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COINCIDENCE AND COMMON FIXED POINT THEOREMS FOR (ψ, φ)- WEAKLY CONTRACTIVE MAPPINGS IN RECTANGULAR B-METRIC SPACES

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Abstract. We establish some coincidence and common fixed point theorems for mappings satisfying a (ψ, φ)-weakly contractive condition in rectangular b-metric spaces. Our results extend very recent results of H. S. Ding, V. Ozturk, S. Radenovic [8] and extend and generalize many existing results inthe literature.

1. Introduction

In 2000, Branciari [1] introduced a concept of generalized metric space where the triangle inequality of a metric space has been replaced by an inequality involving three terms instead of two. As such, any metric space is a generalized metric space but the converse is not true [1]. He proved the Banach's fixed point theorem in such a space.
After that, many fixed point results were established for this interesting space. For more, the reader can refer to [6, 2].
It is also known that common fixed point theorems are generalizations of fixed point theorems. Recently, there have been many researchers who have interested in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems.

R.George, S.Radenovic, K.P Reshma and S.Shukla [11] introduced the concept of rectangular b-metric space, which is not necessarily Hausdorff and which generalizes the concepts of metric space, rectangular metric space and b-metric space. Note that spaces with non Hausdorff topology plays an importnat role in Tarskian approach to programming language semantics used in computer science (For some details see [13]. An analog of the Banach contraction principle as well as the Kannan type fixed point theorem in rectangular b-metric spaces are also proved in [11].

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In this paper, we prove coincidence and common fixed point theorems for two mappings satisfying a (ψ, φ)-weakly contractive condition in rectangular b- metric spaces. Presented theorems extend and generalize many existing results in the literature.

2. Mathematical preliminarily

The following definitions are introduced in [4, 1, 12, 11] and [9], respectively:
Definition 1[4, 12]. Let X be a nonempty set and $s \geq 1$ be a given real number. A functional $d : X \times X \longrightarrow \mathbb{R}^+$ is called a b- metric if for $x, y, z \in X$, the following conditions are satisfied:

- **(1):** $d(x, y) = 0$ if and only if $x = y$,
- **(3):** $d(x, y) = d(y, x)$,
- **(4):** $d(x, y) \leq s[d(x, u) + d(u, y)]$ (b-triangular inequality).

A pair (X, d) is called a b-metric space (with constant s).
Definition 2 [1]. Let X be a nonempty set. A functional $d : X \times X \longrightarrow \mathbb{R}^+$ is called a rectangular metric if for all $x, y \in X$ and for all distinct points $u, v \in X$ each of them different from x and y, the following conditions are satisfied:

- **(1):** $d(x, y) = 0$ if and only if $x = y$,
- **(3):** $d(x, y) = d(y, x)$,
- **(4):** $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (rectangular inequality).

A pair (X, d) is called a rectangular metric space or generalized metric space (g.m.s.) or Branciari's space.
For all properties and definitions of notions in Branciari's spaces see [1, 5, 10, 15, 16, 17, 14, 9].
Definition 3 [11, 9]. Let X be a nonempty set, $s \geq 1$ be a given real number and $d : X \times X \longrightarrow \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$ each distinct from x and y:

- **(1):** $d(x, y) = 0$ if and only if $x = y$,
- **(3):** $d(x, y) = d(y, x)$,
- **(4):** $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ (b-rectangular inequality).

Then (X, d) is called a rectangular b-metric space (with constant s) or a b-generalized metric space (RbMS).

Note that every metric space is a rectangular metric space and every rectangular metric space is a rectangular b-metric space (with coefficient $s = 1$). However the converse of the above implication is not necessarily true (See Examples 1.4 and 1.5 [11]).
The following gives some easy examples of RbMS's.
Example 1. Let $X = \mathbb{N}$, deffine $d : X \times X \longrightarrow \mathbb{R}^+$ by

$$\begin{cases} d(x, y) = 0 & if\ x = y, \\ 4\alpha & if\ x, y \in \{1, 2\}\ and\ x \neq y, \\ \alpha & if\ x\ or\ y \notin \{1, 2\}\ and\ x \neq y, \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a rectangular b-metric space with coefficient $s = \frac{4}{3} > 1$, but (X, d) is not a rectangular metric space, as

$$d(1, 2) = 4\alpha > 3\alpha = d(1, 3) + d(3, 4) + d(4, 2).$$

Example 2. Let (X, ρ) be a g.m.s., and $p \geq 1$ be a real number. Let $d(x, y) = (\rho(x, y))^p$. Evidently, from the convexity of function $f(x) = x^p$ for $x \geq 0$ and by Jensen inequality [18] we have

$$(a + b + c)^p \leq 3^{p-1}(a^p + b^p + c^p)$$

for nonnegative real numbers a, b, c. So, it is easy to obtain that (X, d) is a b-g.m.s with $s \leq 3^{p-1}$.

Note that every b-metric space with coefficient s is a RbMS with coefficient s^2 but the converse is not necessarily true. (See Example 1.7 [11]).

For any $x \in X$ we define the open ball with center x and radius $r > 0$ by $B_r(x) = \{y \in X : d(x, y) < r\}$.
The open balls in RbMS are not necessarily open (See Example 1.7 [11]). Let U be the collection of all subsets A of X satisfying the condition that for each $x \in A$ there exist $r > 0$ such that $B_r(x) \subseteq A$. Then U defines a topology for the RbMS (X, d), which is not necessarily Hausdorff (See Example 1.7 [11]).
Definition 4 . Let (X, d) be a rectangular b-metric space, (x_n) be a sequence in X and $x \in X$. Then

- **(a):** The sequence (x_n) is said to be convergent in (X, d) and converges to x, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim\limits_{n \to +\infty} x_n = x$ or $x_n \longrightarrow x$ as $n \longrightarrow +\infty$.
- **(b):** The sequence (x_n) is said to be Cauchy sequence in (X, d) if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > n_0$, or equivalently, if
$\lim\limits_{n,m \to +\infty} d(x_n, x_m) = 0$.
- **(c):** (X, d) is said to be a complete rectangular b-metric space if every Cauchy sequence in X converges to some $x \in X$.

Note that limit of sequence in a rectangular b-metric space (the same as in a rectangular metric space(g.m.s)) is not necessarily unique and also every rectangular b-metric convergent sequence in a rectangular b-metric space is not necessarily rectangular b-metric-Cauchy ([11], Example 2.7).
Lemma 1[8]. Let (X, d) be a b-rectangular metric space with $s \geq 1$ and let (x_n) be a sequence in X such that $x_n \neq x_m$ whenever $n \neq m$ and

$$\lim_{n \longrightarrow +\infty} d(x_n, x_{n+1}) = 0$$

If (x_n) is not a b-rectangular-Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences (x_{m_k}) and (x_{n_k}) of positive integers such that for the following sequences of real numbers $d(x_{m_k}, x_{n_k}), d(x_{m_k+1}, x_{n_k-1})$ and $d(x_{m_k}, x_{n_k-2})$ hold:

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon, \quad \frac{\varepsilon}{s} \leq \limsup d(x_{m_k+1}, x_{n_k-1}) \leq \varepsilon$$

and

$$\frac{\varepsilon}{s} \leq \limsup d(x_{m_k}, x_{n_k-2}) \leq \varepsilon$$

Lemma 2[9]. Let (X, d) be a b-rectangular metric space with $s \geq 1$ and let (x_n) be a b-rectangular-Cauchy sequence in X such that $x_n \neq x_m$ whenever $n \neq m$:

Then (x_n) can converge to at most one point.

Definition 5. Let X be a non-empty set and $g, f : X \longrightarrow X$.

- **(i):** A point $y \in X$ is called a point of coincidence of f and g if there exists a point $x \in X$ such that $y = fx = gx$.
  The point x is called coincidence point of f and g.
- **(ii):** The mappings f, g are said to be weakly compatible if they commute at their coincidence point (that is, $fgx = gfx$ whenever $fx = gx$).

Let Ψ denote all functions $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ which satisfy ψ is continuous, non-decreasing and $\psi(t) = 0$ if and only if $t = 0$
and let Φ denote all the functions $\varphi : [0, +\infty) \longrightarrow (0, +\infty)$ which satisfy $\liminf\limits_{t \longrightarrow r^+} \varphi(t) > 0$ for all $r > 0$ and $\varphi(t) = 0$ if and only if $t = 0$.
For example, functions $\psi_1(t) = kt$ where $k > 0$, $\psi_2(t) = \frac{t}{t+1}$, $\psi_3(t) = \ln(t + 1)$ and $\psi_4(t) = \min(t, 1)$ are in Ψ.
$\varphi_1(t) = kt$ where $k > 0$, $\varphi_2(t) = \frac{\ln(2t+1)}{2}$ are in Φ.

3. Main results

Theorem 1. Let (X, d) be a b-rectangular metric space with coefficient $s > 1$ and let $f, g : X \longrightarrow X$ be two self maps such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. Suppose that there exist $\varphi \in \Phi$, $\psi \in \Psi$ such that:

$$\psi(sd(fx, fy)) \le \psi(\frac{1}{s}d(gx, gy)) - \varphi(d(gx, gy)) \tag{1}$$

holds for all $x, y \in X$, then f and g have a unique point of coincidence. If, moreover, f and g are weakly compatible, then they have a unique common fixed point.

Proof. Take an arbitrary point $x_0 \in X$ and, using that $f(X) \subseteq g(X)$, choose sequences (x_n) and(y_n) in X such that

$$y_n = fx_n = gx_{n+1}, \quad for \;\; n = 0, 1, 2, ... \tag{2}$$

If $y_k = y_{k+1}$ for some $k \in \mathbb{N}$, then $gx_{k+1} = y_k = y_{k+1} = fx_{k+1}$ and f and g have a point of coincidence.
Suppose, further, that $y_n \ne y_{n+1}$ for all $n \in \mathbb{N}$. Putting $x = x_{n+1}, y = x_n$ in (1) we obtain that

$$\begin{array}{rcl} \psi(sd(fx_{n+1}, fx_n)) & \le & \psi(\frac{1}{s}d(gx_{n+1}, gx_n)) - \varphi(d(gx_{n+1}, gx_n)) \\ & = & \psi(\frac{1}{s}d(y_n, y_{n-1})) - \varphi(d(y_n, y_{n-1})) \end{array}$$

By properties of φ and ψ and $s > 1$ we have

$$\begin{array}{rcl} \psi(d(y_{n+1}, y_n)) & = & \psi(d(fx_{n+1}, fx_n)) \\ & \le & \psi(sd(fx_{n+1}, fx_n)) \\ & \le & \psi(\frac{1}{s}d(y_n, y_{n-1})) - \varphi(d(y_n, y_{n-1})) \\ & < & \psi(d(y_n, y_{n-1})) \end{array} \tag{3}$$

which implies, since ψ is a non-decreasing function,

$$d(y_{n+1}, y_n) < d(y_n, y_{n-1}) \tag{4}$$

Set $\delta_n = d(y_n, y_{n-1})$.
Now we would like to show that $\delta_n \longrightarrow 0$ as $n \longrightarrow +\infty$. It is clear that the sequence (δ_n) is decreasing. Therefore, there is some $\delta \ge 0$ such that $\lim\limits_{n \longrightarrow +\infty} \delta_n = \delta$.
We shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then taking the

limit as $n \longrightarrow +\infty$ (equivalently, $\delta_n \longrightarrow \delta$) of both sides of (3) and remembering $\liminf_{t \longrightarrow r} \varphi(t) > 0$ for all $r > 0$ and ψ is continuous, we have

$$\begin{array}{rcl}\psi(\delta) & \leq & \psi(\frac{1}{s}\delta) - \liminf_{\delta_n \longrightarrow \delta} \varphi(d(y_n, y_{n-1})) \\ & < & \psi(\delta)\end{array}$$

a contradiction. Thus $\delta = 0$, that is

$$\lim_{n \longrightarrow +\infty} \delta_n = \lim_{n \longrightarrow +\infty} d(y_n, y_{n-1}) = 0 \tag{5}$$

Then the sequence (y_n) satisfies all conditions of Lemma 1.

Let us prove that (y_n) is a b-rectangular-Cauchy sequence in b-rectangular metric space (X, d). If (y_n) is not a b-rectangular-Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences (y_{m_k}) and (y_{n_k}) of positive integers such that for the following sequences of real numbers $d(y_{m_k}, y_{n_k}), d(y_{m_k+1}, y_{n_k-1})$ and $d(y_{m_k}, y_{n_k-2})$ hold:

$$d(y_{m_k}, y_{n_k}) \geq \varepsilon, \ \ \frac{\varepsilon}{s} \leq \limsup d(y_{m_k+1}, y_{n_k-1}) \leq \varepsilon$$

and

$$\frac{\varepsilon}{s} \leq \limsup d(y_{m_k}, y_{n_k-2}) \leq \varepsilon$$

Putting $x = x_{m_k+1}, y = x_{n_k-1}$ in (1) we have

$$\begin{array}{rcl}\psi(sd(y_{m_k+1}, y_{n_k-1})) & = & \psi(sd(fx_{m_k+1}, fx_{n_k-1})) \\ & \leq & \psi(\frac{1}{s}d(gx_{m_k+1}, gx_{n_k-1})) - \varphi(d(gx_{m_k+1}, gx_{n_k-1})) \\ & = & \psi(\frac{1}{s}d(y_{m_k}, y_{n_k-2})) - \varphi(d(y_{m_k}, y_{n_k-2}))\end{array}$$

Now, using Lemma 1, properties of the functions $\psi,\ \varphi$, since $s > 1$ and taking the upper limit as $k \longrightarrow +\infty$, we get

$$\psi(\varepsilon) = \psi(s.\frac{\varepsilon}{s}) \leq \psi(\frac{1}{s}.\varepsilon) < \psi(\varepsilon)$$

which is a contradiction. Hence, the sequence $y_n = fx_n = gx_{n+1}$ is a b-rectangular-Cauchy.

Suppose, that the subspace $g(X)$ is complete (the proof when $f(X)$ is complete is similar), then (y_n) tends to some $z \in g(X)$, where $z = gw$ for some $w \in X$, which implies

$$\lim_{n \longrightarrow +\infty} y_n = gw \tag{6}$$

To prove that $fw = gw$, suppose that $fw \neq gw$, then, by Lemma 2, it follows that $y_n = fx_n = gx_{n+1}$ differs from both fw and gw for n sufficiently large.

Now, applying the inequality (1) with $x = x_n$ and $y = w$

$$\begin{array}{rcl}\psi(d(fx_n, fw)) & \leq & \psi(sd(fx_n, fw)) \\ & \leq & \psi(\frac{1}{s}d(gx_n, gw)) - \varphi(d(gx_n, gw)) \\ & = & \psi(\frac{1}{s}d(y_{n-1}, gw)) - \varphi(d(y_{n-1}, gw)) \\ & \leq & \psi(\frac{1}{s}d(y_{n-1}, gw)) \\ & < & \psi(d(y_{n-1}, gw))\end{array}$$

Hence, using that ψ is nondecreasing, we have

$$d(fx_n, fw) < d(y_{n-1}, gw)$$

Letting $n \longrightarrow +\infty$ in the above inequality, using (6) and (5), we get

$$\lim_{n \longrightarrow +\infty} d(y_n, fw) = \lim_{n \longrightarrow +\infty} d(fx_n, fw) = 0 \tag{7}$$

From the b-rectangular inequality,

$$d(gw, fw) \leq s\big(d(gw, y_{n-1}) + d(y_{n-1}, y_n) + d(y_n, fw)\big)$$

Letting $n \longrightarrow +\infty$ in the above inequality, using (6), (7) and (5), we get $d(gw, fw) = 0$ which implies $gw = fw$. Thus we proved that $z = gw = fw$ is a point of coincidence of f and g.
If f and g are weakly compatible, then from $gw = fw = z$ we have $fgw = gfw$, that is, $fz = gz$. Let $v = fz = gz$. Since z is a unique point of coincidence of f and g, then $v = z$. Therefore, we have $z = fz = gz$. Thus we proved that z is a unique common fixed point of f and g. Now we show that z is a unique point of coincidence. Let z_1 be another point of coincidence in X, that is, let $z_1 = gv = fv$. Suppose that $z_1 \neq z$. Then $gv \neq gw$ and so
$\varphi(d(gv, gw)) > 0$. From (1)we have

$$\begin{array}{rcl} \psi(d(gv, gw)) & = & \psi(d(fv, fw)) \\ & \leq & \psi(sd(fv, fw)) \\ & \leq & \psi(\frac{1}{s}d(gv, gw)) - \varphi(d(gv, gw)) \\ & < & \psi(d(gv, gw)) \end{array}$$

a contradiction. Therefore, $z_1 = z$. This complete the proof.
Remark. Theorem 1 extends Theorem 3.1 from [7] from rectangular to rectangular b-metric spaces.

From Theorem 1, if we choose $g = I_X$ the identity mapping on X, we obtain the following fixed point result.
Corollary 1. Let (X, d) be a complet rectangular b-metric space with coefficient $s > 1$ and let $f : X \longrightarrow X$ be a self-mapping on X such that the following condition holds:

$$\psi(sd(fx, fy)) \leq \psi(\frac{1}{s}d(x, y)) - \varphi(d(x, y)) \tag{8}$$

holds for all $x, y \in X$, where $\varphi \in \Phi$ and $\psi \in \Psi$. Then f has a unique fixed point.
Example. Let $X = \{0, 1, 2, 3\}$. Define $d : X \times X \to \mathbb{R}_+$ as follows:
$d(x, y) = d(y, x)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $y = x$. Further, let $d(0, 3) = d(2, 3) = d(0, 2) = 1$, $d(1, 3) = 3$, $d(0, 1) = 6$, $d(1, 2) = 5$.
Then it is easy to show that (X, d) is a rectangular b- metric space with $s = \frac{6}{5}$, but (X, d) is not a b-metric space with the same coefficient s because the triangle inequality does not hold for all $x, y, z \in X$:

$$6 = d(0, 1) > \frac{6}{5}(d(0, 3) + d(3, 1)) = \frac{6}{5}(1 + 3) = \frac{24}{5}.$$

Note that (X, d) is not a rectangular metric space because the rectangular inequality does not hold for all $x, y, u, v \in X$:

$$6 = d(0, 1) > d(0, 2) + d(2, 3) + d(3, 1) = 1 + 1 + 3 = 5$$

Now define a mappings $T, f : X \to X$ as follows:
$T0 = T1 = T2 = 0$, $T3 = 2$
$f(0) = 0$, $f(1) = 2$, $f(2) = 3$, $f(3) = 1$. Then, T and f satisfy (1) with $\psi(t) = 2t$

and $\varphi(t) = t/2$. Indeed, $d(Tx, Ty) > 0$ only if $x \in \{0, 1, 2\}$ and $y = 3$. We have

$$\begin{array}{rcl} \frac{12}{5} = \psi(sd(T(0), T(3))) & < & \frac{2}{s}.d(f(0), f(3)) - \frac{1}{2}d(f(0), f(3)) \\ & = & 10 - 3 = 7 \\ \frac{12}{5} = \psi(d(T(1), T(3))) & < & \frac{2}{s}.d(f(1), f(3)) - \frac{1}{2}d(f(1), f(3)) \\ & = & \frac{50}{6} - \frac{5}{2} = \dfrac{35}{6} \\ \frac{12}{5} = \psi(d(T(2), T(3))) & < & \frac{2}{s}.d(f(2), f(3)) - \frac{1}{2}d(f(2), f(3)) \\ & = & 5 - \dfrac{3}{2} = \frac{7}{2}. \end{array}$$

Therefore, T and f satisfy the inequality (1).Clearly, $T(X) \subseteq f(X)$ and T and f are weakly compatible. So we can apple our Theorem 1 and T and f have a unique fixed point $z = 0$.

The following theorem extends Theorem 2.7 from H. S. Ding, V. Ozturk, S. Radenovic [8].

Theorem 2. Let (X, d) be a b-rectangular metric space with coefficient $s > 1$ and let $f, g : X \to X$ be two self maps such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. Suppose that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $c \in [0, 1)$ such that:

$$\psi(sd(fx, fy)) \leq c.\psi(d(gx, gy)) - \varphi(d(gx, gy)) \tag{9}$$

holds for all $x, y \in X$, then f and g have a unique point of coincidence. If, moreover, f and g are weakly compatible, then they have a unique common fixed point.

Proof. The proof follows the lines of proof of Theorem 1, Putting $x = x_{n+1}, y = x_n$ in (9) we obtain that

$$\begin{array}{rcl} \psi(sd(fx_{n+1}, fx_n)) & \leq & c.\psi(d(gx_{n+1}, gx_n)) - \varphi(d(gx_{n+1}, gx_n)) \\ & = & c.\psi(d(y_n, y_{n-1})) - \varphi(d(y_n, y_{n-1})) \end{array}$$

By properties of φ and ψ and $c < 1$ we have

$$\begin{array}{rcl} \psi(d(y_{n+1}, y_n)) & = & \psi(d(fx_{n+1}, fx_n)) \\ & \leq & \psi(sd(fx_{n+1}, fx_n)) \\ & \leq & c.\psi(d(y_n, y_{n-1})) - \varphi(d(y_n, y_{n-1})) \\ & < & \psi(d(y_n, y_{n-1})) \end{array} \tag{10}$$

which implies, since ψ is a non-decreasing function,

$$d(y_{n+1}, y_n) < d(y_n, y_{n-1}) \tag{11}$$

Set $\delta_n = d(y_n, y_{n-1})$.

Now we would like to show that $\delta_n \longrightarrow 0$ as $n \longrightarrow +\infty$. It is clear that the sequence (δ_n) is decreasing. Therefore, there is some $\delta \geq 0$ such that $\lim\limits_{n \longrightarrow +\infty} \delta_n = \delta$.

We shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then taking the limit as $n \longrightarrow +\infty$ of both sides of (10) we have

$$\begin{array}{rcl} \psi(\delta) & \leq & c.\psi(\delta) - \liminf\limits_{\delta_n \longrightarrow \delta} \varphi(d(y_n, y_{n-1})) \\ & < & \psi(\delta) \end{array}$$

a contradiction. Thus $\delta = 0$, that is

$$\lim_{n \longrightarrow +\infty} \delta_n = \lim_{n \longrightarrow +\infty} d(y_n, y_{n-1}) = 0 \tag{12}$$

Then the sequence (y_n) satisfies all conditions of Lemma 1.

Let us prove that (y_n) is a b-rectangular-Cauchy sequence in b-rectangular metric space (X, d). If (y_n) is not a b-rectangular-Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences (y_{m_k}) and (y_{n_k}) such that $d(y_{m_k}, y_{n_k}), d(y_{m_k+1}, y_{n_k-1})$ and $d(y_{m_k}, y_{n_k-2})$ hold:

$$d(y_{m_k}, y_{n_k}) \geq \varepsilon, \;\; \frac{\varepsilon}{s} \leq \limsup d(y_{m_k+1}, y_{n_k-1}) \leq \varepsilon$$

and

$$\frac{\varepsilon}{s} \leq \limsup d(y_{m_k}, y_{n_k-2}) \leq \varepsilon$$

Putting $x = x_{m_k+1}, y = x_{n_k-1}$ in (9) we have

$$\begin{array}{rcl} \psi(sd(y_{m_k+1}, y_{n_k-1})) & = & \psi(sd(fx_{m_k+1}, fx_{n_k-1})) \\ & \leq & c.\psi(d(gx_{m_k+1}, gx_{n_k-1})) - \varphi(d(gx_{m_k+1}, gx_{n_k-1})) \\ & = & c.\psi(d(y_{m_k}, y_{n_k-2})) - \varphi(d(y_{m_k}, y_{n_k-2})) \end{array}$$

Now, using Lemma 1, properties of the functions ψ, φ, since $c < 1$ and taking the upper limit as $k \longrightarrow +\infty$, we get

$$\psi(\varepsilon) = \psi(s.\frac{\varepsilon}{s}) \leq c.\psi(\varepsilon) < \psi(\varepsilon)$$

which is a contradiction. The rest of the proof is omitted.

Taking $g = I_X$ the identity mapping on X in Theorem 2 , we obtain the following fixed point result.

Corollary 2. Let (X, d) be a complet rectangular b-metric space with coefficient $s > 1$ and let $f : X \longrightarrow X$ be a self-mapping on X. Suppose that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $c \in [0, 1)$ such that:

$$\psi(sd(fx, fy)) \leq c.\psi(d(x, y)) - \varphi(d(x, y)) \tag{13}$$

holds for all $x, y \in X$. Then f has a unique fixed point.

From Theorem 1, we can derive many interesting fixed point results in rectangular b-metric spaces involving contractive conditions of integrable type.

Denote by Λ the set of functions $\gamma : [0, +\infty) \to [0, +\infty)$ which are Lebesgue integrable on each compact subset of $[0, +\infty)$ such that for every $\varepsilon > 0$, we have $\int_0^{\varepsilon} \gamma(s)ds > 0$.

Theorem 3. Let (X, d) be a b-rectangular metric space with coefficient $s > 1$ and let $f, g : X \longrightarrow X$ be two self maps such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. Suppose that there exist $\alpha, \beta \in \Lambda$ such that:

$$\int_0^{sd(fx,fy)} \alpha(s)ds \leq \int_0^{\frac{1}{s}d(gx,gy)} \alpha(s)ds - \int_0^{d(gx,gy)} \beta(s)ds \tag{14}$$

holds for all $x, y \in X$, then f and g have a unique point of coincidence. If, moreover, f and g are weakly compatible, then they have a unique common fixed point.

Proof. The function $t \in [0, +\infty) \to \int_0^t \alpha(s)ds$ belongs to Ψ and the function $t \in [0, +\infty) \to \int_0^t \beta(s)ds$ belongs to Φ. Then the result follows immediately from Theorem 1.

Taking $g = I_X$ the identity mapping on X in Theorem 3, we obtain the following fixed point result.

Corollary 3.Let (X, d) be a complet rectangular b-metric space with coefficient $s > 1$ and let $f : X \longrightarrow X$ be a self-mapping on X such that the following condition holds:

$$\int_0^{sd(fx,fy)} \alpha(s)ds \leq \int_0^{\frac{1}{s}d(x,y)} \alpha(s)ds - \int_0^{d(x,y)} \beta(s)ds \tag{15}$$

holds for all $x, y \in X$, where $\alpha, \beta \in \Lambda$. Then f has a unique fixed point.

Theorem 4. Let (X, d) be a b-rectangular metric space with coefficient $s > 1$, $c \in [0, 1)$ and let $f, g : X \longrightarrow X$ be two self maps such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. Suppose that there exist $\alpha, \beta \in \Lambda$ such that:

$$\int_0^{sd(fx,fy)} \alpha(s)ds \leq c. \int_0^{d(gx,gy)} \alpha(s)ds - \int_0^{d(gx,gy)} \beta(s)ds \tag{16}$$

holds for all $x, y \in X$, then f and g have a unique point of coincidence. If, moreover, f and g are weakly compatible, then they have a unique common fixed point.

Taking $g = I_X$ in Theorem 4, we obtain the following fixed point result.

Corollary 4. Let (X, d) be a complet rectangular b-metric space with coefficient $s > 1$, $c \in [0, 1)$ and let $f : X \longrightarrow X$ be a self-mapping on X such that the following condition holds:

$$\int_0^{sd(fx,fy)} \alpha(s)ds \leq c. \int_0^{d(x,y)} \alpha(s)ds - \int_0^{d(x,y)} \beta(s)ds \tag{17}$$

holds for all $x, y \in X$, where α, $\beta \in \Lambda$. Then f has a unique fixed point.

Remark. Corollary 4 extends Theorem 2 from [3] from rectangular to rectangular b-metric spaces.

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