*Electronic Journal of Mathematical Analysis and Applications*
Vol. 6(2) July 2018, pp. 185-194.
ISSN: 2090-729X(online)
http://fcag-egypt.com/Journals/EJMAA/

# CERTAIN PROPERTIES OF A NEW SUBCLASS OF $p$-VALENTLY CLOSE TO CONVEX FUNCTIONS

PREM PRATAP VYAS AND SHASHI KANT

Abstract. In the present paper we introduce and investigate an intresting subclass $\mathcal{K}_p^{(k)}(\alpha,\beta)$ analytic and $p$-valently close to convex functions in the open unit disk $\mathbb{U}$. For functions belonging to $\mathcal{K}_p^{(k)}(\alpha,\beta)$, we derive several properties coefficient estimates, sufficient condition, distortion theorem and inclusion relationships.

## 1. Introduction and definitions

Let $\mathcal{A}_p$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N) \tag{1}$$

which are analytic in the open unit disk, $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. In particular, we write $\mathcal{A}_1 = \mathcal{A}$.

For any two analytic functions $f$ and $g$ in $\mathbb{U}$, we say that $f$ is subordinate to $g$ in $\mathbb{U}$, written as $f(z) \prec g(z)$ if there exist a schwarz function $w(z)$ such that $f(z) = g(w(z))$, for $z \in \mathbb{U}$. In particular, if $g$ is univalent in $\mathbb{U}$, then $f$ is subordinate to $g$ iff $f(0) = g(0)$ and $f(U) \subset g(U)$.

A function $f \in \mathcal{A}_p$, is said to be $p$-valently starlike of order $\gamma$ $(0 \le \gamma < p)$ in $\mathbb{U}$ if it satisfies the inequality [5]

$$Re\Big(\frac{zf^{'}(z)}{f(z)}\Big) > \gamma \qquad (z \in \mathbb{U})$$

or equivalently

$$\frac{zf^{'}(z)}{f(z)} \prec \frac{p+(p-2\gamma)z}{1-z} \qquad (z \in \mathbb{U}).$$

The class of all $p$-valent starlike functions of order $\gamma$ in $\mathbb{U}$ is denoted by $\mathcal{S}_p^*(\gamma)$. Also, we denote that

$$\mathcal{S}_p^*(0) = \mathcal{S}_p^*, \quad \mathcal{S}_1^*(\gamma) = \mathcal{S}^*(\gamma) \quad and \quad \mathcal{S}_1^*(0) = \mathcal{S}^*.$$

2010 *Mathematics Subject Classification.* 30C45.
*Key words and phrases.* Analytic function, close to convex functions, Coefficient estimates, sufficient condition, distortion theorem, inclusion relationships.
Submitted Dec. 18, 2017.

A function $f \in \mathcal{A}_p$, is said to be $p$-valently close-to-convex of order $\gamma$ $(0 \leq \gamma < p)$ in $\mathbb{U}$ if $g \in \mathcal{S}_p^*(\gamma)$ and satisfies the inequality [9]

$$Re\Big(\frac{zf^{'}(z)}{g(z)}\Big) > \gamma \qquad (z \in \mathbb{U})$$

or equivalently

$$\frac{zf^{'}(z)}{g(z)} \prec \frac{p+(p-2\gamma)z}{1-z} \qquad (z \in \mathbb{U}).$$

The class of all $p$-valent close-to-convex functions of order $\gamma$ in $\mathbb{U}$ is denoted by $\mathcal{K}_p(\gamma)$. Also, we denote that

$$\mathcal{K}_p(0)=\mathcal{K}_p, \quad \mathcal{K}_1(\gamma)=\mathcal{K}(\gamma) \quad and \quad \mathcal{K}_1(0)=\mathcal{K}.$$

Recently, Bulut [3] discussed a class $\mathcal{K}_s^{(k)}(\gamma,p)$ for analytic and $p$-valently close-to-convex functions. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{K}_s^{(k)}(\gamma,p)$ if there exist a function $g \in S_p^*(\frac{(k-1)p}{k})$ $(k \in N$ is a fixed integer), such that

$$Re\Big(\frac{z^{(k-1)p+1}f^{'}(z)}{g_k(z)}\Big) > \gamma \quad (z \in \mathbb{U};\ 0 \leq \gamma < p),$$

where $g_k$ is defined by the equality

$$g_k(z)=\prod_{v=0}^{k-1}\varepsilon^{-vp}g(\varepsilon^v z); \quad \varepsilon=e^{\frac{2\pi\iota}{k}}. \tag{2}$$

Here assuming $g \in S_p^*(\frac{(k-1)p}{k})$ makes $\frac{g_k(z)}{z^{(k-1)p}}$ a p-valant starlike function which in turn implies the close-to-convexity of $f$. By simple calculution, we see that $f(z) \in \mathcal{K}_s^{(k)}(\gamma,p)$ if and only if

$$\Big|\frac{z^{(k-1)p+1}f^{'}(z)}{g_k(z)}-p\Big| < \Big|\frac{z^{(k-1)p+1}f^{'}(z)}{g_k(z)}+p-2\gamma\Big| \tag{3}$$

Recently several similar classes of $\mathcal{K}_s^{(k)}(\gamma,p)$ for analytic and univalent function have been defined and investigated, some of them we refer to [4, 7, 11, 12, 13, 14, 15, 17]. Motivated essentially by the above mentioned class $\mathcal{K}_s^{(k)}(\gamma,p)$ and the above refered works for analytic and univalent functions, we now introduce a new class for $p$-valent analytic function in the following manner:

**Definition 1.** For $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$, a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{K}_p^{(k)}(\alpha,\beta)$, if there exist a function $g \in S_p^*(\frac{(k-1)p}{k})$ $(k \in N$ is a fixed integer), such that

$$\Big|\frac{z^{(k-1)p+1}f^{'}(z)}{g_k(z)}-p\Big| < \beta\Big|\frac{\alpha z^{(k-1)p+1}f^{'}(z)}{g_k(z)}+p\Big| \tag{4}$$

where $g_k$ is defined by the equality (2).

**Remark.**(i) For $p$=1, we get the class $\mathcal{K}_1^{(k)}(\alpha,\beta)$ studied by Wang [16].

(ii) For $p$=1 and $k$=2, we get the class $\mathcal{K}_1^{(2)}(\alpha,\beta)$ studied by Wang [15].

In the present paper, we derive several properties including coefficient estimates, sufficient condition, distortion theorem and inclusion relationships for function belonging to the class $\mathcal{K}_p^{(k)}(\alpha,\beta)$.

In order to prove our main result for the function class $\mathcal{K}_p^{(k)}(\alpha,\beta)$, we need the

following lemmas:
**Lemma 1.** [3] If

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \in S_p^*(\frac{(k-1)p}{k}),$$

then

$$G(z) = \frac{g_k(z)}{z^{(k-1)p}} = z^p + \sum_{n=1}^{\infty} B_{p+n} z^{p+n} \in S_p^*, \tag{5}$$

where $g_k$ is given by (2).
**Lemma 2.** [2] Let $G(z) \in S_p^*$ given by (5) and $\mu$ be a complex number, then

$$\Big| B_{p+2} - \mu B_{p+1}^2 \Big| \leq \ p\Big( max\{1, |1 + 2p(1 - 2\mu)|\} \Big).$$

Let $\Omega$ be class of analytic functions of the form:

$$w(z) = w_1 z + w_2 z^2 + ... \qquad (z \in \mathbb{U}), \tag{6}$$

in the unit disk $\mathbb{U}$ satisfying the condition $|w(z)| < 1$.
**Lemma 3.** ([6], p.10) If $w(z) \in \Omega$, then for any complex number $\mu$:

$$|w_1| \leq 1, |w_2 - \mu w_1^2| \leq 1 + (|\mu| - 1)|w_1^2| \leq max\{1, |\mu|\}.$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.
**Lemma 4.** Let the function $K(z) = p + k_1 z + k_2 z^2 + k_3 z^3 + ... \quad (z \in \mathbb{U})$ be analytic in the unit disk $\mathbb{U}$, and satisfies the condition

$$\Big| \frac{K(z) - p}{\alpha K(z) + p} \Big| < \beta \qquad (z \in \mathbb{U}),$$

for $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$, if and only if there exist an analytic function $\phi$ in the unit disk $\mathbb{U}$, such that $|\phi(z)| \leq \beta \quad (z \in \mathbb{U})$, and

$$K(z) = \frac{p - pz\phi(z)}{1 + \alpha z\phi(z)}, \qquad (z \in \mathbb{U}).$$

**Proof.** Assume that the function

$$\frac{zf^{'}(z)}{G(z)} = p + k_1 z + k_2 z^2 + k_3 z^3 + ... = K(z) \quad (z \in \mathbb{U}),$$

satisfies the condition

$$\Big| \frac{K(z) - p}{\alpha K(z) + p} \Big| < \beta \qquad (z \in \mathbb{U}).$$

Setting

$$k(z) = \frac{p - K(z)}{p + \alpha K(z)},$$

we see that the function $k(z)$ is analytic in $\mathbb{U}$, satisfies the inequality $|k(z)| < \beta$ for $z \in \mathbb{U}$ and $k(0) = 0$. Now, by using schwarz's lemma, we get that the function $k(z)$ has of the form $k(z) = z\phi(z)$, where $\phi(z)$ is analytic in $\mathbb{U}$ and satisfies $|\phi(z)| \leq \beta$ for $z \in \mathbb{U}$. Thus, we obtain

$$K(z) = \frac{p - pk(z)}{1 + \alpha k(z)} = \frac{p - pz\phi(z)}{1 + \alpha z\phi(z)} \ .$$

Conversely, if

$$K(z) = \frac{p - pz\phi(z)}{1 + \alpha z\phi(z)}$$

and $|\phi(z)| \leq \beta$ for $z \in \mathbb{U}$, then $K$ is analytic in the unit disk $\mathbb{U}$. so we get

$$\Big|\frac{K(z) - p}{\alpha K(z) + p}\Big| = |z\phi(z)| \leq \beta|z| < \beta \qquad (z \in \mathbb{U}),$$

which completes the proof of our lemma.

**Lemma 5.** [8] Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$. Then

$$\frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z}.$$

Let $f(z) = \Sigma_{n=1}^{\infty} a_n z^n$ and $g(z) = \Sigma_{n=1}^{\infty} b_n z^n$ be two analytic functions defined in $\mathbb{D}$. Then there Hadamard product (or convolution) is the function $(f * g)(z)$ defined by

$$(f * g)(z) = \Sigma_{n=1}^{\infty} a_n b_n z^n.$$

The classes of starlike and convex functions are closed under convolution with convex function. The following lemma is required for our next result.

**Lemma 6.** [10] Let $\psi$ and $\phi$ be convex in $\mathbb{U}$ and suppose $f \prec \psi$, then

$$f * \phi = \psi * \phi.$$

## 2. Main results

First of all, we show in which way our class is associated with the appropriate subordination.

**Theorem 1.** A function $f(z) \in \mathcal{K}_p^{(k)}(\alpha, \beta)$ if and only if there exits $g_k(z)$ satisfying the condition (2) such that

$$\frac{1}{p}\frac{zf'(z)}{G(z)} \prec \frac{1 + \beta z}{1 - \alpha\beta z} \qquad (z \in \mathbb{U}), \tag{7}$$

where G(z) is given by (5).

**Proof.** Let $f(z) \in \mathcal{K}_p^{(k)}(\alpha, \beta)$. Then, for $\alpha \neq 1$ and $\beta \neq 1$, squaring and expanding both sides of (4), we see that the region of $\frac{1}{p}\frac{zf'(z)}{G(z)}$ for $z \in \mathbb{U}$ is contained in the disk $\mathbf{C}$ whose center is $\frac{(1+\alpha\beta^2)}{(1-\alpha^2\beta^2)}$ and radius is $\frac{[\beta(1+\alpha)]}{(1-\alpha^2\beta^2)}$. Since $q(z) = \frac{1+\beta z}{1-\alpha\beta z}$ maps the unit disk $\mathbb{U}$ to the disk $\mathbf{C}$ and $q(z)$ is univalent in $\mathbb{U}$, we obtain the relation (7). Conversely, assume that the relation (7) holds true. Then we have

$$\frac{1}{p}\frac{zf'(z)}{G(z)} \prec \frac{1 + \beta z}{1 - \alpha\beta z},$$

$$(0 \leq \alpha \leq 1, 0 < \beta \leq 1; z \in \mathbb{U}),$$

where $w(z)$ is analytic in $\mathbb{U}$, $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$. Therefore from the above equation, we obtain the inequality (4), that is, $f(z) \in \mathcal{K}_p^{(k)}(\alpha, \beta)$.

**Theorem 2.** Let $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $f$ given by (1) and $g \in \mathcal{S}_p^*\left(\frac{(k-1)p}{k}\right)$ are such that the condition (4) holds. Then, for $n \geq 1$, we have

$$|ma_m - pB_m|^2 - (1+\alpha)^2\beta^2 p^2 \leq \sum_{n=1}^{m-1}\Big\{(\alpha^2\beta^2 - 1)n^2|a_n|^2 + (\beta^2 - 1)p^2|B_n|^2 + 2p(\alpha\beta^2 + 1)n|a_n B_n|\Big\} \tag{8}$$

where the coefficients $B_n$ are given in (5).
**Proof.** Suppose that the condition (4) is satisfied then by lemma 4, we have

$$\frac{zf'(z)}{G(z)} = \frac{p - pz\phi(z)}{1 + \alpha z\phi(z)} \qquad (z \in \mathbb{U}),$$

where $\phi$ is an analytic functions in $\mathbb{U}$, $\phi(z) \leq 1$ for $z \in \mathbb{U}$ and G(z) is given by (5). From the above equality, we obtain that

$$[\alpha zf'(z) + pG(z)]\, z\phi(z) = pG(z) - zf'(z)\ . \tag{9}$$

Now, we put

$$z\phi(z) = \sum_{n=1}^{\infty} t_n z^n \quad (z \in \mathbb{U}).$$

Thus from (9), we find that

$$\left((1+\alpha)p + \sum_{n=1}^{\infty} \alpha(p+n)a_{p+n}z^n + p\sum_{n=1}^{\infty} B_{p+n}z^n\right)\sum_{n=1}^{\infty} t_n z^n$$

$$= p\sum_{n=1}^{\infty} B_{p+n}z^n - \sum_{n=1}^{\infty}(p+n)a_{p+n}z^n\ . \tag{10}$$

Equating the coefficient of $z^m$ in (10), we have

$$pB_{p+m}-(p+m)a_{p+m} = (1+\alpha)pt_m+(\alpha(p+1)a_{p+1}+pB_{p+1})t_{m-1}+...+(\alpha(p+m-1)a_{p+m-1}+pB_{p+m-1})t_1$$

which shows that $pB_{p+n} - (p+n)a_{p+n}$ on the right hand side of (10) depends only on $a_{p+1}, B_{p+1}, a_{p+2}, B_{p+2}, ..., a_{p+n-1}, B_{p+n-1}$, of left-hand side. Hence , for $n \geq 1$, we can write as

$$\left((1+\alpha)p + \sum_{n=1}^{m-1}(\alpha na_n + pB_n)z^n\right) z\phi(z) = \sum_{n=1}^{m}(pB_n - na_n)z^n + \sum_{n=m+1}^{\infty} c_n z^n\ . \tag{11}$$

Using the fact that $|z\phi(z)| \leq \beta|z| < \beta$ for all $z \in \mathbb{U}$ in (11), this reduce to inequality

$$\left|(1+\alpha)p + \sum_{n=1}^{m-1}(\alpha na_n + pB_n)z^n\right|\beta > \left|\sum_{n=1}^{m}(pB_n - na_n)z^n + \sum_{n=m+1}^{\infty} c_n z^n\right|.$$

Then squaring the above inequality and integrating along $|z| = r < 1$, we obtain

$$\beta^2 \int_0^{2\pi}\left|(1+\alpha)p + \sum_{n=1}^{m-1}(\alpha na_n + pB_n)r^n e^{in\theta}\right|^2 d\theta$$

$$> \int_0^{2\pi}\left|\sum_{n=1}^{m}(pB_n - na_n)r^n e^{in\theta} + \sum_{n=m+1}^{\infty} c_n r^n e^{in\theta}\right|^2 d\theta\ .$$

Using now the Paraseval's inequality, we obtain

$$\beta^2\left((1+\alpha)^2p^2 + \sum_{n=1}^{m-1}|\alpha na_n + pB_n|^2 r^{2n}\right) > \sum_{n=1}^{m}|pB_n - na_n|^2 r^{2n} + \sum_{n=m+1}^{\infty}|c_n|^2 r^{2n}\ .$$

Letting $r \to 1$ in this inequality, we get

$$\sum_{n=1}^{m}|na_n-pB_n|^2 \le \beta^2\left((1+\alpha)^2p^2+\sum_{n=1}^{m-1}|\alpha na_n+pB_n|^2\right) .$$

Hence we deduce that

$$|ma_m-pB_m|^2-(1+\alpha)^2\beta^2p^2 \le \sum_{n=1}^{m-1}\left\{(\alpha^2\beta^2-1)n^2|a_n|^2+(\beta^2-1)p^2|B_n|^2+2p(\alpha\beta^2+1)n|a_nB_n|\right\} ,$$

and thus we obtain the inequality (8). Which completes the proof of Theorem 2.

**Theorem 3.** Let $0 \le \alpha \le 1$, $0 < \beta \le 1$, $f$ given by (1) and $g \in \mathcal{S}_p^*\left(\frac{(k-1)p}{k}\right)$ such that

$$(1+\alpha\beta)\sum_{n=1}^{\infty}(p+n)|a_{p+n}|+(1+\beta)p\sum_{n=1}^{\infty}|B_{p+n}|<(1+\alpha)\beta p , \qquad (12)$$

where the coefficients $B_{p+n}$ are given by (5), then $f \in \mathcal{K}_p^{(k)}(\alpha,\beta)$.

**Proof.** For $f$ given by (1)) and $g_k$ defined by (2), we set

$$\Lambda = \left|zf'(z)-p\frac{g_k(z)}{z^{(k-1)p}}\right|-\beta\left|\alpha zf'(z)+p\frac{g_k(z)}{z^{(k-1)p}}\right|$$

$$= \Big|\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}-p\sum_{n=1}^{\infty}B_{p+n}z^{p+n}\Big|-\beta\Big|(1+\alpha)pz^p+\alpha\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}+p\sum_{n=1}^{\infty}B_{p+n}z^{p+n}\Big|$$

$$\Lambda \le \sum_{n=1}^{\infty}(p+n)|a_{p+n}||z|^{p+n}+p\sum_{n=1}^{\infty}|B_{p+n}||z|^{p+n}$$

$$-\beta\Big((1+\alpha)p|z|^p-\alpha\sum_{n=1}^{\infty}(p+n)|a_{p+n}||z|^{p+n}-p\sum_{n=1}^{\infty}|B_{p+n}||z|^{p+n}\Big)$$

$$= -(1+\alpha)\beta p|z|^p+(1+\alpha\beta)\sum_{n=1}^{\infty}(p+n)|a_{p+n}||z|^{p+n}+(1+\beta)p\sum_{n=1}^{\infty}|B_{p+n}||z|^{p+n}$$

$$= \Big(-(1+\alpha)\beta p+(1+\alpha\beta)\sum_{n=1}^{\infty}(p+n)|a_{p+n}|+(1+\beta)p\sum_{n=1}^{\infty}|B_{p+n}|\Big)|z|^p.$$

From the inequality (12), we obtain that $\Lambda < 0$.

Thus we have

$$\left|zf'(z)-p\frac{g_k(z)}{z^{(k-1)p}}\right|<\beta\left|\alpha zf'(z)+p\frac{g_k(z)}{z^{(k-1)p}}\right|$$

which is equivalent to (4). Hence $f \in \mathcal{K}_p^{(k)}(\alpha,\beta)$. This completes the proof of Theorem 3.

**Theorem 4.** If $f \in \mathcal{K}_p^{(k)}(\alpha,\beta)$, then for $|z| = r$ $(0 \le r < 1)$, we have

$$(i) \quad \frac{p(1-\beta r)r^{p-1}}{(1+\alpha\beta r)(1+r)^{2p}} \le |f'(z)| \le \frac{p(1+\beta r)r^{p-1}}{(1-\alpha\beta r)(1-r)^{2p}} \qquad (13)$$

$$(ii) \quad \int_0^r \frac{p(1-\beta\tau)\tau^{p-1}}{(1+\alpha\beta\tau)(1+\tau)^{2p}}d\tau \le |f(z)| \le \int_0^r \frac{p(1+\beta\tau)\tau^{p-1}}{(1-\alpha\beta\tau)(1-\tau)^{2p}}d\tau \qquad (14)$$

**Proof.** If $f \in \mathcal{K}_p^{(k)}(\alpha, \beta)$, then there exist a function $g \in S_p^*(\frac{(k-1)p}{k})$ such that (4) holds. (*i*) From Lemma 1 it follows that the function $G(z)$ given by (5) is $p-$valently starlike function.Hence from [1, Theorem 1] we have

$$\frac{r^p}{(1+r)^{2p}} \leq |G(z)| \leq \frac{r^p}{(1-r)^{2p}} \qquad (|z| = r\ (0 \leq r < 1)). \tag{15}$$

Let us define $\Psi(z)$ by

$$\Psi(z) = \frac{zf'(z)}{G(z)} \qquad (z \in \mathbb{U}),$$

then by (7), we have

$$\frac{(p - p\beta r)}{(1 + \alpha\beta r)} \leq |\Psi(z)| \leq \frac{(p + p\beta r)}{(1 - \alpha\beta r)} \qquad (z \in \mathbb{U}). \tag{16}$$

Thus from (15) and (16), we get the inequalities (13).
(*ii*) Let $z = re^{\iota\theta}$ $(0 < r < 1)$. If $l$ denotes the closed line-segment in the complex $\zeta-$plane from $\zeta = 0$ and $\zeta = z$, i.e. $l = [0, re^{\iota\theta}]$, then we have

$$f(z) = \int_l f'(\zeta)d\zeta = \int_0^r f'(\tau e^{\iota\theta})e^{\iota\theta}d\tau \qquad (|z| = r\ (0 \leq r < 1)).$$

Thus, by using the upper estimate in (13), we have

$$|f(z)| = \Big|\int_l f'(\zeta)d\zeta\Big| \leq \int_0^r |f'(\tau e^{\iota\theta})|d\tau \leq \int_0^r \frac{p(1+\beta\tau)\tau^{p-1}}{(1-\alpha\beta\tau)(1-\tau)^{2p}}d\tau \qquad (|z| = r\ (0 \leq r < 1)),$$

which yields the right hand of the inequality in (14).
In order to prove the lower bound in (14), let $z_0 \in \mathbb{U}$ with $|z_0| = r$ $(0 < r < 1)$, such that
$|f(z_0)| = \min\{|f(z)| : |z| = r\}$.
It is sufficient to prove that the left-hand side inequality holds for this point $z_0$. Moreover, we have

$$|f(z)| \geq |f(z_0)| \qquad (|z| = r\ (0 \leq r < 1)).$$

The image of the closed line-segment $l_0 = [0, f(z_0)]$ by $f^{-1}$ is a piece of arc $\Gamma$ included in the closed disk $\mathbb{U}_r$ given by

$$\mathbb{U}_r = \{z : z \in \mathbb{C} \ \ and \ \ |z| \leq r \ \ (0 \leq r < 1)\},$$

that is, $\Gamma = f^{-1}(l_0) \subset \mathbb{U}_r$. Hence, in accordance with (13),we obtain

$$|f(z_0)| = \int_{l_0} |dw| = \int_\Gamma |f'(\zeta)||d\zeta| \geq \int_0^r \frac{p(1-\beta\tau)\tau^{p-1}}{(1+\alpha\beta\tau)(1+\tau)^{2p}}d\tau.$$

This finishes the proof of the inequality (14).
**Theorem 5.** Let $-1 \leq -\alpha_2\beta_2 \leq -\alpha_1\beta_1 < \beta_1 \leq \beta_2 \leq 1$ Then.

$$\mathcal{K}_p^{(k)}(\alpha_1, \beta_1) \subset \mathcal{K}_p^{(k)}(\alpha_2, \beta_2)$$

**Proof.** Suppose that $f \in \mathcal{K}_p^{(k)}(\alpha_1, \beta_1)$ Then

$$\frac{1}{p}\frac{zf'(z)}{G(z)} \prec \frac{1+\beta_1 z}{1-\alpha_1\beta_1 z}$$

since $-1 \leq -\alpha_2\beta_2 \leq -\alpha_1\beta_1 < \beta_1 \leq \beta_2 \leq 1$. By Lemma 5, we have

$$\frac{1}{p}\frac{zf'(z)}{G(z)} \prec \frac{1+\beta_1 z}{1-\alpha_1\beta_1 z} \prec \frac{1+\beta_2 z}{1-\alpha_2\beta_2 z}$$

it follows that $f(z) \in \mathcal{K}_p^{(k)}(\alpha_2, \beta_2)$, which implies the inclusion result.

**Theorem 6.** For a function $f(z)$ given by (1) is in the class $\mathcal{K}_p^{(k)}(\alpha, \beta)$ and $\mu \in \mathbb{C}$, the following estimates holds.

$$|a_{p+2} - \mu a_{p+1}^2| \leq \; 2(1+\alpha)\beta p\Big|\frac{p}{p+2} - \frac{2\mu p^2}{(p+1)^2}\Big| + \frac{p^2}{p+2}\mu_1 + \frac{(1+\alpha)(1+\alpha\beta)\beta p}{p+2}\mu_2 \tag{17}$$

where

$$\mu_1 = max\Big\{1, \big|1 + 2p\big(1 - \frac{2\mu p(p+2)}{(p+1)^2}\big)\big|\Big\} \tag{18}$$

and

$$\mu_2 = max\Big\{1, \big|\frac{(1+\alpha)(p+2)\beta\mu p - (1+\alpha\beta)(p+1)^2}{(1+\alpha\beta)(1+p)^2}\big|\Big\}. \tag{19}$$

**Proof.** Let $f \in \mathcal{K}_p^{(k)}(\alpha, \beta)$, then

$$\frac{1}{p}\frac{zf'(z)}{G(z)} = \frac{1+\beta w(z)}{1-\alpha\beta w(z)} \qquad (z \in \mathbb{U}), \tag{20}$$

where G(z) is given by (5) and $w(z)$ is schwarz function given by (6) which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$.

Using the series expansions in (20), we have

$$1 + \Big(\frac{p+1}{p}a_{p+1} - B_{p+1}\Big)z + \Big(\frac{p+2}{p}a_{p+2} - \frac{p+1}{p}a_{p+1}B_{p+1} + (B_{p+1}^2 - B_{p+2})\Big)z^2 + ...$$

$$= 1 + (1+\alpha)\beta w_1 z + (1+\alpha)(1+\alpha\beta)\beta(w_1^2 + w_2)z^2 + ... \,. \tag{21}$$

Equating of coefficients in (21) gives us

$$a_{p+1} = \frac{p}{p+1}\big((1+\alpha)\beta w_1 + B_{p+1}\big),$$

$$a_{p+2} = \frac{p}{p+2}\big((1+\alpha)\beta w_1 B_{p+1} + (1+\alpha)(1+\alpha\beta)\beta(w_1^2 + w_2) + B_{p+2}\big).$$

Therefore, we have

$$|a_{p+2} - \mu a_{p+1}^2| \leq \; \frac{p}{p+2}\Big|B_{p+2} - \frac{\mu(p^2+2p)}{(p+1)^2}B_{p+1}^2\Big| + (1+\alpha)\beta\Big|\frac{p}{p+2} - \frac{2\mu p^2}{(p+1)^2}\Big||w_1||B_{p+1}|$$

$$+ \frac{p}{p+2}(1+\alpha)(1+\alpha\beta)\beta\Big|w_2 - \Big(\frac{\mu(1+\alpha)\beta(p^2+2p) - (1+\alpha\beta)(p+1)}{(1+\alpha\beta)(p+1)^2}\Big)w_1^2\Big|. \tag{22}$$

Now, the desired result follows upon using lemma 2 and lemma 3 in (22).

**Theorem 7.** If $f(z) \in \mathcal{K}_p^{(k)}(\alpha, \beta)$, then there exists

$$q(z) \prec \frac{1+\beta z}{1-\alpha\beta z}$$

such that for all s and t with $|s| \leq 1$ and $|t| \leq 1$,

$$\frac{t^{p-1}f'(sz)q(tz)}{s^{p-1}f'(tz)q(sz)} \prec \big(\frac{1-tz}{1-sz}\big)^{2p}. \tag{23}$$

**Proof.** Let $f(z) \in \mathcal{K}_p^{(k)}(\alpha, \beta)$, then there exist $g(z) \in \mathcal{S}_p^*\left(\frac{(k-1)p}{k}\right)$.

Suppose

$$q(z) = \frac{1}{p}\frac{zf'(z)}{G(z)} \;, \tag{24}$$

where

$$G(z) = \frac{g_k(z)}{z^{(k-1)p}} \ .$$

Then by (7) ,we have

$$q(z) \prec \frac{1+\beta z}{1-\alpha\beta z}$$

logarithmic derivative of (24), implies

$$\frac{zf^{''}(z)}{f^{'}(z)} - \frac{zq^{'}(z)}{q(z)} + 1 - p = \frac{zG^{'}(z)}{G(z)} - p \ . \tag{25}$$

Since $G(z) \in \mathcal{S}_p^*$,

$$\frac{1}{p}\frac{zG^{'}(z)}{G(z)} \prec \frac{1+z}{1-z},$$

so

$$\frac{zG^{'}(z)}{G(z)} - p \prec \frac{2pz}{1-z}. \tag{26}$$

From (25) and (26), we have

$$\frac{zf^{''}(z)}{f^{'}(z)} - \frac{zq^{'}(z)}{q(z)} + 1 - p \ \prec \frac{2pz}{1-z} \ . \tag{27}$$

For s and t such that $|s| \leq 1$ and $|t| \leq 1$, the function

$$h(z) = \int_0^z \frac{s}{1-su} - \frac{t}{1-tu} du \tag{28}$$

is convex in $\mathbb{U}$.

Applying Lemma 6, we have

$$\big(\frac{zf^{''}(z)}{f^{'}(z)} - \frac{zq^{'}(z)}{q(z)} + 1 - p\big) * h(z) \prec \ \frac{2pz}{1-z} * h(z).$$

Given any function $k(z)$ analytic in $\mathbb{U}$, with $k(0) = 0$, we have

$$(k * h)(z) = \int_{tz}^{sz} k(u)\frac{du}{u} \qquad (z \in \mathbb{U}),$$

which implies that

$$log\Big[\frac{(sz)^{1-p}f^{'}(sz)q(tz)}{(tz)^{1-p}f^{'}(tz)q(sz)}\Big] \prec \ log\Big[\frac{1-tz}{1-sz}\Big]^{2p}$$

which is equivalent to (23). This completes the proof of Theorem 7.

## References


[1] M. K. Aouf, On a class of $p$-valent starlike functions of order $\alpha$, Internat. J. Math. Sci. 10(4), 733-744, 1987.

[2] M. K. Aouf, R.M. El-Ashwah and H.M. Zayed, Feket-Szego inequalities for $p$-valent starlike and convex functions of complex order, J. Egyptian Math. Soc. 22, 190-196, 2014.

[3] S. Bulut, Certain properties of a new subclass of analytic and $p$-valently close-to-convex functions, arXiv:1612.08735, 24 Dec 2016.

[4] N. E. Cho, Oh S. Kwon and V. Ravichandran, Coefficient, distortion and growth inequalities for certain close-to-convex functions, J. Ineq. Appl. 1, 100, 2011.

[5] E. G. Goluzina, On the coefficients of a class of functions, regular in a disk and having an integral representation in it, J. of Soviet Math. 6, 606-617, 1974.

[6] F. R. Keogh and E.P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. 20, 8-12, 1969.

[7] J. Kowalczyk and E. Les-Bomba, On a subclass of close-to-convex functions, Appl. Math. Lett. 23, 1147-1151, 2010.
[8] M. S. Liu, On a subclass of $p$-valent close-to-convex functions of order $\beta$ and type $\alpha$, J. Math. Study, 30, 102-104, 1997.
[9] A. E. Livingston, $p$-valent close-to-convex functions, Trans. Amer. Math. Soc. 115, 161-179, 1965.
[10] S. T. Ruschewehy and T. Sheil-Small, Hadamard products of schlicht functions and the Polya-Schoenberg conjecture , Comment. Math. Helv. 48, 119-135, 1973.
[11] A. Soni and S. Kant, A new subclass of close-to-convex functions with fekete-szego problem, J. Rajasthan Acad. Phy. Sci. 12(2), 1-14, 2013.
[12] B. Seker, On certain new subclass of close-to-convex functions, Appl. Math. Comput. 218, 1041-1045, 2011.
[13] B. Seker and N. E. Cho, A subclass of close-to-convex functions, Hacettepe J. Math. Stat. 42(4), 373-379, 2013.
[14] T. V. Sudharsan, P. Balasubrahmanyam and K. G. subramanian, On functions starlike with respect to symmetric and conjugate points, Twiwanese J. Math. 2(1), 57-68, 1998.
[15] Z. G. Wang, C. Y. Gao and S. M. Yuan, On certain subclass of close-to-convex functions, Acta Math. Acad. Paedagog. Nyhazi (N.S.), 22, 171-177, 2006.
[16] Z. G. Wang, C. Y. Gao and S. M. Yuan, On certain new subclass of close-to-convex functions, Matematicki Vesnik 58, 119-124, 2006.
[17] Q. H. Xu, H. M. Srivastava and Z. Li, A certain subclass of analytic and close-to-convex functions, Appl. Math. Lett. 24, 396-401, 2011.

Prem Pratap Vyas and Shashi Kant
Department of Mathematics, Government Dungar College, Bikaner-334001, INDIA
*E-mail address*: prempratapvyas@gmail.com
*E-mail address*: drskant.2007@yahoo.com