_1(u_2,\upsilon_2)\right]w'(t)dx+\frac{1}{2}\|w'(t)\|_2^2 \\
&+\int_\Omega\left[f_2(u_1,\upsilon_1)-f_2(u_2,\upsilon_2)\right]q'(t)dx+\frac{1}{2}\|q'(t)\|_2^2.
\end{aligned}
\tag{126}
$$

As in [15], we estimate the right hand side of (126) as follows

$$
\left|\int_\Omega\left[f_1(u_1,\upsilon_1)-f_1(u_2,\upsilon_2)\right]w'dx\right|\le C\left(\|w'(t)\|_2^2+\|\nabla w(t)\|_2^2+\|\nabla q(t)\|_2^2\right), \tag{127}
$$

$$
\left|\int_\Omega\left[f_2(u_1,\upsilon_1)-f_2(u_2,\upsilon_2)\right]w'dx\right|\le C\left(\|q'(t)\|_2^2+\|\nabla w(t)\|_2^2+\|\nabla q(t)\|_2^2\right). \tag{128}
$$

We set

$$
\begin{aligned}
Y(t)&=\|w'(t)\|_2^2+\|q'(t)\|_2^2+\|\nabla w(t)\|_2^2+\|\nabla q(t)\|_2^2 \\
&\quad+\int_0^1\left(\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|\tilde{w}(t)\|_2^2\right)dk_1+\int_0^1\left(\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|\tilde{q}(t)\|_2^2\right)dk_2,
\end{aligned}
$$

then the equality (126) becomes

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}Y(t)+\mu_1\|\omega'(t)\|_2^2+\alpha_1\|q'(t)\|_2^2+\frac{1}{2}\|\tilde{q}(x,1,t)\|_2^2+\frac{1}{2}\|\tilde{w}(x,1,t)\|_2^2 \\
&\le\|\tilde{w}(t)\|_2\|\nabla w'(t)\|_2+\|\tilde{q}(t)\|_2\|\nabla q'(t)\|_2+\frac{1}{2}\|w'(t)\|_2^2 \\
&+\|q'(t)\|_2^2+\|\nabla w(t)\|_2^2+\|\nabla q(t)\|_2^2,
\end{aligned}
\tag{129}
$$

then

$$
\frac{1}{2}\frac{d}{dt}Y(t)\le Y(t). \tag{130}
$$

Integrating the last equality and using the Gronwall's Lemma we get

$$
\begin{aligned}
&\|w'(t)\|_2^2+\|q'(t)\|_2^2+a_{01}l_1\|\nabla w(t)\|_2^2+a_{02}l_2\|\nabla q(t)\|_2^2 \\
&+\int_0^1\left(\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_1}\right)\|\tilde{w}(t)\|_2^2\right)dk_1+\int_0^1\left(\left(\frac{\tau_2(t)}{1-\tau_2'(t)k_2}\right)\|\tilde{q}(t)\|_2^2\right)dk_2=0.
\end{aligned}
$$

This completes our proof of existence and uniqueness of the weak solution.
Remark 2 By virtue of the theory of ordinary differential equations, the system (19)-(24) has a local solution which is extended to a maximal interval $[0,T_k[$ with ($0<T_k\le+\infty$).

Now we will prove that the solution obtained above is global and bounded in time, for this purpose, we define

$$
\begin{aligned}
I(t) &= \xi_1(t)e^{-k_1\tau_2(t)}\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx + (g_1\circ u)(t) + (g_2\circ \upsilon)(t)\\
&+ \xi_2(t)e^{-k\tau_2(t)}\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx + (p+1)\int_\Omega F(u,\upsilon)dx \qquad (131)\\
&+ \left(1-\int_0^t g_1(s)ds\right)a_1(u(t),u(t)) + \left(1-\int_0^t g_2(s)ds\right)a_2(\upsilon(t),\upsilon(t)),
\end{aligned}
$$

and

$$
\begin{aligned}
J(t) &= \frac{\xi_1(t)e^{-k_1\tau_2(t)}}{2}\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx + \frac{1}{2}(g_1\circ u)(t) + \frac{1}{2}(g_2\circ \upsilon)(t)\\
&+ \frac{\xi_2(t)e^{-k_2\tau_2(t)}}{2}\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx + \int_\Omega F(u,\upsilon)dx \qquad (132)\\
&+ \frac{1}{2}\left(1-\int_0^t g_1(s)ds\right)a_1(u(t),u(t)) + \frac{1}{2}\left(1-\int_0^t g_2(s)ds\right)a_2(\upsilon(t),\upsilon(t)).
\end{aligned}
$$

Remark 3 From the definition of $E(t)$ and taking into account (132), we observe that

$$E(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \|\upsilon_t(t)\|_2^2\right) + J(t). \qquad (133)$$

Definition 1 Let $(u_0,\upsilon_0)\in(H_0^1(\Omega))^2$, $(u_1,\upsilon_1)\in(L^2(\Omega))^2$ and $(\phi_0,\phi_1)\in(L^2(\Omega\times(0,1)))^2$ be given. We denote by $((u,z_1),(\upsilon,z_2))$ the solution to the problem (18). We define

$$T^* = \sup\Big\{T>0, ((u,z_1),(\upsilon,z_2)) \quad \text{exists} \quad \text{on } [0,T]\Big\}.$$

If $T^*=\infty$, we say that the solution of (18) is global.
Lemma 3 Let $((u,z_1),(\upsilon,z_2))$, be the solution of problem (18). Assume further that $I(0)>0$ and

$$\alpha = \rho\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}} < 1. \qquad (134)$$

Then $I(t)>0$, $\forall\ t$.
Proof. Since $I(0)>0$, then there exists (by continuity of $u(t)$), a time $t_1>0$ such that

$$I(t)\geq 0, \quad \forall\ t\in(0,t_1). \qquad (135)$$

Let

$$\mathfrak{F} = \Big\{I(t_0)=0 \quad \text{and} \quad I(t)>0,\ 0\leq t<t_0\Big\}.$$

From (33) and (34), we have for $\forall\ t \in [0, t_0]$,

$$\begin{aligned} J(t) &\geq \frac{p-1}{2(p+1)}\left[l_1 a_1(u(t),u(t)) + l_2 a_2(\upsilon(t),u(t))\right] \\ &+ \frac{p-1}{2(p+1)}\left[\xi_1(t)e^{-k_1\tau_2(t)}\int_0^1\int_\Omega z_1^2(x,k_1,t)dk_1dx\right] \\ &+ \frac{p-1}{2(p+1)}\left[\xi_2(t)e^{-k_2\tau_2(t)}\int_0^1\int_\Omega z_2^2(x,k_2,t)dk_2dx\right] \\ &+ \frac{p-1}{2(p+1)}\left[(g_1\circ u)(t) + (g_2\circ \upsilon)(t)\right] + \frac{1}{p+1}I(t) \\ &\geq \frac{p-1}{2(p+1)}\left[l_1 a_1(u(t),u(t)) + l_2 a_2(\upsilon(t),\upsilon(t))\right]. \end{aligned} \tag{136}$$

Thus by (133) and (136) and the fact that $(g_1 \circ u)(t) + (g_2 \circ \upsilon)(t) > 0$, we deduce

$$\begin{aligned} l_1 a_1(u(t),u(t)) + l_2 a_2(\upsilon(t),\upsilon(t)) &\leq \frac{2(p+1)}{p-1}J(t) \leq \frac{2(p+1)}{(p-1)}E(t) \\ &\leq \frac{2(p+1)}{(p-1)}E(0),\ \forall\ t\in[0,t_0]. \end{aligned} \tag{137}$$

Employing Lemma 1, we obtain

$$\begin{aligned} &(p+1)\int_\Omega F(u(t_0),\upsilon(t_0))dx \leq \rho(l_1\|\nabla u(t_0)\|_2^2 + l_2\|\nabla\upsilon(t_0)\|_2^2)^{\frac{p+1}{2}} \\ &\leq \rho\left(\frac{2(p+1)}{p-1}\right)^{\frac{p-1}{2}}(l_1\|\nabla u(t_0)\|_2^2 + l_2\|\nabla\upsilon(t_0)\|_2^2) \\ &= \alpha(l_1\|\nabla u(t_0)\|_2^2 + l_2\|\nabla\upsilon(t_0)\|_2^2) \\ &< (l_1\|\nabla u(t_0)\|_2^2 + l_2\|\nabla\upsilon(t_0)\|_2^2) \\ &\leq \frac{l_1}{a_{01}}a_1(u(t_0),u(t_0)) + \frac{l_2}{a_{02}}a_2(\upsilon(t_0),\upsilon(t_0)). \end{aligned} \tag{138}$$

By exploiting Lemma 2, we conclude from (138) that $I(t) > 0$ on $[0, t_0]$, which contradicts that $I(t) > 0$ on $[0, T]$, which completes the proof.

Theorem 2 Let$(u_0, \upsilon_0) \in (H_0^1(\Omega))^2$, $(u_1, \upsilon_1) \in (L^2(\Omega))^2$ and $(\phi_0, \phi_1) \in (L^2(\Omega \times (0, 1)))^2$ be given. Suppose that (134) and $I(0) > 0$ are fulfilled. Then the solution of (18) is global and bounded.

Proof. To prove Theorem 2, using the definition of T^*, we have to verify that

$$a_1(u(t),u(t)) + a_2(\upsilon(t),\upsilon(t)),$$

is uniformly bounded in time. To do this, we use (133) to get

$$\begin{aligned} &E(0) \geq E(t) = J(t) + \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\upsilon_t(t)\|^2 \\ &\geq \left(\frac{2(p+1)}{p-1}\right)\left[a_1(u(t),u(t)) + a_2(\upsilon(t),\upsilon(t))\right] + \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\upsilon_t(t)\|_2^2. \end{aligned} \tag{139}$$

Therefore

$$a_1(u(t),u(t)) + a_2(\upsilon(t),\upsilon(t)) \leq CE(0),$$

where C is a positive constant, which depends only on p. Thus, we obtain the global existence result.

4. Asymptotic stability

In this section, we will prove the asymptotic stability result by constructing a suitable Lyapunov functional. Now we define the following functional

$$L(t) = ME(t) + \epsilon\psi(t) + \varphi(t) + \epsilon I(t), \tag{140}$$

$$\psi(t) = \int_\Omega u u_t dx + \int_\Omega \upsilon \upsilon_t dx, \tag{141}$$

$$\begin{aligned} \varphi(t) = &- \int_\Omega u_t \int_0^t g_1(t-s)(u(t)-u(s))dsdx \\ &- \int_\Omega \upsilon_t \int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx, \end{aligned} \tag{142}$$

$$I(t) = \int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)} u_t^2(x,s)dxds + \int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)} \upsilon_t^2(x,s)dxds. \tag{143}$$

Remark 4 We can easily see that

$$\int_{t-\tau_2(t)}^t \int_\Omega e^{\lambda(s-t)} u_t^2(x,s)dxds = \int_0^1 \int_\Omega e^{-k_1\tau_2(t)} z_1^2(x,k_1,t)dxdk_1,$$

after using a change of variables, $t - \tau_2(t) = s$. We use the same way for the second term in (143).

In order to show our stability result, we need the following Lemmas.

Lemma 4([24]) Let $((u, z_1), (\upsilon, z_2))$ be the solution of problem (18) and assume that (134) holds. Then, for $\gamma \geq 0$, we have

$$\begin{aligned} \int_\Omega \left(\int_0^t g_1(t-s)(u(t)-u(s))ds \right)^{\gamma+1} dx \leq& \left(\frac{4(p+1)E(0)}{l_1(p-1)} \right)^{\frac{\gamma}{2}} \\ &\times (1-l_1)^{\gamma+1} c_s^{\gamma+1} (g_1\ o\ u)(t), \end{aligned} \tag{144}$$

and

$$\begin{aligned} \int_\Omega \left(\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))ds \right)^{\gamma+1} dx \leq& \left(\frac{4(p+1)E(0)}{l_2(p-1)} \right)^{\frac{\gamma}{2}} \\ &\times (1-l_2)^{\gamma+1} c_s^{\gamma+1} (g_2\ o\ \upsilon)(t). \end{aligned} \tag{145}$$

Lemma 5([24]) Suppose that $(A_0)-(A_5)$ hold. Let $(u_0, \upsilon_0) \in (H_0^1(\Omega))^2$, $(u_1, \upsilon_1) \in (L^2(\Omega))^2$ be given and satisfying (134). Then there exist two positive constants η_1 and η_2 such that for any $\delta > 0$ and for all $t \geq 0$, we have

$$\begin{aligned} \int_\Omega f_1(u,\upsilon) \int_0^t g_1(t-s)(u(t)-u(s))dsdx \leq& \eta_1\delta(l_1\|\nabla u\|_2^2 + l_2\|\nabla\upsilon\|_2^2) \\ &+ \frac{(1-l_1)c_s^2}{4\delta}(g_2\ o\ u)(t), \end{aligned} \tag{146}$$

and

$$\begin{aligned} \int_\Omega f_2(u,\upsilon) \int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx \leq& \eta_2\delta(l_1\|\nabla u\|_2^2 + l_2\|\nabla\upsilon\|_2^2) \\ &+ \frac{(1-l_2)c_s^2}{4\delta}(g_2\ o\ \upsilon)(t). \end{aligned} \tag{147}$$

Lemma 6 There exists two positive constants λ_1, λ_2 depending on ϵ and M such that for all $t > 0$ and M sufficiently large, we have

$$\lambda_1 E(t) \leq L(t) \leq \lambda_2 E(t). \tag{148}$$

Proof. Thank's to the Holder and Young's inequalities, we have

$$|\psi(t)| \leq \omega \|u\|_2^2 + \frac{1}{4\omega}\|u_t\|_2^2 + \omega\|\upsilon\|_2^2 + \frac{1}{4\omega}\|\upsilon_t\|_2^2, \tag{149}$$

and

$$\begin{aligned}
&\varphi(t) = | - \int_\Omega u_t \int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&- \int_\Omega \upsilon_t \int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx| \\
&\leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)^2 dx \\
&+ \frac{1}{2}\|\upsilon_t\|_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))ds\right)^2 dx \\
&\leq \frac{1}{2}\left(\|u_t\|_2^2 + (1-l_1)c_s^2 \int_0^t g_1(t-s)a_1(u(t)-u(s), u(t)-u(s))ds\right) \\
&+ \frac{1}{2}\left(\|\upsilon_t\|_2^2 + (1-l_2)c_s^2 \int_0^t g_2(t-s)a_2(\upsilon(t)-\upsilon(s), \upsilon(t)-\upsilon(s))ds\right) \\
&\leq \frac{1}{2}\left(\|u_t\|_2^2 + (1-l_1)c_s^2 \left(\frac{2\beta E(0)}{l_1}\right)(g_1 o u)(t)\right) \\
&+ \frac{1}{2}\left(\|\upsilon_t\|_2^2 + (1-l_2)c_s^2 \left(\frac{2\beta E(0)}{l_2}\right)(g_2 o \upsilon)(t)\right).
\end{aligned} \tag{150}$$

It follows from (143) that forall $c > 0$, we have

$$\begin{aligned}
|I(t)| &\leq \left|\xi_1(t)\int_\Omega\int_0^1 e^{-k_1\tau_2(t)} z_1^2(x,k_1,t)dk_1dx\right| \\
&+ \left|\xi_2(t)\int_\Omega\int_0^1 e^{-k_2\tau_2(t)} z_2^2(x,k_2,t)dk_2dx\right| \\
&\leq c\xi_1(t)e^{-k_1\tau_2(t)}\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx \\
&+ c\xi_2(t)e^{-k_2\tau_2(t)}\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx.
\end{aligned} \tag{151}$$

Hence, combining (149)-(151). This yields

$$
\begin{aligned}
&|L(t)-ME(t)| = \epsilon\psi(t)+\varphi(t)+\epsilon I(t) \leq \\
&\epsilon\omega c_s^2 a_1(u(t),u(t)) + \epsilon\omega c_s^2 a_2(\upsilon(t),\upsilon(t)) \\
&+\frac{(1-l_1)c_s^2}{2}\left(\frac{2\beta E(0)}{l_1}\right)(g_1 o u)(t) + \frac{(1-l_2)c_s^2}{2}\left(\frac{2\beta E(0)}{l_2}\right)(g_2 o \upsilon)(t) \\
&+c\xi_1(t)e^{-k_1\tau_2(t)}\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx + \left(\frac{\epsilon}{4\omega}+\frac{1}{2}\right)\|u_t\|_2^2 \\
&+c\xi_2(t)e^{-k_2\tau_2(t)}\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx + \left(\frac{\epsilon}{4\omega}+\frac{1}{2}\right)\|\upsilon_t\|_2^2.
\end{aligned} \tag{152}
$$

Where

$$
c_1 = \epsilon\omega c_s^2, \quad c_2 = \epsilon\omega c_s^2, \quad c_3 = \left(\frac{\epsilon}{4\omega}+\frac{1}{2}\right), \quad c_4 = \left(\frac{\epsilon}{4\omega}+\frac{1}{2}\right),
$$

$$
c_5 = \frac{(1-l_1)c_s^2}{2}\left(\frac{2\beta E(0)}{l_1}\right), \quad c_6 = \frac{(1-l_2)c_s^2}{2}\left(\frac{2\beta E(0)}{l_2}\right), \quad c_7 = c_8 = c.
$$

Finally we obtain

$$
|L(t)-ME(t)| \leq c_9 E(t), \tag{153}
$$

where $c_9 = max(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$. Thus, from the definition of $E(t)$ and selecting M sufficiently large, we arrive easily at

$$
\lambda_2 E(t) \leq L(t) \leq \lambda_1 E(t), \tag{154}
$$

where $\lambda_1 = (M - c_9)$, $\lambda_2 = (M + c_9)$. This completes the proof.
Lemma 7 The functional defined in(143) satisfies

$$
\begin{aligned}
&\frac{dI(t)}{dt} \leq \frac{\xi_1(t)}{2\tau_0}\|u_t\|_2^2 - \xi_1(t)\frac{c}{2\tau_1}\int_\Omega z_1^2(x,1,t)dx \\
&-\xi_1(t)k_1e^{-\tau_2(t)k_1}\tau_0\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx + \frac{\xi_2(t)}{2\tau_0}\|\upsilon_t\|_2^2 \\
&-\xi_2(t)\frac{c}{2\tau_1}\int_\Omega z_2^2(x,1,t)dx - \xi_2(t)k_2e^{-\tau_2(t)k_2}\tau_0\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx.
\end{aligned} \tag{155}
$$

Where τ_0 and τ_1 are positive constants.
Proof. Taking derivative of (155) and using the same technique as (35) produces

$$\begin{aligned}
\frac{dI(t)}{dt} &= \frac{d}{dt}\left[\xi_1(t)e^{-k_1\tau_2(t)}\int_\Omega\int_0^1 z_1^2(x,k_1,t))dk_1dx\right] \\
&+ \frac{d}{dt}\left[\xi_2(t)e^{-k_2\tau_2(t)}\int_\Omega\int_0^1 z_2^2(x,k_2,t))dk_2dx\right] \\
&= \xi_1'(t)e^{-\tau_2(t)k_1}\int_\Omega\int_0^1 z_1^2(x,k_1,t))dkdx \\
&- \xi_1(t)k_1e^{-\tau_2(t)k_1}\tau_2'(t)\int_\Omega\int_0^1 z_2^2(x,k_1,t))dk_1dx \\
&+ \xi_2'(t)e^{-\tau_2(t)k_2}\int_\Omega\int_0^1 z_2^2(x,k_2,t))dk_2dx \\
&- \xi_2(t)k_2e^{-\tau_2(t)k_2}\tau_2'(t)\int_\Omega\int_0^1 z_2^2(x,k_2,t))dk_2dx \\
&+ \frac{1}{\tau_2(t)}e^{-\tau_2(t)k_1}\tau_2(t)\xi_1(t)\int_\Omega\int_0^1 \frac{d}{dt}z_1^2(x,k_1,t))dk_1dx \\
&+ \frac{1}{\tau_2(t)}e^{-\tau_2(t)k_2}\tau_2(t)\xi_2(t)\int_\Omega\int_0^1 \frac{d}{dt}z_2^2(x,k_2,t))dk_2dx \\
&= \xi_1'(t)e^{-\tau_2(t)k_1}\int_\Omega\int_0^1 z_1^2(x,k_1,t))dk_1dx \\
&- \xi_1(t)k_1e^{-\tau_2(t)k_1}\tau_2'(t)\int_\Omega\int_0^1 z_1^2(x,k_1,t))dk_1dx \\
&+ \xi_2'(t)e^{-\tau_2(t)k_2}\int_\Omega\int_0^1 z_2^2(x,k_2,t))dk_2dx \\
&- \xi_2(t)k_2e^{-\tau_2(t)k_2}\tau_2'(t)\int_\Omega\int_0^1 z_2^2(x,k_2,t))dk_2dx \\
&+ \frac{1}{\tau_2(t)}e^{-\tau_2(t)k_1}\xi_1(t)\int_\Omega\int_0^1 \frac{\partial}{\partial k_1}(1-\tau_2'(t)k_1)z_1^2(x,k_1,t))dk_1dx \\
&+ \frac{1}{\tau_2(t)}e^{-\tau_2(t)k_2}\xi_2(t)\int_\Omega\int_0^1 \frac{\partial}{\partial k_2}(1-\tau_2'(t)k_2)z_2^2(x,k_2,t))dk_2dx \\
&\le -\xi_1(t)k_1e^{-\tau_2(t)k_1}\tau_2'(t)\int_\Omega\int_0^1 z_1^2(x,k_1,t))dk_1dx \\
&- \xi_2(t)k_1e^{-\tau_2(t)k_1}\tau_2'(t)\int_\Omega\int_0^1 z_2^2(x,k_1,t))dk_1dx \\
&+ \frac{1}{\tau_2(t)}\xi_1(t)\int_\Omega[z_1^2(x,0,t))-z_1^2(x,1,t)]dx + \xi_1(t)\frac{\beta}{\tau_2(t)}\int_\Omega z_1^2(x,1,t)dx \\
&+ \frac{1}{\tau_2(t)}\xi_2(t)\int_\Omega[z_2^2(x,0,t))-z_2^2(x,1,t)]dx + \xi_1(t)\frac{\beta}{\tau_2(t)}\int_\Omega z_1^2(x,1,t)dx \\
&\le \frac{\xi_1(t)}{2\tau_0}\|u_t\|_2^2 - \xi_1(t)\frac{c}{2\tau_1}\int_\Omega z_1^2(x,1,t)dx \\
&- \xi_1(t)k_1e^{-\tau_2(t)k_1}\tau_0\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx + \frac{\xi_2(t)}{2\tau_0}\|\upsilon_t\|_2^2 \\
&- \xi_2(t)\frac{c}{2\tau_1}\int_\Omega z_2^2(x,1,t)dx - \xi_2(t)k_2e^{-\tau_2(t)k_2}\tau_0\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx.
\end{aligned} \tag{156}$$

Lemma 8 The functional defined in (141) satisfies

$$\begin{aligned}\frac{d\psi(t)}{dt} &\leq \left(1+\frac{\mu_1}{4\beta}\right)\|u_t\|_2^2 + \left(1+\frac{\alpha_1}{4\beta}\right)\|\upsilon_t\|_2^2 - (p+1)\int_\Omega F(u(t),\upsilon(t))dx \\ &+ \left[\mu_1\left(1+\frac{2\beta c_s^2}{a_{01}}\right) - l_1\right] a_1(u(t),u(t)) + \frac{N}{4a_{02}\mu}(1-l_2)(g_2\circ\upsilon)(t) \\ &+ \left[\alpha_2\left(1+\frac{2\beta c_s^2}{a_{01}}\right) - l_2\right] a_2(\upsilon(t),\upsilon(t)) + \frac{\mu_1}{4\beta}\|z_1(x,1,t)\|_2^2 \\ &+ \frac{\alpha_2}{4\beta}\|z_2(x,1,t)\|_2^2 + \frac{N}{4a_{01}\mu}(1-l_1)(g_1\circ u)(t).\end{aligned} \tag{157}$$

Proof. Taking the derivative of (141) and using the system (18), we get

$$\begin{aligned}\frac{d\psi(t)}{dt} &= \int_\Omega u_{tt}udx + \int_\Omega u_{tt}udx + \|u_t\|_2^2 + \|\upsilon_t\|_2^2 \\ &= \|u_t\|_2^2 + \|\upsilon_t\|_2^2 - a_1(u(t),u(t)) - a_2(\upsilon(t),\upsilon(t)) \\ &+ \int_\Omega\int_0^t g_1(t-s)A_1\nabla u(s)\nabla u(t)dsdx \\ &+ \int_\Omega\int_0^t g_2A_2(t-s)\nabla\upsilon(s)\nabla\upsilon(t)dsdx - \mu_2\int_\Omega z_1(x,1,t)udx - \mu_1\int_\Omega u_tudx \\ &- \alpha_2\int_\Omega z_2(x,1,t)\upsilon dx - \alpha_1\int_\Omega \upsilon_t\upsilon dx - (p+1)\int_\Omega F(u(t),\upsilon(t))dx,\end{aligned} \tag{158}$$

following [31]], yields

$$\begin{aligned}&\int_\Omega A_1\int_0^t g_1(t-s)(\nabla u(t)\nabla u(s)dsdx \\ &= \sum_{i,j=1}^N \int_0^t g_1(t-s)\int_\Omega a_{1ij}(x)\frac{\partial u(t)}{\partial x_j}\left(\frac{\partial u(s)}{\partial x_i} - \frac{\partial u(t)}{\partial x_i} + \frac{\partial u(t)}{\partial x_i}\right)dxds \\ &= \sum_{i,j=1}^N \int_\Omega\int_0^t a_{1ij}(x)\frac{\partial u(t)}{\partial x_j}\frac{\partial u(t)}{\partial x_i}dsdx \\ &+ \sum_{i,j=1}^N \int_\Omega\int_0^t \left(g_1(t-s)a_{1ij}(x)\frac{\partial u(t)}{\partial x_j}\left(\frac{\partial u(s)}{\partial x_i} - \frac{\partial u(t)}{\partial x_i}\right)\right)dsdx \\ &\leq (1-l_1)a_1(u(t),u(t)) + \mu\sum_{i,j=1}^N\int_\Omega\left(a_{1ij}(x)\frac{\partial u(s)}{\partial x_j}ds\right)^2dx \\ &+ \frac{1}{\mu}\sum_{i,j=1}^N\int_\Omega\left(\int_0^t g_1(t-s)\left(\frac{\partial u(s)}{\partial x_i} - \frac{\partial u(t)}{\partial x_i}\right)ds\right)^2dx \\ &\leq \left[(1-l_1) + \frac{\mu a_{11}}{a_{01}}\right]a_1(u(t),u(t)) + \frac{N}{4a_{01}\mu}(1-l)(g_1ou)(t),\end{aligned} \tag{159}$$

in the same way

$$\int_\Omega A_2 \int_0^t g_2(t-s)(\nabla \upsilon(t)\nabla \upsilon(s)dsdx \leq \left[(1-l_2)+\frac{\mu a_{22}}{a_{02}}\right] a_{22}(u(t),u(t)) + \frac{N}{4a_{02}\mu}(1-l_2)(g_2 o \upsilon)(t), \tag{160}$$

for the seventh, eight, ninth and tenth term in (158), we use Holder and Young's inequalities to get

$$\left|\int_\Omega u_t u dx\right| \leq \frac{\beta c_s^2}{a_{01}} a_1(u(t),u(t)) + \frac{1}{4\beta}\|u_t\|_2^2, \tag{161}$$

$$\left|\int_\Omega \upsilon_t \upsilon dx\right| \leq \frac{\beta c_s^2}{a_{02}} a_2(\upsilon(t),\upsilon(t)) + \frac{1}{4\beta}\|\upsilon_t\|_2^2, \tag{162}$$

$$\left|\int_\Omega z_1(x,1,t) u dx\right| \leq \frac{\beta c_s^2}{a_{01}} a_1(u(t),u(t)) + \frac{1}{4\beta}\int_\Omega z_1^2(x,1,t)dx, \tag{163}$$

$$\left|\int_\Omega z_2(x,1,t) \upsilon dx\right| \leq \frac{\beta c_s^2}{a_{02}} a_1(\upsilon(t),\upsilon(t)) + \frac{1}{4\beta}\int_\Omega z_2^2(x,1,t)dx. \tag{164}$$

Inserting (159)-(164), we get finally

$$\begin{aligned}\frac{d\psi(t)}{dt} &\leq \left(1+\frac{\mu_1}{4\beta}\right)\|u_t\|_2^2 + \left(1+\frac{\alpha_1}{4\beta}\right)\|\upsilon_t\|_2^2 - (p+1)\int_\Omega F(u(t),\upsilon(t))dx \\ &+ \left[\mu_1\left(1+\frac{2\beta c_s^2}{a_{01}}\right)-l_1\right] a_1(u(t),u(t)) + \frac{N}{4a_{02}\mu}(1-l_2)(g_2\circ\upsilon)(t) \\ &+ \left[\alpha_2\left(1+\frac{2\beta c_s^2}{a_{01}}\right)-l_2\right] a_2(\upsilon(t),\upsilon(t)) + \frac{\mu_1}{4\beta}\|z_1(x,1,t)\|_2^2 \\ &+ \frac{\alpha_2}{4\beta}\|z_2(x,1,t)\|_2^2 + \frac{N}{4a_{01}\mu}(1-l_1)(g_1\circ u)(t).\end{aligned} \tag{165}$$

Lemma 9 The functional defined in (142) satisfies

$$\begin{aligned}\frac{d\varphi(t)}{dt} &\leq (\beta+\mu_1-g_{10})\|u_t\|_2^2 + (\beta+\alpha_1-g_{20})\|\upsilon_t\|_2^2 \\ &+ \left\{\frac{\beta}{a_{01}} + \frac{a_{11}\beta}{a_{01}}(1-l_1^2) + \left(\frac{2\lambda\delta}{a_{01}} - \frac{2a_{11}\beta}{a_{01}}\right)\right\} a_1(u(t),u(t)) \\ &+ \left\{\frac{\beta}{a_{02}} + \frac{a_{22}\beta}{a_{02}}(1-l_2^2) + \left(\frac{2\lambda\delta}{a_{02}} - \frac{2a_{22}\beta}{a_{02}}\right)\right\} a_2(\upsilon(t),\upsilon(t)) \\ &+ (1-l_1)\left\{\left\{\frac{1}{a_{01}\beta}\left(\frac{1}{4}+2\beta a_{11}+\frac{N}{4}\right)+\frac{c_s^2}{4\delta}\right\} + (1-l_1)c_s^2(\mu_1+\mu_2)\right\} \\ &\times (g_1\circ u)(t) + \mu_2\|z_1(x,1,t)\|_2^2 + \alpha_2\|z_2(x,1,t)\|_2^2 \\ &+ (1-l_2)\left\{\left\{\frac{1}{a_{02}\beta}\left(\frac{1}{4}+2\beta a_{22}+\frac{N}{4}\right)+\frac{c_s^2}{4\delta}\right\} + (1-l_2)c_s^2(\alpha_1+\alpha_2)\right\} \\ &\times (g_2\circ\upsilon)(t) + \frac{g_1(0)c_s^2}{4\beta}(-g_1'\circ u)(t) + \frac{g_2(0)c_s^2}{4\beta}(-g_2'\circ\upsilon)(t).\end{aligned} \tag{166}$$

Proof. Taking the derivative of (142) and using (18), we obtain

$$\begin{aligned}
\tfrac{d\varphi(t)}{dt} &= -\int_\Omega u_{tt}\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&-\int_\Omega u_t\int_0^t g_1'(t-s)(u(t)-u(s))dsdx-\left(\int_0^t g_1(s)ds\right)\int_\Omega u_t^2dx \\
&-\int_\Omega \upsilon_{tt}\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx \\
&-\int_\Omega \upsilon_t\int_0^t g_2'(t-s)(\upsilon(t)-\upsilon(s))dsdx-\left(\int_0^t g_2(s)ds\right)\int_\Omega \upsilon_t^2dx \\
&=\sum_{i,j=1}^N\int_\Omega a_{1ij}(x)\frac{\partial u(t)}{\partial x_j}\left(\int_0^t g_1(t-s)\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u(s)}{\partial x_i}\right)ds\right)dx \\
&+\sum_{i,j=1}^N\int_\Omega a_{2ij}(x)\frac{\partial \upsilon(t)}{\partial x_j}\left(\int_0^t g_2(t-s)\left(\frac{\partial \upsilon(t)}{\partial x_i}-\frac{\partial \upsilon(s)}{\partial x_i}\right)ds\right)dx \\
&-\textstyle\sum_{i,j=1}^N\int_\Omega\left(\int_0^t g_1(t-s)\frac{\partial u(s)}{\partial x_i}ds\right)\left(\int_0^t g_1(t-s)\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u(s)}{\partial x_i}\right)ds\right)dx \\
&-\textstyle\sum_{i,j=1}^N\int_\Omega\left(\int_0^t g_2(t-s)\frac{\partial \upsilon(s)}{\partial x_i}ds\right)\left(\int_0^t g_2(t-s)\left(\frac{\partial \upsilon(t)}{\partial x_i}-\frac{\partial \upsilon(s)}{\partial x_i}\right)ds\right)dx \\
&-\int_\Omega f_1(u(t),\upsilon(t))\left(\int_0^t g_1(t-s)u(t)-u(s)ds\right)dx \\
&-\int_\Omega f_2(u(t),\upsilon(t))\left(\int_0^t g_2(t-s)\upsilon(t)-\upsilon(s)ds\right)dx \\
&+\int_\Omega \mu_1u(t)\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&+\int_\Omega \alpha_1\upsilon(t)\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx \\
&+\int_\Omega \mu_2z_1(x,1,t)\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&+\int_\Omega \alpha_2z_2(x,1,t)\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx \\
&-\int_\Omega u_t\int_0^t g_1'(t-s)(u(t)-u(s))dsdx-\left(\int_0^t g_1(s)ds\right)\int_\Omega u_t^2dx \\
&-\int_\Omega \upsilon_t\int_0^t g_2'(t-s)(\upsilon(t)-\upsilon(s))dsdx-\left(\int_0^t g_2(s)ds\right)\int_\Omega \upsilon_t^2dx.
\end{aligned} \tag{167}$$

Using Young's inequality and the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we infer

$$\begin{aligned}
&\sum_{i,j=1}^{N} \int_\Omega a_{1ij}(x) \frac{\partial u(t)}{\partial x_j} \left(\int_0^t g_1(t-s) \left(\frac{\partial u(t)}{\partial x_i} - \frac{\partial u(s)}{\partial x_i} \right) ds \right) dx \\
&+ \sum_{i,j=1}^{N} \int_\Omega a_{2ij}(x) \frac{\partial \upsilon(t)}{\partial x_j} \left(\int_0^t g_2(t-s) \left(\frac{\partial \upsilon(t)}{\partial x_i} - \frac{\partial \upsilon(s)}{\partial x_i} \right) ds \right) dx \\
&\leq \frac{\beta}{a_{01}} a_1(u(t), u(t)) + \frac{\beta}{a_{02}} a_2(\upsilon(t), \upsilon(t)) + \frac{(1-l_1)}{4a_{01}\beta} (g_1 o u)(t) \\
&+ \frac{(1-l_2)}{4a_{02}\beta} (g_2 o \upsilon)(t),
\end{aligned} \tag{168}$$

and

$$\begin{aligned}
&\left| \sum_{i,j=1}^{N} \int_\Omega \left(\int_0^t g_1(t-s) \frac{\partial u(s)}{\partial x_i} ds \right) \left(\int_0^t g_1(t-s) \left(\frac{\partial u(t)}{\partial x_i} - \frac{\partial u(s)}{\partial x_i} \right) ds \right) dx \right| \\
&+ \left| \sum_{i,j=1}^{N} \int_\Omega \left(\int_0^t g_2(t-s) \frac{\partial \upsilon(s)}{\partial x_i} ds \right) \left(\int_0^t g_2(t-s) \left(\frac{\partial \upsilon(t)}{\partial x_i} - \frac{\partial \upsilon(s)}{\partial x_i} \right) ds \right) dx \right| \\
&\leq \beta \sum_{i,j=1}^{N} \int_\Omega \left(\int_0^t g_1(t-s) \frac{\partial u(s)}{\partial x_i} ds \right)^2 dx \\
&+ \tfrac{1}{\beta} \sum_{i,j=1}^{N} \int_\Omega \left(\int_0^t g_1(t-s) \left(\frac{\partial u(t)}{\partial x_i} - \frac{\partial u(s)}{\partial x_i} \right) ds \right)^2 dx \\
&+ \beta \sum_{i,j=1}^{N} \int_\Omega \left(\int_0^t g_2(t-s) \frac{\partial \upsilon(s)}{\partial x_i} ds \right)^2 dx \\
&+ \tfrac{1}{\beta} \sum_{i,j=1}^{N} \int_\Omega \left(\int_0^t g_2(t-s) \left(\frac{\partial \upsilon(t)}{\partial x_i} - \frac{\partial \upsilon(s)}{\partial x_i} ds \right) \right)^2 dx \\
&\leq \tfrac{a_{11}\beta}{a_{01}} (1-l_1)^2 a_1(u(t), u(t)) + \tfrac{a_{22}\beta}{a_{01}} (1-l_2)^2 a_2(\upsilon(t), \upsilon(t)) \\
&+ \tfrac{(1-l_1)}{a_{01}} \left[2\beta a_{11} + \tfrac{N}{4\beta} \right] (g_1 \circ u)(t) + \tfrac{(1-l_2)}{a_{02}} \left[2\beta a_{22} + \tfrac{N}{4\beta} \right] (g_2 \circ \upsilon)(t),
\end{aligned} \tag{169}$$

from the Lemma 5, we deduce

$$\begin{aligned}
&\int_\Omega f_1(u, \upsilon) \int_0^t g_1(t-s)(u(t) - u(s)) ds dx \leq \lambda\delta \left(l_1 \|\nabla u\|_2^2 + l_2 \|\nabla \upsilon\|_2^2 \right) \\
&+ \frac{(1-l_1)c_s^2}{4\delta} (g_2 o u)(t) \leq \frac{\lambda\delta l_1}{a_{01}} a_1(u(t), u(t)) + \frac{\lambda\delta l_2}{a_{02}} a_2(\upsilon(t), \upsilon(t)) \\
&+ \frac{(1-l_1)c_s^2}{4\delta} (g_1 o u)(t),
\end{aligned} \tag{170}$$

also

$$\begin{aligned}&\int_\Omega f_2(u,\upsilon)\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx \le \lambda\delta\left(l_1\|\nabla u\|_2^2 + l_2\|\nabla \upsilon\|_2^2\right)\\ &+\frac{(1-l_2)c_s^2}{4\delta}(g_2o\upsilon)(t) \le \frac{\lambda\delta l_2}{a_{02}}a_2(\upsilon(t),\upsilon(t)) + \frac{\lambda\delta l_1}{a_{01}}a_1(u(t),u(t))\\ &+\frac{(1-l_2)c_s^2}{4\delta}(g_2o\upsilon)(t).\end{aligned} \tag{171}$$

Since $g_1,\ g_2$ are positive, continuous and $g_1(0)>0$, $g_2(0)>0$ for any t_0, we have

$$\int_0^t g_1(s)ds \ge \int_0^{t_0} g_1(s)ds = g_{10}, \quad \forall t\ge t_0, \tag{172}$$

$$\int_0^t g_2(s)ds \ge \int_0^{t_0} g_2(s)ds = g_{20}, \quad \forall t\ge t_0, \tag{173}$$

then we use (172) and (173) to get

$$\begin{aligned}&\int_\Omega u_t\int_0^t g_1'(t-s)(u(t)-u(s))dsdx - \left(\int_0^t g_1(s)ds\right)\int_\Omega u_t^2dx\\ &\le \beta\|u_t\|_2^2 + \frac{g_1(0)c_s^2}{4\beta}(-g_1'ou)(t) - g_{10}\|u_t\|_2^2,\end{aligned} \tag{174}$$

$$\begin{aligned}&\int_\Omega \upsilon_t\int_0^t g_2'(t-s)(\upsilon(t)-\upsilon(s))dsdx - \left(\int_0^t g_2(s)ds\right)\int_\Omega \upsilon_t^2dx\\ &\le \beta\|\upsilon_t\|_2^2 + \frac{g_2(0)c_s^2}{4\beta}(-g_2'o\upsilon)(t) - g_{20}\|\upsilon_t\|_2^2,\end{aligned} \tag{175}$$

from the Lemma 4 with $\gamma=0$, we have, for $\delta>0$,

$$\begin{aligned}&\left|-\int_\Omega \mu_1u_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx\right|\\ &\le \mu_1\|u_t\|_2^2 + \mu_1(1-l_1)^2c_s^2(g_1ou)(t),\end{aligned} \tag{176}$$

$$\begin{aligned}&\left|-\int_\Omega \alpha_2\upsilon_t\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx\right|\\ &\le \alpha_2\|\upsilon_t\|_2^2 + \alpha_2(1-l_2)^2c_s^2(g_2o\upsilon)(t),\end{aligned} \tag{177}$$

$$\begin{aligned}&\left|-\int_\Omega \mu_2z_1(x,1,t)\int_0^t g_1(t-s)(u(t)-u(s))dsdx\right|\\ &\le \mu_2\int_\Omega z_1^2(x,1,t)dx + \mu_2(1-l_1)^2c_s^2(g_1ou)(t),\end{aligned} \tag{178}$$

and

$$\begin{aligned}&\left|-\int_\Omega \alpha_2z_2(x,1,t)\int_0^t g_2(t-s)(\upsilon(t)-\upsilon(s))dsdx\right|\\ &\le \alpha_2\int_\Omega z_2^2(x,1,t)dx + \alpha_2(1-l_1)^2c_s^2(g_2o\upsilon)(t).\end{aligned} \tag{179}$$

A substitution of (174)-(179) into (167) yields

$$\begin{aligned}
&\tfrac{d\varphi(t)}{dt} \leq (\beta+\mu_1-g_{10})\|u_t\|_2^2+(\beta+\alpha_1-g_{20})\|\upsilon_t\|_2^2\\
&+\left\{\tfrac{\beta}{a_{01}}+\tfrac{a_{11}\beta}{a_{01}}(1-l_1^2)+\left(\tfrac{2\lambda\delta}{a_{01}}-\tfrac{2a_{11}\beta}{a_{01}}\right)\right\}a_1(u(t),u(t))\\
&+\left\{\tfrac{\beta}{a_{02}}+\tfrac{a_{22}\beta}{a_{02}}(1-l_2^2)+\left(\tfrac{2\lambda\delta}{a_{02}}-\tfrac{2a_{22}\beta}{a_{02}}\right)\right\}a_2(\upsilon(t),\upsilon(t))\\
&+(1-l_1)\left\{\left\{\tfrac{1}{a_{01}\beta}\left(\tfrac14+2\beta a_{11}+\tfrac{N}{4}\right)+\tfrac{c_s^2}{4\delta}\right\}+(1-l_1)c_s^2(\mu_1+\mu_2)\right\}\\
&\times(g_1\circ u)(t)+\mu_2\|z_1(x,1,t)\|_2^2+\alpha_2\|z_2(x,1,t)\|_2^2\\
&+(1-l_2)\left\{\left\{\tfrac{1}{a_{02}\beta}\left(\tfrac14+2\beta a_{22}+\tfrac{N}{4}\right)+\tfrac{c_s^2}{4\delta}\right\}+(1-l_2)c_s^2(\alpha_1+\alpha_2)\right\}\\
&\times(g_2\circ \upsilon)(t)+\tfrac{g_1(0)c_s^2}{4\beta}(-g_1'\circ u)(t)+\tfrac{g_2(0)c_s^2}{4\beta}(-g_2'\circ \upsilon)(t).
\end{aligned} \tag{180}$$

Theorem 3 Let $(u_0,,\upsilon_0)\in(H_0^1(\Omega)\cap H^2(\Omega))^2,(u_1,\upsilon_1)\in(H_0^1(\Omega))^2$ be given. Assume that $(A_0)-(A_5)$ hold. Then, for each $t_0>0$, there exist strictly positive constants K and κ such that the solution of (18) satisfies

$$E(t)\leq Ke^{-\alpha\int_{t_0}^t\zeta(s)ds},\quad for\quad t\geq t_0. \tag{181}$$

Proof. Taking the derivative of (140) and using the Lemmas 7, 8 and 9, we infer

$$\begin{aligned}
&\frac{dL(t)}{dt}\leq-\left(a_1M-\epsilon\left(1+\frac{\mu_1}{4\beta}-\frac{\xi_1(t)}{2\tau_0}\right)+g_{10}-\mu_1-\beta\right)\|u_t\|_2^2\\
&-\left(a_2M-\epsilon\left(1+\frac{\alpha_1}{4\beta}-\frac{\xi_2(t)}{2\tau_0'}\right)+g_{20}-\alpha_1-\beta\right)\|\upsilon_t\|_2^2\\
&-\left\{\frac{a_{11}\beta}{a_{01}}(1+l_1^2)-\frac{\beta}{a_{01}}-\frac{2\lambda\delta}{a_{01}}-\epsilon\left\{\mu_2\left(1+\frac{2\beta c_s^2}{a_{01}}\right)+l_1\right\}\right\}a_1(u(t),u(t))\\
&-\left\{\frac{a_{22}\beta}{a_{01}}(1+l_2^2)-\frac{\beta}{a_{02}}-\frac{2\lambda\delta}{a_{02}}-\epsilon\left\{\alpha_2\left(1+\frac{2\beta c_s^2}{a_{02}}\right)+l_2\right\}\right\}a_2(\upsilon(t),\upsilon(t))\\
&+(1-l_1)\left\{\frac{1}{a_{01}\beta}\left(\frac14+2\beta a_{11}+\frac{N}{4}+\frac{\epsilon}{4\mu}\right)+\frac{c_s^2}{4\delta}\right\}(g_1\circ u)(t)\\
&+(1-l_1)^2(\mu_1+\mu_2)(g_1\circ u)(t)-(Ma_3+\mu_1)\|z_1(x,1,t)\|_2^2\\
&+(1-l_2)\left\{\frac{1}{a_{02}\beta}\left(\frac14+2\beta a_{22}+\frac{N}{4}+\frac{\epsilon}{4\mu}\right)+\frac{c_s^2}{4\delta}\right\}(g_2\circ \upsilon)(t)\\
&+(1-l_2)^2(\alpha_1+\alpha_2)(g_2\circ \upsilon)(t)+\left(\frac{M}{2}-\frac{g_2(0)c_s^2}{4\beta}\right)(g_2'o\upsilon)(t)\\
&+\left(\epsilon\left(\frac{\mu_2}{4\beta}-\xi_1(t)\frac{c}{2\tau_1}\right)\right)\|z_1(x,1,t)\|_2^2-\epsilon(p+1)\int_\Omega F(u(t),\upsilon(t))dx\\
&-\left((Ma_4+\alpha_1)-\epsilon\left(\frac{\alpha_2}{4\beta}-\xi_2(t)\frac{c}{2\tau_1}\right)\right)\|z_2(x,1,t)\|_2^2\\
&-\epsilon\xi_1(t)k_1\tau_0e^{-\tau_2(t)k_1}\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx+\left(\frac{M}{2}-\frac{g_1(0)c_s^2}{4\beta}\right)(g_1'ou)(t)\\
&-\epsilon\xi_2(t)k_1\tau_0e^{-\tau_2(t)k_2}\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx.
\end{aligned} \tag{182}$$

At this point, we choose M so large such that

$$\eta_1 = \left(a_1 M - \epsilon\left(1 + \frac{\mu_1}{4\beta} - \frac{\xi_1(t)}{2\tau_0}\right) + g_{10} - \mu_1 - \beta\right) > 0,$$
$$\eta_2 = \left(a_2 M - \epsilon\left(1 + \frac{\alpha_1}{4\beta} - \frac{\xi_2(t)}{2\tau_0'}\right) + g_{20} - \alpha_1 - \beta\right) > 0,$$
$$\eta_3 = \left(\frac{M}{2} - \frac{g_1(0)c_s^2}{4\beta}\right) > 0,$$
$$\eta_4 = \left(\frac{M}{2} - \frac{g_2(0)c_s^2}{4\beta}\right) > 0.$$

Then we choose ϵ sufficiently small such that

$$\eta_5 = \left\{\frac{a_{11}\beta}{a_{01}}(1 + l_1^2) - \frac{\beta}{a_{01}} - \frac{2\lambda\delta}{a_{01}} - \epsilon\left\{\mu_2\left(1 + \frac{2\beta c_s^2}{a_{01}}\right) + l_1\right\}\right\} > 0,$$
$$\eta_6 = \left\{\frac{a_{22}\beta}{a_{01}}(1 + l_2^2) - \frac{\beta}{a_{02}} - \frac{2\lambda\delta}{a_{02}} - \epsilon\left\{\alpha_2\left(1 + \frac{2\beta c_s^2}{a_{02}}\right) + l_2\right\}\right\} > 0,$$

and (182) remains valid. Hence for all $t \geq t_0$, we arrive at

$$\begin{aligned}\frac{dL(t)}{dt} &\leq -\eta_1\|u_t\|_2^2 - \eta_2\|\upsilon_t\|_2^2 - \eta_3(g_1' o u)(t) + \eta_4(g_2' o \upsilon)(t) \\ &- \epsilon(p+1)\int_\Omega F(u(t),\upsilon(t))dx - \eta_5 a_1(u(t),u(t)) - \eta_6 a_2(\upsilon(t),\upsilon(t)) \\ &+ \eta_7(g_1 \circ u)(t) + \eta_8(g_2 \circ \upsilon)(t) - \eta_9\|z_1(x,1,t)\|_2^2 - \eta_{10}\|z_2(x,1,t)\|_2^2 \\ &- \eta_{11}\int_\Omega\int_0^1 z_1^2(x,k_1,t)dk_1dx - \eta_{12}\int_\Omega\int_0^1 z_2^2(x,k_2,t)dk_2dx,\end{aligned} \tag{183}$$

which yields

$$\frac{dL(t)}{dt} \leq -\eta_{14}E(t) + \eta_{15}\left(g_1 \circ u)(t) + g_2 \circ \upsilon)(t)\right), \quad \forall t \geq t_0, \tag{184}$$

where $\eta_i, i = 5, 6, 7..$ are some positive constants. Multiplying the above inequality by $\zeta = \min\{\zeta_1, \zeta_2\}$ and exploiting (A_0), we get, for all $t \geq t_0$

$$\zeta(t)L'(t) \leq -\eta_{14}\zeta(t)E(t) + \zeta(t)\eta_{14}((g_1 \circ u)(t) + g_2 \circ \upsilon)(t)).$$

Since $g_1'(t) \leq -\zeta(t)g_1(t)$ and $g_2'(t) \leq -\zeta(t)g_2(t)$ and using the fact that

$$-(g_1' \circ u(t) + (g_2' \circ \upsilon(t)) \leq -2E'(t),$$

by (41), we get

$$\begin{aligned}\zeta(t)L'(t) &\leq -\eta_{14}\zeta(t)E(t) - \eta_{15}\left(g_1' \circ u)(t) + g_2' \circ \upsilon)(t)\right) \\ &\leq -\eta_{14}\zeta(t)E(t) - 2\eta_{15}E'(t), \forall\ t \geq t_0.\end{aligned} \tag{185}$$

Define $\chi(t) = \zeta(t)L(t) + 2\eta_{15}E(t)$, which is equivalent to $E(t)$ and $\zeta'(t) \leq 0$, $\forall t \geq 0$. We obtain, $\forall\ t \geq t_0$

$$\begin{aligned}\chi'(t) &\leq \zeta'(t)L(t) - \eta_{14}\zeta(t)E(t) - \eta_{14}\zeta(t)E(t) \\ &\leq -\alpha\zeta(t)E(t).\end{aligned} \tag{186}$$

Integrating the last inequality over (t_0, t), we conclude that

$$\chi(t)(t) \leq \chi(0)e^{-\alpha\int_{t_0}^t \zeta(s)ds}. \tag{187}$$

Then, the equivalent relation between $\chi(t)$ and $E(t)$ yields

$$E(t) \leq Ke^{-\alpha \int_{t_0}^{t} \zeta(s)ds}. \tag{188}$$

This completes the proof.

Remark 5 We illustrate the energy decay rate given by Theorem 3 through the following examples which are introduced in ([23], [24]).

(1) If $g_1(t) = a_1 e^{-b_1(1+t)^{\nu_1}}$, $g_2(t) = \frac{a_2}{(1+t)^{\nu_2}}$, for $a_i > 0$ and $\nu_i > 0$, then $\zeta_1(t) = b_1\nu_1(1+t)^{\nu_1-1}$ and $\zeta_2(t) = \frac{\nu_2}{1+t}$ satisfy the condition (6). Thus (188) gives the estimate

$$E(t) \leq K(1+t)^{-\alpha}$$

(2) If $g_1(t) = a_1 e^{-(1+t)^{\nu_1}}$, $g_2(t) = a_2 e^{-(1+t)^{\nu_2}}$ for $a_i, \nu_i > 0 (i = 1, 2)$, then $\zeta_i(t) = \nu_i(1+t)^{\min(0,\nu_i-1)}$ satisfies the condition (6). Thus (188) gives the estimate

$$E(t) \leq Ke^{-\alpha(1+t)^{\min(1,\nu_1,\nu_2)}}$$

(3) If $g_1(t) = a_1 e^{-(\ln(1+t))^{\nu_1}}$, $g_2(t) = a_2 e^{-(\ln(1+t))^{\nu_2}}$ for $a_i > 0$ and $\nu_i > 1(i = 1, 2)$, then $\zeta_i(t) = \frac{\nu_i(\ln(1+t))^{\nu_i-1}}{1+t}(i = 1, 2)$ satisfies the condition (6) . Thus (188) gives the estimate

$$E(t) \leq Ke^{-\alpha(\ln(1+t))^{\min(\nu_1,\nu_2)}}$$

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