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SOME CONNECTIONS BETWEEN VARIOUS SUBCLASSES OF PLANAR HARMONIC MAPPINGS INVOLVING POISSON DISTRIBUTION SERIES

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Abstract. In the present paper, we established connections between various subclasses of harmonic univalent functions by applying certain convolution operator involving Poisson distribution series. To be more precise, we investigate such connections with harmonic γ-uniformly convex and harmonic γ-uniformly starlike mappings in the plane.

1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$.

Let H be the family of all harmonic functions of the form $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n,\ g(z) = \sum_{n=1}^{\infty} B_n z^n,\ |B_1| < 1,\ (z \in U), \tag{1.2}$$

are in the class A.

Denote by S_H the subclass of H that are univalent and sense-preserving in U. We also let the subclass S_H^0 of S_H as

$$S_H^0 = \left\{f = h + \overline{g} \in S_H : g'(0) = B_1 = 0\right\}.$$

The classes S_H^0 and S_H were first studied in [6]. Also, we let K_H^0, $S_H^{*,0}$ and C_H^0 denote the subclasses of S_H^0 of harmonic functions which are, respectively, convex, starlike and close-to-convex in U. For definitions and properties of these classes, one may refer to ([1], [6]) or [7].

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For $0 \leq \beta < 1$, let

$$N_H(\beta) = \left\{ f \in H : \Re\left(\frac{f'(z)}{z'} \right) \geq \beta,\ z = re^{i\theta} \in U \right\},$$

$$R_H(\beta) = \left\{ f \in H : \Re\left(\frac{f''(z)}{z''} \right) \geq \beta,\ z = re^{i\theta} \in U \right\},$$

where

$$z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right),\ z'' = \frac{\partial}{\partial\theta}(z'),\ f'(z) = \frac{\partial}{\partial\theta} f\left(re^{i\theta}\right), f''(z) = \frac{\partial}{\partial\theta}\left(f'(z)\right).$$

Define

$$TN_H(\beta) \equiv N_H(\beta) \cap T \text{ and } TR_H(\beta) \equiv R_H(\beta) \cap T,$$

where T consists of the functions $f = h + \overline{g}$ in S_H so that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |A_n| z^n,\ g(z) = \sum_{n=1}^{\infty} |B_n| z^n. \tag{1.3}$$

The classes T, $N_H(\beta)$, $TN_H(\beta)$, $R_H(\beta)$ and $TR_H(\beta)$ were initially introduced and studied, respectively, in ([21], [4], [4], [5] and [5]).

Let $HUC(\gamma, \alpha)$ be a subclass of the functions $f = h + \overline{g}$ in H which satisfy the condition

$$\Re\left\{ 1 + (1 + \gamma e^{i\eta}) \frac{z^2 h''(z) + \overline{2zg'(z) + z^2 g''(z)}}{zh'(z) - \overline{zg'(z)}} \right\} \geq \alpha,$$

for some γ $(0 \leq \gamma < \infty)$, α $(0 \leq \alpha < 1)$, $\eta \in R$ and $z \in U$.

Define $THUC(\gamma, \alpha) \equiv HUC(\gamma, \alpha) \cap T$.

A mapping in $HUC(\gamma, \alpha)$ or $THUC(\gamma, \alpha)$ is called γ-uniformly harmonic convex in U. These classes were studied in by Kim *et al.* in [9]. For $g \equiv 0$, $\gamma = 1$ and $\alpha = 0$ the class $HUC(\gamma, \alpha)$ reduces to the class UCV of analytic uniformly convex functions studied by Goodman [8].

Analogues to $HUC(\gamma, \alpha)$ is the class $HUS^*(\gamma, \alpha)$ consisting of harmonic functions $f = h + \overline{g}$ in H which satisfy the condition

$$\Re\left\{ \frac{zf'(z)}{z'f(z)} - \alpha \right\} \geq \gamma \left| \frac{zf'(z)}{z'f(z)} - 1 \right|$$

for some γ $(0 \leq \gamma < \infty)$, $\alpha (0 \leq \alpha < 1)$ and $z \in U$. Also define $THUS^*(\gamma, \alpha) \equiv HUS^*(\gamma, \alpha) \cap T$. The mappings in $HUS^*(\gamma, \alpha)$ or $THUS^*(\gamma, \alpha)$ are called γ-harmonic uniformly starlike in U. For $\alpha = 0$, these classes were studied in [19]. For $g \equiv 0$, $\gamma = 1$ and $\alpha = 0$, $HUS^*(\gamma, \alpha)$ reduces to the family US^* of analytic uniformly starlike functions defined by Rønning [11].

Very recently, Porwal [12] introduced a Poisson distribution series as

$$K(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n.$$

Now, for $m_1, m_2 > 0$, we introduce the operator $\Omega(m_1, m_2)$ for $f(z) \in S_H$ as

$$\Omega(m_1, m_2) f(z) = K(m_1, z) * h(z) + \overline{K(m_2, z) * g(z)} = H(z) + \overline{G(z)},$$

where

$$H(z) = z + \sum_{n=2}^{\infty} \frac{m_1^{n-1}}{(n-1)!} e^{-m_1} A_n z^n,\ G(z) = B_1 z + \sum_{n=2}^{\infty} \frac{m_2^{n-1}}{(n-1)!} e^{-m_2} B_n z^n. \tag{1.4}$$

for any function $f = h + \overline{g}$ in H.

Throughout this paper, we will frequently use the notation

$$\Omega(f) = \Omega(m_1, m_2) f.$$

Motivated by results on connections between various subclasses of analytic and harmonic univalent functions by using hypergeometric functions and generalized Bessel functions (see [2], [11], [13]-[18] and [20]), we establish a number of connections between the classes $HUC(\gamma, \alpha)$, $HUS^*(\gamma, \alpha)$, K_H^0, $S_H^{*,0}$, C_H^0, $N_H(\beta)$ and $R_H(\beta)$ by applying the convolution operator Ω.

2. Connections with harmonic uniformly convex mappings

In order to establish connections between harmonic convex mappings and harmonic uniformly convex mappings, we need following results in Lemma 2.1 [6] and Lemma 2.2 [9].

Lemma 2.1. *If $f = h + \overline{g} \in K_H^0$ where h and g are given by (1.2) with $B_1 = 0$, then*

$$|A_n| \leq \frac{n+1}{2}, \ |B_n| \leq \frac{n-1}{2}.$$

Lemma 2.2. *Let $f = h + \overline{g}$ be given by (1.2). If $0 \leq \gamma < \infty$, $0 \leq \alpha < 1$ and*

$$\sum_{n=2}^{\infty} n\left\{n(\gamma+1) - (\gamma+\alpha)\right\}|A_n| + \sum_{n=1}^{\infty} n\left\{n(\gamma+1) + (\gamma+\alpha)\right\}|B_n| \leq 1-\alpha, \quad (2.1)$$

then f is harmonic, sense-preserving univalent functions in U and $f \in HUC(\gamma, \alpha)$.

Remark 2.1. In [9], it is also shown that $f = h + \overline{g}$ given by (1.3) is in the family $THUC(\gamma, \alpha)$, if and only if the coefficient condition (2.1) holds. Moreover, if $f \in THUC(\gamma, \alpha)$, then

$$|A_n| \leq \frac{1-\alpha}{n\left\{n(\gamma+1) - (\gamma+\alpha)\right\}}, \ n \geq 2,$$

$$|B_n| \leq \frac{1-\alpha}{n\left\{n(\gamma+1) + (\gamma+\alpha)\right\}}, \ n \geq 1.$$

Theorem 2.3. *If $0 \leq \alpha < 1$, $0 \leq \gamma < \infty$, $m_1, m_2 > 0$ and the inequality*

$$\begin{aligned} e^{m_1} \big[&(\gamma+1)m_1^3 + (6\gamma+7-\alpha)m_1^2 + (6\gamma+10-4\alpha)m_1 \\ &+ (\gamma+1)m_2^3 + (6\gamma+5+\alpha)m_2^2 + (6\gamma+4+2\alpha)m_2 \big] \leq 2(1-\alpha), \end{aligned}$$

is satisfied then $\Omega\left(K_H^0\right) \subset HUC(\gamma, \alpha)$.

Proof. Let $f = h + \overline{g} \in K_H^0$ where h and g are of the form (1.2) with $B_1 = 0$. We need to show that $\Omega(f) = H + \overline{G} \in HUC(\gamma, \alpha)$, where H and G defined by (1.4) with $B_1 = 0$ are analytic functions in U.

In view of Lemma 2.2, we need to prove that

$$P_1 \leq 1 - \alpha,$$

where

$$
\begin{aligned}
P_1 =& \sum_{n=2}^{\infty} n\left\{n(\gamma+1)-(\gamma+\alpha)\right\}\left|\frac{e^{-m_1}m_1^{n-1}}{(n-1)!}A_n\right| \\
&+\sum_{n=2}^{\infty} n\left\{n(\gamma+1)+(\gamma+\alpha)\right\}\left|\frac{e^{-m_2}m_2^{n-1}}{(n-1)!}B_n\right| \\
&\leq \frac{1}{2}\left[\sum_{n=2}^{\infty} n(n+1)\left\{n(\gamma+1)-(\gamma+\alpha)\right\}\frac{e^{-m_1}m_1^{n-1}}{(n-1)!}\right. \\
&\left.+\sum_{n=2}^{\infty} n(n-1)\left\{n(\gamma+1)+(\gamma+\alpha)\right\}\frac{e^{-m_2}m_2^{n-1}}{(n-1)!}\right] \\
&=\frac{1}{2}\left[\sum_{n=2}^{\infty}\left[(\gamma+1)(n-1)(n-2)(n-3)+(6\gamma+7-\alpha)(n-1)(n-2)+(6\gamma+10-4\alpha)(n-1)\right.\right. \\
&\left.+2(1-\alpha)\right]\frac{e^{-m_1}m_1^{n-1}}{(n-1)!} \\
&\left.+\sum_{n=2}^{\infty}\left[(\gamma+1)(n-2)(n-3)+(6\gamma+5+\alpha)(n-2)+(6\gamma+4+2\alpha)\right]\frac{e^{-m_2}m_2^{n-1}}{(n-2)!}\right] \\
&=\frac{1}{2}\left[(\gamma+1)\sum_{n=2}^{\infty}\frac{e^{-m_1}m_1^{n-1}}{(n-4)!}+(6\gamma+7-\alpha)\sum_{n=2}^{\infty}\frac{e^{-m_1}m_1^{n-1}}{(n-3)!}\right. \\
&\left.+(6\gamma+10-4\alpha)\sum_{n=2}^{\infty}\frac{e^{-m_1}m_1^{n-1}}{(n-2)!}+2(1-\alpha)\sum_{n=2}^{\infty}\frac{e^{-m_1}m_1^{n-1}}{(n-1)!}\right] \\
&+(\gamma+1)\sum_{n=2}^{\infty}\frac{e^{-m_2}m_2^{n-1}}{(n-4)!}+(6\gamma+5+\alpha)\sum_{n=2}^{\infty}\frac{e^{-m_2}m_2^{n-1}}{(n-3)!}+(6\gamma+4+2\alpha)\sum_{n=2}^{\infty}\frac{e^{-m_2}m_2^{n-1}}{(n-2)!} \\
&=\frac{1}{2}\left[(\gamma+1)m_1^3+(6\gamma+7-\alpha)m_1^2+(6\gamma+10-4\alpha)m_1+2(1-\alpha)(1-e^{-m_1})\right. \\
&\left.+(\gamma+1)m_2^3+(6\gamma+5+\alpha)m_2^2+(6\gamma+4+2\alpha)m_2\right] \\
&\leq (1-\alpha),
\end{aligned}
$$

by given hypothesis.

This completes the proof of Theorem 2.3. □

In order to determine connection between $TN_H(\beta)$ and $HUC(\gamma,\alpha)$, we require the following result obtained in [4].

Lemma 2.4. *Let $f=h+\overline{g}$ where h and g are given by (1.3) and suppose that $0\leq\beta<1$, then*

$$
f\in TN_H(\beta)\Leftrightarrow\sum_{n=2}^{\infty}n\left|A_n\right|+\sum_{n=1}^{\infty}n\left|B_n\right|\leq 1-\beta.
$$

Remark 2.2. If $f\in TN_H(\beta)$, then

$$
\begin{aligned}
\left|A_n\right| &\leq \frac{1-\beta}{n},\ n\geq 2, \\
\left|B_n\right| &\leq \frac{1-\beta}{n},\ n\geq 1.
\end{aligned}
$$

Theorem 2.5. *If $0 \le \beta < 1$, $0 \le \gamma < \infty$, $m_1, m_2 > 0$ and the inequality*

$$(1-\beta)\left[(\gamma+1)m_1 + (1-\alpha)(1-e^{-m_1}) + (\gamma+1)m_2 + (2\gamma+1+\alpha)(1-e^{-m_2})\right] \le 1-\alpha-(2\gamma+\alpha+1)\left|B_1\right|$$

is satisfied, then

$$\Omega(TN_H(\beta)) \subset HUC(\gamma, \alpha).$$

Proof. Let $f = h + \overline{g} \in TN_H(\beta)$ where h and g are given by (1.3). In view of Lemma 2.2, it is enough to show that $P_2 \le 1 - \alpha$, where

$$\begin{aligned} P_2 = & \sum_{n=2}^{\infty} n\left\{n(\gamma+1)-(\gamma+\alpha)\right\}\left|\frac{e^{-m_1}m_1^{n-1}}{(n-1)!}A_n\right| \\ & + (2\gamma+\alpha+1)\left|B_1\right| + \sum_{n=2}^{\infty} n\left\{n(\gamma+1)+(\gamma+\alpha)\right\}\left|\frac{e^{-m_2}m_2^{n-1}}{(n-1)!}B_n\right|. \end{aligned}$$

Using Remark 2.2, it follows that

$$\begin{aligned} P_2 \le & (1-\beta)\left[\sum_{n=2}^{\infty}\left\{n(\gamma+1)-(\gamma+\alpha)\right\}\frac{e^{-m_1}m_1^{n-1}}{(n-1)!}\right. \\ & \left. + \sum_{n=2}^{\infty}\left\{n(\gamma+1)+(\gamma+\alpha)\right\}\frac{e^{-m_2}m_2^{n-1}}{(n-1)!}\right] + (2\gamma+\alpha+1)\left|B_1\right| \\ & = (1-\beta)\left[\sum_{n=2}^{\infty}\left[(\gamma+1)(n-1)+(1-\alpha)\right]\frac{e^{-m_1}m_1^{n-1}}{(n-1)!}\right. \\ & \left. + \sum_{n=2}^{\infty}\left[(\gamma+1)(n-1)+(2\gamma+1+\alpha)\right]\frac{e^{-m_2}m_2^{n-1}}{(n-1)!}\right] + (2\gamma+\alpha+1)\left|B_1\right| \\ & = (1-\beta)\left[(\gamma+1)m_1 + (1-\alpha)(1-e^{-m_1}) + (\gamma+1)m_2 + (2\gamma+1+\alpha)(1-e^{-m_2})\right] \\ & + (2\gamma+\alpha+1)\left|B_1\right| \\ & \le 1-\alpha \end{aligned}$$

by the given hypothesis, this completes the proof of Theorem 2.5. □

For the relationship between the classes $TR_H(\beta)$ and $HUC(\gamma, \alpha)$, we shall require the following lemma which is due to [5].

Lemma 2.6. *Let $f = h + \overline{g}$ where h and g are given by* (1.3), *and suppose that $0 \le \beta < 1$. Then*

$$f \in TR_H(\beta) \Leftrightarrow \sum_{n=2}^{\infty} n^2\left|A_n\right| + \sum_{n=1}^{\infty} n^2\left|B_n\right| \le 1-\beta.$$

Remark 2.3. If $f = h + \overline{g} \in TR_H(\beta)$ where h and g are given by (1.3), then

$$\left|A_n\right| \le \frac{1-\beta}{n^2},\ n \ge 2$$

and

$$\left|B_n\right| \le \frac{1-\beta}{n^2},\ n \ge 1.$$

Theorem 2.7. *If* $m_1, m_2 > 0,\ 0 \leq \beta < 1, \gamma \geq 0$ *and the inequality*

$$\begin{aligned}(1-\beta)&\Bigg[(\gamma+1)\left(1-e^{-m_1}\right)-\frac{(\gamma+\alpha)}{m_1}\left(1-e^{-m_1}-m_1e^{-m_1}\right)\\&+(\gamma+1)\left(1-e^{-m_2}\right)+\frac{(\gamma+\alpha)}{m_2}\left(1-e^{-m_2}-m_2e^{-m_2}\right)\Bigg]\\&\leq 1-\alpha-(2\gamma+\alpha+1)\left|B_1\right|\end{aligned}$$

is satisfied then

$$\Omega\left(TR_H(\beta)\right)\subset HUC(\gamma,\alpha).$$

Proof. Making use of Lemma 2.2 and the definition of P_2 in Theorem 2.5, we need only to prove that $P_2 \leq 1-\alpha$. Using Remark 2.3, it follows that

$$\begin{aligned}P_2 &= \sum_{n=2}^{\infty} n\left\{n(\gamma+1)-(\gamma+\alpha)\right\}\left|\frac{e^{-m_1}m_1^{n-1}}{(n-1)!}A_n\right|\\&+(2\gamma+\alpha+1)\left|B_1\right|+\sum_{n=2}^{\infty} n\left\{n(\gamma+1)+(\gamma+\alpha)\right\}\left|\frac{e^{-m_2}m_2^{n-1}}{(n-1)!}B_n\right|\\&\leq(1-\beta)\left[\sum_{n=2}^{\infty}\frac{\left\{n(\gamma+1)-(\gamma+\alpha)\right\}}{n}\frac{e^{-m_1}m_1^{n-1}}{(n-1)!}\right.\\&\left.+\sum_{n=2}^{\infty}\frac{\left\{n(\gamma+1)+(\gamma+\alpha)\right\}}{n}\frac{e^{-m_2}m_2^{n-1}}{(n-1)!}\right]+(2\gamma+\alpha+1)\left|B_1\right|\\&=(1-\beta)\left[\sum_{n=0}^{\infty}\left\{(\gamma+1)-\frac{(\gamma+\alpha)}{n+2}\right\}\frac{e^{-m_1}m_1^{n+1}}{(n+1)!}\right.\\&\left.+\sum_{n=0}^{\infty}\left\{(\gamma+1)+\frac{(\gamma+\alpha)}{n+2}\right\}\frac{e^{-m_2}m_2^{n+1}}{(n+1)!}\right]+(2\gamma+\alpha+1)\left|B_1\right|\\&=(1-\beta)\Bigg[(\gamma+1)\left(1-e^{-m_1}\right)-\frac{(\gamma+\alpha)}{m_1}\left(1-e^{-m_1}-m_1e^{-m_1}\right)\\&+(\gamma+1)\left(1-e^{-m_2}\right)+\frac{(\gamma+\alpha)}{m_2}\left(1-e^{-m_2}-m_2e^{-m_2}\right)\Bigg]+(2\gamma+\alpha+1)\left|B_1\right|\\&\leq 1-\alpha,\end{aligned}$$

by given hypothesis. □

Theorem 2.8. *If* $m_1, m_2 > 0,\ 0 \leq \alpha < 1,\ \gamma \geq 0$ *and the inequality*

$$e^{-m_1}+e^{-m_2}\geq 1+\frac{(2\gamma+\alpha+1)}{1-\alpha}\left|B_1\right|$$

is satisfied, then $\Omega\left(THUC(\gamma,\alpha)\right)\subset HUC(\gamma,\alpha)$.

Proof. By adopting the technique of the proof of Theorem 2.5, Lemma 2.2 and Remark 2.1, we obtain

$$
\begin{aligned}
P_2 &\leq (1-\alpha)\left[\sum_{n=2}^{\infty}\frac{e^{-m_1}m_1^{n-1}}{(n-1)!}+\sum_{n=2}^{\infty}\frac{e^{-m_2}m_2^{n-1}}{(n-1)!}\right]+(2\gamma+\alpha+1)\left|B_1\right| \\
&=(1-\alpha)\left[\sum_{n=0}^{\infty}\frac{e^{-m_1}m_1^{n+1}}{(n+1)!}+\sum_{n=0}^{\infty}\frac{e^{-m_2}m_2^{n+1}}{(n+1)!}\right]+(2\gamma+\alpha+1)\left|B_1\right| \\
&=(1-\alpha)\left[e^{-m_1}(e^{m_1}-1)+e^{-m_2}(e^{m_2}-1)\right]+(2\gamma+\alpha+1)\left|B_1\right| \\
&=(1-\alpha)\left[1-e^{-m_1}+1-e^{-m_2}\right]+(2\gamma+\alpha+1)\left|B_1\right| \\
&=(1-\alpha)\left[2-e^{-m_1}-e^{-m_2}\right]+(2\gamma+\alpha+1)\left|B_1\right| \\
&\leq 1-\alpha,
\end{aligned}
$$

by the given condition and this completes the proof of the theorem. □

Theorem 2.9. *If $m_1, m_2 > 0$, $0 \leq \alpha < 1$, $\gamma \geq 0$, then $\Omega\left(THUC(\gamma,\alpha)\right) \subset THUC(\gamma,\alpha)$, if and only if*

$$e^{-m_1}+e^{-m_2} \geq 1+\frac{(2\gamma+\alpha+1)}{1-\alpha}|B_1|.$$

Proof. The proof of the above theorem is much akin to that of Theorem 2.8. Therefore we omits the details involved. □

3. Connections with harmonic uniformly starlike mappings

In this section we shall look at the analogous results between various classes of planar harmonic mappings and $HUS^*(\gamma,\alpha)$ by applying the convolution operator Ω.

Lemma 3.1. *Let $f = h + \overline{g} \in H$ be given by (1.2). If $0 \leq \gamma < \infty$, $0 \leq \alpha < 1$ and*

$$\sum_{n=2}^{\infty}\left\{n(\gamma+1)-(\gamma+\alpha)\right\}\left|A_n\right|+\sum_{n=1}^{\infty}\left\{n(\gamma+1)+(\gamma+\alpha)\right\}\left|B_n\right| \leq 1-\alpha, \tag{3.1}$$

then f is harmonic, sense-preserving univalent functions in U and $f \in HUS^*(\gamma,\alpha)$.

Remark 3.1. The result in Lemma 3.1 is a special case of the corresponding result proved in [3]. However, for $\gamma = 1$, Lemma 3.1 reduces to the result found in [19].

Remark 3.2. In [3], it is also shown that $f = h + \overline{g}$ given by (1.2) is in the family $THUS^*(\gamma,\alpha)$, if and only if the coefficient condition (3.1) holds. Moreover, if $f \in THUS^*(\gamma,\alpha)$, then

$$
\begin{aligned}
\left|A_n\right| &\leq \frac{1-\alpha}{\left\{n(\gamma+1)-(\gamma+\alpha)\right\}},\ n \geq 2, \\
\left|B_n\right| &\leq \frac{1-\alpha}{\left\{n(\gamma+1)+(\gamma+\alpha)\right\}},\ n \geq 1.
\end{aligned}
$$

Applying the Lemma 3.1 and using the techniques of the proof of following Theorem 2.3 so we only state the results.

Theorem 3.2. *If $0 \leq \alpha < 1$, $0 \leq \gamma < \infty$, $m_1, m_2 > 0$ and the inequality*

$$e^{m_1}\left[(\gamma+1)\left(m_1^2+m_2^2\right)+(2\gamma+3-\alpha)m_1+(3\gamma+2+\alpha)m_2\right] \leq 2(1-\alpha)$$

is satisfied then $\Omega\left(K_H^0\right) \subset HUS^*(\gamma,\alpha)$.

Analogous to Theorem 3.2, we next find connections of the classes $S_H^{*,0}$, C_H^0 with $HUS^*(\gamma,\alpha)$. However, we first need the following result which may be found in [1] or [7].

Lemma 3.3. *If $f = h+\overline{g} \in S_H^{*,0}$ or C_H^0 with h and g as given by (1.2) with $B_1 = 0$, then*

$$|A_n| \leq \frac{(2n+1)(n+1)}{6},\ |B_n| \leq \frac{(2n-1)(n-1)}{6}.$$

Theorem 3.4. *If $0 \leq \alpha < 1$, $0 \leq \gamma < \infty$ and $m_1, m_2 > 0$ and the inequality*

$$e^{m_1}\left[2(\gamma+1)\left(m_1^3+m_2^3\right)+(13\gamma-2\alpha+15)m_1^2+(15\gamma+9\alpha-24)m_1+(11\gamma+2\alpha+9)m_2^2+(9\gamma+3\alpha+6)m_2\right] \leq 6(1-\alpha)$$

is satisfied then $\Omega\left(S_H^{*,0}\right) \subset HUS^*(\gamma,\alpha)$ and $\Omega\left(C_H^0\right) \subset HUS^*(\gamma,\alpha)$.

Theorem 3.5. *If all the restrictions and coefficient condition in Theorem 2.7 are satisfied then $\Omega\left(TN_H(\beta)\right) \subset HUS^*(\gamma,\alpha)$.*

Analogues to Theorem 2.8, we only state the following result.

Theorem 3.6. *If all the restrictions and coefficient condition in Theorem 2.8 are satisfied then $\Omega\left(THUS^*(\gamma,\alpha)\right) \subset HUS^*(\gamma,\alpha)$.*

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