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TRANSLATION-FACTORABLE SURFACES IN THE 3-DIMENSIONAL EUCLIDEAN AND LORENTZIAN SPACES SATISFYING $\Delta r_i = \lambda_i r_i$

SID AHMED DIFI, HAKEM ALI AND HANIFI ZOUBIR

Abstract. This paper deals with the Translation-Factorable (TF) surfaces in the 3-dimensional Euclidean space and Lorentzian-Minkowski space with the condition $\Delta r_i = \lambda_i r_i$ where Δ denotes the Laplace operator. Our result will be obtained for the complete classification theorems and give an explicit forms of these surfaces.

1. Introduction

In 1983 B-Y Chen introduced the notion of Euclidian immersions of finite type. Basically there are submanifolds whose into $\mathbb{R}^m$ is constructed by making use of finite number of $\mathbb{R}^m$-valued eigenfunctions of their Lapalacian. Many works were done to characterize the classification of submanifolds in terms of finite type. Important results about 2-type spherical closed submanifolds (where spherical means into a sphere) have been obtained see [9].
A well known are the only surfaces in $\mathbb{R}^3$ satisfying the condition

$$\Delta r = \lambda r \quad \lambda \in \mathbb{R}$$

where Δ is the Laplace operator associated with the induced metric.
On the other hand Garay [13] determined the complete surfaces of revolution in $\mathbb{R}^3$ whose component functions are eigenfunctions of their Laplace operator i.e.

$$\Delta r^i = \lambda^i r^i \quad \lambda^i \in \mathbb{R}$$

Later Lopez [16] studied the hypersurfaces in $\mathbb{R}^{n+1}$ verifing

$$\Delta r = \lambda r \quad A \in \mathbb{R}^{n+1*n+1}$$

Kaimakamis and Papantounion [7] studied surfaces of revolution in the 3-dimensional Lorentz-Minkowski space satisfying the condition

$$\Delta^{II} r = Ar$$

where Δ^{II} is the Laplace operator with respect to the second fundamental form and A is a real 3×3 array.

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Zoubir and Bekkar [8] classified the surfaces of revolution with non zero Gaussian curvature K_G in the 3-dimensional Euclidean space $\mathbf{E}^3$ and Lorentzian-Minkowski spaces under the condition

$$\Delta r^i = \lambda^i r^i. \quad \lambda^i \in \mathbb{R}$$

Baba Hamed, Bekkar and Zoubir [4] determined the translation surfaces in the 3-dimensional Lorentz-Minkowski space $\mathbf{E}_1^3$, whose component functions are eigenfunctions of their Laplace operator. Baba Hamed, Bekkar [3] studies the helicoidal surfaces without parabolic points in $\mathbf{E}_1^3$, which satisfy the condition

$$\Delta^{II} r_i = \lambda_i r$$

Bekkar and Senoussi [6] studied the factorable surfaces in the 3-dimensional Minkowski space under the condition

$$\Delta r_i = \lambda_i r$$

where $\lambda_i \in \mathbb{R}$ and dr_i are the coordinate of the surface. There has been classification of factorable surface in the 3-dimensional Lorentz-Minkowski Euclidian and pseudo-Galilean space. Lopezand and Moruz [17] studied translation and homothetical surfaces with constant minimal homothetical non degenerate surfaces in Euclidian in $\mathbf{E}_1^3$

In this paper we classify the factorable surfaces in the 3-dimensional Euclidian space $\mathbf{E}^3$ and lorentzian $\mathbf{E}_1^3$ under the condition

$$\Delta r_i = \lambda_i r_i \tag{1}$$

where $\lambda_i \in \mathbb{R}$

2. Preliminaries

Let $\mathbf{E}^3$ be the 3-dimensional Euclidian space, equipped with the inner product

$$g(X, Y) = x_1 y_1 + x_2 y_2 + x_3 y_3$$

for $X = (x_1, x_2 x_3)$, $Y = (y_1, y_2, y_3) \in \mathbf{E}^3$

Let $\mathbf{E}_1^3$ be the 3-dimensional Minkowski space, with the scalar product given by

$$g_L = -dx^2 + dy^2 + dz^2$$

where (x, y, z) is a standard rectangular coordinate system of $\mathbf{E}_1^3$

Let $r : \mathbf{M}^2 \to \mathbf{E}_1^3$ be an isometric immersion of a surface in the 3-dimensional Lorentzian-Minkowski space.

A surface $\mathbf{M}^2$ is said to be of finite type if every component of its position vector field r can be written as a finite sum of eigenfunction of the Laplace Δ of $\mathbf{M}^2$, if

$$r = r_0 + \sum_{i=1}^{k} r_i$$

Definition 2.1 (4-15)**.** *A surface M is a translation surface if it can be parametrized by*

$$x(u, v) = (u, v, f(u) + g(v)) \tag{2}$$

Definition 2.2 (6-18)**.** *A surface M is a factorable surface if it can be parameterized by*

$$x(u, v) = (u, v, f(u)g(v)) \tag{3}$$

Next, we define an extended surface in $\mathbf{E}^3$ using definitions we call it TF-type surface as follows:

Definition 2.3. *A surface M is a TF-type surface if it can be parameterized by*

$$x(u,v) = (u, v, A(f(u) + g(v)) + Bf(u)g(v)), \tag{4}$$

where A and B are non-zero real numbers.

Remark 2.4. *In [4], we have if $A \neq 0$ and $B = 0$ in, then surface is a translation surface. In [14], we have if $A = 0$ and $B \neq 0$, then surface is a factorable surface.*

For vector $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ in $\mathbf{E}^3_1$, the Lorentz scalar product and the cross product are defined by :

$$g_L = -x_1y_1 + x_2y_2 + x_3y_3$$

The Gauss curvature and the mean curvature are:

$$K_G = g_L(\mathbf{N}, \mathbf{N}) \left(\frac{LN - M^2}{EG - F} \right), \quad H = \frac{EN + GL - 2FM}{2|EG - F^2|}$$

Let x^i, x^j be a local coordinate system of $\mathbf{M}^2$. For the array $(g_{i,j})$ $(i, j = 1, 2)$ consisting of components of the induced metric on $\mathbf{M}^2$, we denote by $(g^{i,j})$ the inverse matrix of the array $(g_{i,j})$. Then the Laplacian operator Δ on $\mathbf{M}^2$ is given by:

$$\Delta = \frac{-1}{\sqrt{|D|}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{|D|} g^{i,j} \frac{\partial}{\partial x^j} \right) \tag{5}$$

A vector V of $\mathbf{E}^3_1$ is said to be timelike if $g_L(V, V) < 0$, spacelike if $g_L(V, V) > 0$ or $V = 0$ and lightlike or null if $g_L(V, V) = 0$ and $V \neq 0$. A surface in $\mathbf{E}^3_1$ is spacelike, timelike or lightlike if the tangent plane at any point is spacelike, timelike or lightlike respectively [19].

3. Translation-factorable surfaces in $\mathbf{E}^3$

In this section, we consider surface in $\mathbf{E}^3$. Assume that $\mathbf{M}^2$ is equivalent to

$$r(u,v) = (u, v, f(u)g(v) + f(u) + g(v)) \tag{6}$$

the coefficients of the first fundamental form are:

$$E = (f'g + f')^2 + 1, \quad F = (f'g + f')(fg' + g'), \quad G = (fg' + g')^2 + 1$$

$$\mathbf{N} = \frac{1}{W}(-f'g - f', -fg' - g', 1) \tag{7}$$

the coefficients of the second fundamental form are:

$$L = \frac{f''g + f''}{W}, \quad M = \frac{(f'g + f')(fg' + g')}{W}, \quad N = \frac{fg'' + g''}{W}$$

where $W = \sqrt{(f'g + f')^2 + (fg' + g')^2 + 1}$

The Laplacian Δ of $\mathbf{M}^2$ is given by:

$$\Delta = \frac{1}{W^2} \left(E\frac{\partial^2}{\partial v^2} + G\frac{\partial^2}{\partial u^2} - 2F\frac{\partial^2}{\partial u \partial v} \right) + \frac{2H}{W} \left((f'g + f')\frac{\partial}{\partial u} + (fg' + g')\frac{\partial}{\partial v} \right) \tag{8}$$

$$\Delta u = \lambda_1 u; \; \Delta v = \lambda_2 v; \; \Delta(f(u)g(v) + f(u) + g(v)) = \lambda_3(f(u)g(v) + f(u) + g(v)) \tag{9}$$

By using (1), (7) we get:

$$2(f'g + f')H = W\lambda_1 u \tag{10}$$

$$2(fg' + g')H = W\lambda_2 v \tag{11}$$

$$2H = -W\lambda_3(fg + f + g) \tag{12}$$

Next, we explore the classification of the Translation-Factorable surfaces $\mathbf{M}^2$ satisfying (1)

Case 1: Let $\lambda_3 \neq 0$.

(i) If $fg + f + g = 0$, then $H = 0$

(ii) If $fg + f + g \neq 0$ we have:

(k_1) If $\lambda_1 = 0$ and $\lambda_2 \neq 0$, equations (10) and (11) imply that:

$$f(u) = a \in \mathbb{R} - \{-1\}, \quad g' \neq 0 \quad and \quad H = \frac{(a+1)g''}{2W^3}$$

The system of equations (10), (11) and (12) becomes

$$(1+a)^2 g'g'' = \lambda_2 v((a+1)^2 g'^2 + 1)^2 \tag{13}$$

$$(1+a)g'' = -\lambda_3(ag + a + g)((a+1)^2 g'^2 + 1)^2 \tag{14}$$

Equation (14) is equivalent to

$$g(v) = \frac{1}{(a+1)}\left(-a \pm \sqrt{\frac{-\lambda_2 v^2 + a^2\lambda_3}{\lambda_3}}\right) \qquad (-1 < -\lambda_2 v^2 + a^2\lambda_3 < 0)$$

Hence, the surface $\mathbf{M}^2$ can be expressed by

$$r\left(u, v, \pm\sqrt{\frac{-\lambda_2 v^2 + a^2\lambda_3}{\lambda_3}}\right) \qquad (-1 < -\lambda_2 v^2 + a^2\lambda_3 < 0)$$

(k_2) If $\lambda_2 = 0$ and $\lambda_1 \neq 0$. Equations (10) and (11) imply that:

$$f(u) = c \in \mathbb{R} - \{-1\}, \quad g' \neq 0 \quad and \quad H = \frac{(c+1)f''}{2W^3}$$

The system of equations (10), (11) and (12), in this case takes the form

$$(1+c)^2 f'f'' = \lambda_2 u((c+1)^2 f'^2 + 1)^2 \tag{15}$$

$$(1+c)f'' = -\lambda_3(cf + c + f)((c+1)^2 f'^2 + 1)^2 \tag{16}$$

Equation (16) is equivalent to

$$f(u) = \frac{1}{(a+1)}\left(-a \pm \sqrt{\frac{-\lambda_2 u^2 + a^2\lambda_3}{\lambda_3}}\right) \qquad (-1 < -\lambda_2 u^2 + a^2\lambda_3 < 0)$$

Hence, the surface $\mathbf{M}^2$ can be expressed by:

$$r\left(u, v, \pm\sqrt{\frac{-\lambda_2 u^2 + a^2\lambda_3}{\lambda_3}}\right) \qquad (-1 < -\lambda_2 u^2 + a^2\lambda_3 < 0)$$

(k_3) If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Equations (10) and (11) imply that:

$$f' \neq 0 \quad and \quad g' \neq 0$$

We multiply Equation (10) by $fg' + g'$ and Equation (10) by $f'g + f'$, we obtain:

$$\frac{(f+1)}{f'}\lambda_1 u = \frac{(g+1)}{g'}\lambda_2 v = e, \qquad e \in \mathbb{R}^* \tag{17}$$

Equations (10) and (12) imply that:

$$\lambda_1 u = -\lambda_3(fg + f + g)(f'g + f) \tag{18}$$

equations (17) and (18) imply that:

$$-\lambda_3(fg + f + g)^2 = e \tag{19}$$

The functions f and g are constants, hence there are no Translation-Factorable surfaces in this cases satisfying (1)
(k_4) If $\lambda_1 = 0$ and $\lambda_2 = 0$ equations (10) and (11) imply that:

$$f' = 0 \qquad and \qquad g' = 0$$

Hence $\lambda_3 = 0$. Therefore, there are no Translation-Factorable surfaces in this cases satisfying (1)
Case 2: Let $\lambda_3 = 0$. Then, the equation (12) gives rise to $H = 0$ which means that the surfaces are minimal. We get also by the equations (10) and (11): $\lambda_1 = \lambda_2 = 0$
Finally:

Theorem 3.1. *Let $\mathbf{M}^2$ be a Translation-Factorable (TF) given by (6) in $\mathbf{E}^3$. Then $\mathbf{M}^2$ satisfies $\Delta r_i = \lambda_i r_i$, (i=1;2;3) if and only if the following statements hold*
(1) $\mathbf{M}^2$ has zero mean curvature
(2) $\mathbf{M}^2$ is parameterized as

$$\left(u, v, \pm\sqrt{\frac{-\lambda_2 v^2 + a^2\lambda_3}{\lambda_3}}\right) \qquad (-1 < -\lambda_2 v^2 + a^2\lambda_3 < 0)$$

(3) $\mathbf{M}^2$ is parameterized as

$$\left(u, v, \pm\sqrt{\frac{-\lambda_2 u^2 + a^2\lambda_3}{\lambda_3}}\right) \qquad (-1 < -\lambda_2 u^2 + a^2\lambda_3 < 0)$$

Translation-Factorable surfaces in $\mathbf{E}^3_1$. In this section, we consider surfaces in $\mathbf{E}^3_1$ and we investigate the classification of the Translation-Factorable satisfying (1). We distinguish $EG - F^2 > 0$ or $EG - F^2 < 0$.
Suppose that $\mathbf{M}^2$ is given by (6), the coefficients of the first and second fundamental forms are:

$$E = (f'g + f')^2 - 1; \qquad F = (f'g + f')(fg' + g'); \qquad G = 1 + (fg' + g)^2 \tag{20}$$

and

$$L = \frac{f''g + f''}{W}; \qquad M = \frac{(f'g + f)(fg' + g')}{W}; \qquad N = \frac{fg'' + g''}{W}$$

The mean curvature H is

$$H = 1/2W^{-3}H_1$$

Where $H_1 = (1 + (fg' + g')^2)(f''g + f'') + (f'g + f')^2 - 1)(fg'' + g'') - 2(f + 1)(g + 1)f'^2 g'^2$

Spacelike Translation-Factorable surfaces in $\mathbf{E}_1^3$. We investigate the spacelike translation and factorable surfaces in $\mathbf{E}_1^3$.
If we use (5), the Laplacian Δ of $\mathbf{M}^2$ is given by:

$$\Delta = \frac{1}{W^2}\left(E\frac{\partial^2}{\partial v^2} + G\frac{\partial^2}{\partial u^2} - 2F\frac{\partial^2}{\partial u \partial v}\right) - \frac{2H}{W}\left((fg'+g')\frac{\partial}{\partial v} - (f'g+f')\frac{\partial}{\partial u}\right) \tag{21}$$

where $W = \sqrt{EG - F^2}$.
Assume that $EG - F^2 = (f'g+f')^2 - (fg'+g')^2 - 1 > 0$, the metric of $\mathbf{M}^2$ is spacelike.
Then using (1), (20) and (21) we have:

$$W^{-4}(f'g+f')H_1 = \lambda_1 u \tag{22}$$

$$W^{-4}(fg'+g')H_1 = -\lambda_2 v \tag{23}$$

$$W^{-4}H_1 = \lambda_3(fg+f+g) \tag{24}$$

First, we examine the classification of the spacelike Translation-Factorable surfaces $\mathbf{M}^2$ satisfying (1).
Case 1: Let $\lambda_3 = 0$, then, the equation (24) gives rise to $H_1 = 0$ meaning that the surface are minimal. We get also by the equations (22) and (23) $\lambda_1 = \lambda_2 = 0$.
Case 2: Let $\lambda_3 \neq 0$, then $H_1 \neq 0$ and hence we have necessarily by equation (22) $\lambda_1 \neq 0$.
i) If $\lambda_2 = 0$ we get (23) , so $g(v) = a$, $a \in \mathbb{R} - \{-1\}$
In this case, the system of equations (22),(23) and (24) takes the form:

$$(a+1)^2 f'f'' = \lambda_1 u((a+1)^2 f'^2 - 1)^2 \tag{25}$$

$$(a+1)f'' = \lambda_3(af+f+a)((a+1)^2 f'^2 - 1)^2 \tag{26}$$

Using equation (26)

$$f(u) = \frac{1}{(a+1)}\left(-a \pm \sqrt{\frac{\lambda_2 u^2 + a^2\lambda_3}{\lambda_3}}\right) \quad \textit{which such that } \lambda_2 u^2 + a^2\lambda_3 < 1$$

So the parametrization of the surfaces can be written in the form:

$$r(u,v) = \left(u, v, \pm\sqrt{\frac{\lambda_2 u^2 + a^2\lambda_3}{\lambda_3}}\right)$$

ii) If $\lambda_2 \neq 0$ we can rewrite the system as follow:

$$\begin{cases} \lambda_2 v(fg+g+f) = -a(fg'+g') \\ \lambda_1 u(fg+g+f) = a(f'g+f') \end{cases} \tag{27}$$

Equation (21) and (22) $(a \neq -1)$ imply that:

$$\lambda_1 u = \lambda_3(fg+f+g)(f'g+f') \tag{28}$$

From (27) and (28) we obtain:

$$a = \lambda_3(fg+f+g)^2$$

Therefore the functions f and g are constants assuming that there are no Translation-Factorable surfaces in this case satisfying (1). Thus, we can give the following result:

Theorem 3.2. *Let $\mathbf{M}^2$ be a spacelike Translation-Factorable (TF) given by (6) in $\mathbf{E}_1^3$. Then $\mathbf{M}^2$ satisfies $\Delta r_i = \lambda_i r_i$, (i=1;2;3) if and only if the following statements hold*
(1) $\mathbf{M}^2$ has zero mean curvature
(2) $\mathbf{M}^2$ is parameterized as

$$r(u,v) = \left(u, v, \pm\sqrt{\frac{\lambda_2 u^2 + a^2\lambda_3}{\lambda_3}}\right) \quad (0 < \lambda_2 u^2 + a^2\lambda_3 < 1)$$

Timelike Translation-Factorable surfaces in $\mathbf{E}_1^3$. In this section, we deal with the spacelike translation-factorable surfaces in $\mathbf{E}_1^3$.
If we use (5) , the Laplacian Δ of $\mathbf{M}^2$ is given by:

$$\Delta = \frac{1}{W^2}\left(E\frac{\partial^2}{\partial v^2} + G\frac{\partial^2}{\partial u^2} - 2F\frac{\partial^2}{\partial u \partial v}\right) - \frac{2H}{W}\left((fg' + g')\frac{\partial}{\partial v} - (f'g + f')\frac{\partial}{\partial u}\right) \tag{29}$$

where $W = \sqrt{F^2 - EG}$.
Assuming that $EG - F^2 = (f'g + f')^2 - (fg' + g')^2 - 1 < 0$, the metric of $\mathbf{M}^2$ is timelike.
Then using (29) and (20) we get

$$\begin{cases} \Delta(u) = -W^{-4}(f'g + f')H_1 \\ \Delta(v) = W^{-4}(fg' + g')H_1 \\ \Delta(fg + f + g) = -W^{-4}H_1 \end{cases} \tag{30}$$

hence

$$\Delta r = W^{-4}H_1(-f'g + f', fg' + g', -1) \tag{31}$$

By (1) and (30) we obtain the following system of differential equations

$$W^{-4}(f'g + f')H_1 = -\lambda_1 u \tag{32}$$

$$W^{-4}(fg' + g')H_1 = \lambda_2 v \tag{33}$$

$$W^{-4}H_1 = \lambda_3(fg + f + g) \tag{34}$$

We explore the classification of the timelike Translation-Factorable surfaces $\mathbf{M}^2$ satisfying (1.1) .
Case 1: Let $\lambda_3 = 0$, then, the equation (35) gives rise to $H_1 = 0$, which means that the surfaces are minimal. We have also by the equations (33) and (34) $\lambda_1 = \lambda_2 = 0$.
Cases 2: Let $\lambda_3 \neq 0$.
i) If $fg + f + g = 0$, then $H_1 = 0$
ii) If $fg + f + g \neq 0$, in this case we have:
a) If $\lambda_1 = 0$ and $\lambda_2 \neq 0$ equations (33) and (34) imply that:

$$f' = 0, \quad g' \neq 0, \quad \textit{and } g'' \neq 0$$

It follows that $f(u) = a$, $a \in \mathbb{R} - \{-1\}$ and $g'(v)$ is a non constant function.
The system (33), (34) and (35) is reduced to be equivalent to

$$-(a+1)^2 g'g'' = \lambda_2 v((a+1)^2 g'^2 + 1)^2 \tag{35}$$

$$(a+1)g'' = \lambda_3(ag + g + a)((a+1)^2 f'^2 + 1)^2 \tag{36}$$

Equation (36) implies

$$g(v)=\frac{1}{(a+1)}\left(-a\pm\sqrt{\frac{-\lambda_2 v^2+a^2\lambda_3}{\lambda_3}}\right)\quad \textit{which such that } 0<-\lambda_2 v^2+a^2\lambda_3<1$$

So the parametrization of the surfaces can be written in the form:

$$r(u,v)=\left(u,v,\pm\sqrt{\frac{-\lambda_2 v^2+a^2\lambda_3}{\lambda_3}}\right)\quad \textit{which such that } 0<-\lambda_2 v^2+a^2\lambda_3<1$$

ii) If $\lambda_2=0$ and $-\lambda_2 v^2+a^2\lambda_3\neq 0$ then

$$g'=0,\qquad f'\neq 0,\qquad f''\neq 0$$

The system (33), (34) and (35) is reduce equivalently to

$$-(a+1)^2 g'g''=\lambda_1 u((a+1)^2 g'^2+1)^2 \tag{37}$$

$$-(a+1)f''=\lambda_3(af+f+a)(1-(a+1)^2 f'^2)^2 \tag{38}$$

Hence

$$f(u)=\frac{1}{(a+1)}\left(-a\pm\sqrt{\frac{\lambda_2 u^2+a^2\lambda_3}{2\lambda_3}}\right)\quad \textit{which such that } 0<\lambda_2 u^2+a^2\lambda_3<1$$

So the parametrization of the surfaces can be written in the form

$$r(u,v)=\left(u,v,\pm\sqrt{\frac{\lambda_2 u^2+a^2\lambda_3}{\lambda_3}}\right)\quad \textit{which such that } \lambda_2 u^2+a^2\lambda_3<1$$

c) If $\lambda_1=\lambda_2=0$ we have:
i) If $f'=g'=0$ imply $H_1=0$. From (35) we obtain $\lambda_3=0$ which is a contradiction.
ii) If $f'=0$ and $g'\neq 0$, then (34) gives $g=0$, which is a contradiction.
iii) If $f'\neq 0$ and $g'=0$, then (33) gives $g=0$, which is a contradiction.
3) If $\lambda_1\neq 0$ and $\lambda_2\neq 0$, then

$$f'\neq 0\qquad g'\neq 0$$

We multiply Equation (33) by $g'f+g'$ and (34) by $f'g+f'$, and we obtain:

$$\begin{cases}\lambda_2 v(fg+g+f)=a(fg'+g')\\ \lambda_1 u(fg+g+f)=-a(f'g+f')\end{cases} \tag{39}$$

Equation (33) and (33) $(a\neq -1)$ imply

$$\lambda_1 u=\lambda_3(fg+f+g)(f'g+f') \tag{40}$$

From (39) and (40) we obtain:

$$-a=\lambda_3(fg+f+g)^2$$

The functions f and g are constants and hence there are no Translation-Factorable surfaces in this case satisfying (1). Thus we can give the following result:

Theorem 3.3. *Let $\mathbf{M}^2$ be a timelike Translation-Factorable (TF) given by (6) in $\mathbf{E}_1^3$. Then $\mathbf{M}^2$ satisfies $\Delta r_i = \lambda_i r_i$, (i=1;2;3) if and only if the following statements hold*
(1) $\mathbf{M}^2$ has zero mean curvature
(2) $\mathbf{M}^2$ is parameterized as

$$r(u,v) = \left(u, v, \pm\sqrt{\frac{-\lambda_2 v^2 + a^2\lambda_3}{\lambda_3}} \right) \quad (0 < -\lambda_2 v^2 + a^2\lambda_3 < 1)$$

(3) $\mathbf{M}^2$ is parameterized as

$$r(u,v) = \left(u, v, \pm\sqrt{\frac{\lambda_2 u^2 + a^2\lambda_3}{\lambda_3}} \right) \quad (0 < \lambda_2 u^2 + a^2\lambda_3 < 1)$$

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Sid Ahmed Difi

Djillali Liabes University, Faculty of Technology, Department of Basic Teaching in Science and Technology, 22000 Sidi Bel Abbes, Algeria

E-mail address: sidahmedmt@yahoo.fr

Hakem Ali

Djillali Liabes University, Faculty of Technology, Department of Basic Teaching in Science and Technology, 22000 Sidi Bel Abbes, Algeria

E-mail address: hakemali@yahoo.com

Hanifi Zoubir

Polytechnical School of Oran, Faculty of Technology, Department of Basic Teaching in Science and Technology, 31000 Oran, Algeria

E-mail address: zoubirhanifi@yahoo.fr