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SOME GEOMETRIC PROPERTIES OF ANALYTIC SERIES WHOSE COEFFICIENTS ARE RECIPROCAL OF FUSS-CATALAN NUMBERS

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Abstract. In the present investigation we first introduce two parameter family of function namely $\mathbb{E}_{p,r}(z)$ and then find sufficient conditions so that the function $\mathbb{E}_{p,r}(z)$ have certain geometric properties like close-to-convexity and starlikeness in the open unit disk. Some interesting consequences of main results are also pointed out in the form of corollaries.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions inside the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and consider the subclass $\mathcal{A} = \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}$ which consist functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We denote by $\mathcal{S}$, the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{D}$ i. e.

$$\mathcal{S} = \{f \in \mathcal{A} |\ f \text{ is one-to-one in } \mathbb{D}\}.$$

A function $f \in \mathcal{A}$ is called starlike (with respect to 0), denoted by $f \in \mathcal{S}^*$ if $tw \in f(\mathbb{D})$ for all $w \in f(\mathbb{D})$ and $t \in [0, 1]$. A function $f \in \mathcal{A}$ that maps $\mathbb{D}$ onto a convex domain is called convex function and class of such functions is denoted by $\mathcal{K}$. For a given $0 < \alpha \leq 1$, a function $f \in \mathcal{A}$ is called starlike function of order α, denoted by $\mathcal{S}^*(\alpha)$, if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in \mathbb{D}.$$

For a given $0 < \alpha \leq 1$, a function $f \in \mathcal{A}$ is called convex function of order α, denoted by $\mathcal{K}(\alpha)$, if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in \mathbb{D}.$$

It is well known that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$. We recall [5] that the function $zg'(z)$ is starlike if and only if the function $g(z)$ is convex.

Given a convex function $g \in \mathcal{K}$ with $g(z) \neq 0$ and $0 < \alpha \leq 1$, a function $f \in \mathcal{A}$,

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is called close-to-convex of order α with respect to convex function g, denoted by $\mathcal{C}_g(\alpha)$, if

$$\Re\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha,\ z \in \mathbb{D} \tag{1.2}$$

The class $\mathcal{C}_g(0)$ is the class of functions close-to-convex with respect to g. Geometrically a function $f \in \mathcal{A}$ belongs to $\mathcal{C}$ if the complement E of the image-region $F = \{f(z) : |z| < 1\}$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays). The Noshiro-Warschawski theorem implies that close-to-convex functions are univalent in $\mathbb{D}$, but not necessarily the converse. It is easy to verify that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C}$. For more details see [5].

If $f, g \in \mathcal{H}$ where $\mathcal{H}$ denote the class of all holomorphic functions, then the function f is said to be subordinate to g, written as $f(z) \prec g(z)$ $(z \in \mathbb{D})$, if there exists a Schwarz function $w \in \mathcal{H}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{D})$ such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{D}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

It is always interesting to find sufficient conditions such that certain class of analytic functions becomes close-to-convex, starlike or convex function. In the present investigation, we are interested in some geometric properties of analytic power series whose coefficients are reciprocal of Fuss-Catalan numbers.

A definition of the Fuss-Catalan numbers: Catalan numbers $\{c_n\}_{n\geq 0}$ are said to be the sequence satisfying the recursive relation

$$c_{n+1} = c_0c_n + c_1c_{n-1} + ... + c_nc_0, \quad c_0 = 1. \tag{1.3}$$

It is well known that the nth term of Catalan numbers is

$$c_n = \frac{1}{2n+1}\binom{2n+1}{n}$$

i.e. $\{c_n\}_{n\geq 0} = \{1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ ...\}$. Also, one of many combinatorial interpretations of Catalan numbers is that c_n is the number of shortest lattice paths from $(0, 0)$ to (n, n) on the 2-dimensional plane such that those paths lie beneath the line $y = x$. Further generalization of Catalan numbers is Fuss-Catalan numbers $\{c_n^{(p)}\}_{p,n\geq 0}$, which were investigated by Fuss [7] and studied by several other authors [1, 2, 3, 4]. The following proposition gives some characteristic properties of Fuss-Catalan numbers:

If n and p are nonnegative integers, the following statements are equivalent:

(1) $c_n^{(p)} = \frac{1}{pn+1}\binom{pn+1}{n}$
(2) $c_{n+1}^{(p)} = \sum_{r_1+r_2+...+r_p=n} c_{r_1}^{(p)} \times c_{r_2}^{(p)} \times ... \times c_{r_p}^{(p)}, \quad c_0^{(p)} = 1$
(3) $c_n^{(p)}$ *is the number of shortest lattice paths from* $(0, 0)$ *to* $(n, (p-1)n)$ *on the 2-dimensional plane such that those paths lie beneath* $y = (p-1)x$.

Catalan numbers $\{c_n\}$ are special case of Fuss-Catalan numbers $\{c_n^{(2)}\}$ for $p = 2$. In combinatorial mathematics and statistics, the two parameter Fuss-Catalan numbers

$A_n(p,r)$ are defined in [7] as numbers of the form

$$\begin{aligned} A_n(p,r) &= \frac{r\Gamma(np+r)}{\Gamma(n+1)\Gamma[n(p-1)+r+1]} \\ &= \frac{r}{np+r}\binom{np+r}{n} \ (n \geq 0, p \in \{2,3,\cdots\}, r \in \{1,2,3,\cdots\}). \end{aligned} \tag{1.4}$$

The Fuss Catalan numbers $A_n(p,r)$ can also be written in the following form

$$A_n(p,r) = \frac{r}{n!}\prod_{i=1}^{n-1}(np+r-i). \tag{1.5}$$

It is easy to see that

$$A_n(p,r) = A_n(p,r-1) + A_{n-1}(p,p+r-1), \tag{1.6}$$

under convention that $A_{-1}(p,r) := 0$, and

$$A_n(p,p) = A_{n+1}(p,1). \tag{1.7}$$

In the present paper, we study geometric properties of two parameter family of functions of the form:

$$E_{p,r}(z) : = \sum_{n=1}^{\infty}\frac{1}{A_n(p,r)}z^n \quad (z \in \mathbb{D},\ p \in \{2,3,\cdots\}, r \in \{1,2,3,\cdots\}). \tag{1.8}$$

Observe that, the function $E_{p,r}(z)$ does not belong to the family $\mathcal{A}$. Thus, it is natural to consider the following normalization of function $E_{p,r}(z)$ in $\mathbb{D}$:

$$\begin{aligned} \mathbb{E}_{p,r}(z) &= A_1(p,r)E_{p,r}(z) \\ &= z + \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma[n(p-1)+r+1]}{\Gamma(np+r)}z^n, \quad (\text{as } A_1(p,r) = r) \end{aligned} \tag{1.9}$$

Using (1.5), (1.9) can be written as

$$\mathbb{E}_{p,r}(z) := z + \sum_{n=2}^{\infty}\frac{n!}{\prod\limits_{i=1}^{n-1}(np+r-i)}z^n. \tag{1.10}$$

$$(z \in \mathbb{D},\ p \in \{2,3,\cdots\}, r \in \{1,2,3,\cdots\})$$

To prove our main results we need following Definition and Lemmas:

Lemma 1.1. (Ozaki [8]). *Let* $f(z) = z + \sum\limits_{n=2}^{\infty} A_n z^n$. *Suppose*

$$1 \geq 2A_2 \geq \cdots \geq nA_n \geq (n+1)A_{n+1} \geq \cdots \geq 0 \tag{1.11}$$

or

$$1 \leq 2A_2 \leq \cdots \leq nA_n \leq (n+1)A_{n+1} \leq \cdots \leq 2. \tag{1.12}$$

then f *is close-to-convex with respect to convex function* $-log(1-z)$ *in* $\mathbb{D}$.

Lemma 1.2. (Fejer [6]). *Let* $\{a_n\}_{n\geq 1}$ *be a sequence of non negative real numbers such that* $a_1 = 1$. *If the quantities*

$$\underline{\Delta} a_n = na_n - (n+1)a_{n+1} \quad \textit{and} \quad \underline{\Delta} a_n^2 = na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2}$$

are non negative, then the function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *is starlike in* $\mathbb{D}$.

Lemma 1.3. (Fejer [6]). *Let* $\{a_n\}_{n\geq 1}$ *be a sequence of non negative real numbers such that* $a_1 = 1$. *If* $\{a_n\}_{n\geq 2}$ *is convex decreasing, i.e.* $0 \geq a_{n+2} - a_{n+1} \geq a_{n+1} - a_n$, *then*

$$\Re\left\{\sum_{n=1}^{\infty} a_n z^{n-1}\right\} > \frac{1}{2}, \quad (z \in \mathbb{D}).$$

Definition 1.1. *An infinite sequence* $\{b_n\}_1^{\infty}$ *of complex numbers will be called a subordinating factor sequence if whenever*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{1.13}$$

is analytic, univalent and convex in $\mathbb{U}$, *then*

$$\sum_{n=1}^{\infty} a_n b_n z^n \subseteq f(z) \; (z \in \mathbb{D}, a_1 = 1). \tag{1.14}$$

Lemma 1.4. (Wilf [9]). *The sequence* $\{b_n\}_1^{\infty}$ *is a subordinating factor sequence if and only if*

$$\Re\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \; (z \in \mathbb{D}). \tag{1.15}$$

2. Close-to-convexity and Starlikeness

Theorem 2.1. *For all* $p \geq 2$, $r \geq 1$, $\mathbb{E}_{p,r}(z)$ *is close-to-convex with respect to convex function* $-log(1-z)$ *in* $\mathbb{D}$.

Proof. Using (1.10), we have

$$\begin{aligned} \underline{\Delta} a_n &= na_n - (n+1)a_{n+1} \\ &= \frac{n\, n!}{\prod\limits_{i=1}^{n-1}(np+r-i)} - \frac{(n+1)\,(n+1)!}{\prod\limits_{i=1}^{n}((n+1)p+r-i)} \\ &= \frac{n!}{\left(\prod\limits_{i=1}^{n-1}(np+r-i)\right)\left(\prod\limits_{i=1}^{n}((n+1)p+r-i)\right)} X(n), \end{aligned} \tag{2.1}$$

where

$$X(n) = n((n+1)p+r-n)\prod_{i=1}^{n-1}((n+1)p+r-i) - (n+1)^2\prod_{i=1}^{n-1}(np+r-i). \tag{2.2}$$

It is easy to see from (2.2) that each term in the first finite product is greater than the corresponding term in the second finite product, also we observe that

$$n\,[(n+1)p+r-n] \geq (n+1)^2, \quad (\text{ for all } n \geq 1; \; p \geq 2 \text{ and } r \geq 1).$$

Thus the sequence $\{na_n\}$ is non-increasing and applying Lemma 1.1, we get $\mathbb{E}_{p,r}(z)$ is close-to-convex with respect to convex function $-log(1-z)$ in $\mathbb{D}$. □

Theorem 2.2. *For all* $p \geq 2$ *and* $r \geq \max\{1, 9-2p\}$, $\mathbb{E}_{p,r}(z)$ *is starlike in* $\mathbb{D}$.

Proof. From (1.10), we have

$$\underline{\Delta} a_n^2 = na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2}$$

$$= n!\left[\frac{n}{\prod_{i=1}^{n-1}(np+r-i)} - \frac{2(n+1)^2}{\prod_{i=1}^{n}((n+1)p+r-i)} + \frac{(n+2)^2(n+1)}{\prod_{i=1}^{n+1}((n+2)p+r-i)}\right].$$

The Numerator of the difference of first two term is equal to

$$\begin{aligned} Nr. &= n\prod_{i=1}^{n}((n+1)p+r-i) - 2(n+1)^2\prod_{i=1}^{n-1}(np+r-i) \\ &= n((n+1)p+r-1)\prod_{i=2}^{n}((n+1)p+r-i) - 2(n+1)^2\prod_{i=1}^{n-1}(np+r-i) \end{aligned} \tag{2.3}$$

Again each term in the first finite product is greater than the corresponding term in the second finite product, also

$$n((n+1)p+r-1) \geq 2(n+1)^2$$
$$\iff n^2(p-2) + n(p+r-5) - 2 \geq 0,$$

if the above inequality is true for $n=1$ then it will be true for all n provided $p \geq 2$ and hence we get the condition that $r \geq max\{1, 9-2p\}$. Thus the sequence $\{na_n\}$ is convex decreasing sequence and hence using Lemma 1.2, we get the required result. □

Theorem 2.3. *For all $p \geq 2$, $r \geq 1$, we have*

$$\Re\left\{\frac{\mathbb{E}_{p,r}(z)}{z}\right\} > \frac{1}{2}\ (z \in \mathbb{D}) \tag{2.4}$$

Proof. We first prove that

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{n!}{\prod_{i=1}^{n-1}(np+r-i)}\right\}_{n=2}^{\infty};\ (a_1 = 1)$$

is a non-increasing sequence. Since

$$\begin{aligned} a_n - a_{n+1} &= \frac{n!\left[\prod_{i=1}^{n}((n+1)p+r-i) - (n+1)\prod_{i=1}^{n-1}(np+r-i)\right]}{\prod_{i=1}^{n-1}(np+r-i)\prod_{i=1}^{n}((n+1)p+r-i)} \\ &= \frac{n!\left[((n+1)p+r-n)\prod_{i=1}^{n-1}((n+1)p+r-i) - (n+1)\prod_{i=1}^{n-1}(np+r-i)\right]}{\prod_{i=1}^{n-1}(np+r-i)\prod_{i=1}^{n}((n+1)p+r-i)}. \end{aligned} \tag{2.5}$$

which is greater than equal to zero, as in numerator of (2.5), each term in first finite product is greater than the corresponding term in the second finite product, also

$$\begin{aligned} & (n+1)p + r - n \geq n+1 \\ \iff & n(p-2) + p + r - 1 \geq 0, \end{aligned}$$

which is true for all $n \geq 1$, $p \geq 2$ and $r \geq 1$. Next we prove that $\{a_n\}_{n=2}^{\infty}$ is a convex decreasing sequence for this we show $a_{n+2} - a_{n+1} \geq a_{n+1} - a_n$ for all $n \geq 2$. That is

$$a_n - 2a_{n+1} + a_{n+2} = n! \left[\frac{1}{\prod_{i=1}^{n-1}(np+r-i)} - \frac{2(n+1)}{\prod_{i=1}^{n}((n+1)p+r-i)} + \frac{(n+2)(n+1)}{\prod_{i=1}^{n+1}((n+2)p+r-i)} \right].$$

Again the Numerator of the difference of first two term is equal to

$$\begin{aligned} Nr. &= \prod_{i=1}^{n}((n+1)p+r-i) - 2(n+1)\prod_{i=1}^{n-1}(np+r-i) \\ &= [(n+1)p+r-1]\prod_{i=2}^{n}((n+1)p+r-i) - 2(n+1)\prod_{i=1}^{n-1}(np+r-i) \end{aligned} \tag{2.6}$$

Again each term in the first finite product is greater than the corresponding term in the second finite product, also

$$\begin{aligned} & (n+1)p + r - 1 \geq 2(n+1) \\ \iff & (n+1)(p-2) + r - 1 \geq 0, \end{aligned}$$

which is true for all $n \geq 2$, $p \geq 2$ and $r \geq 1$. Thus the sequence $\{a_n\}$ is convex decreasing sequence and in view of Lemma 1.3, we have

$$\Re\left\{\sum_{n=1}^{\infty} a_n z^{n-1}\right\} > \frac{1}{2}, \quad (z \in \mathbb{D}).$$

Which is equivalent to

$$\Re\left\{\frac{\mathbb{E}_{p,r}(z)}{z}\right\} > \frac{1}{2}, \ (z \in \mathbb{D}).$$

This proves the Theorem 2.3. □

Corollary 2.1. *For all $p \geq 2$, $r \geq 1$, then the sequence*

$$\left\{\frac{\Gamma(n+2)\Gamma[(n+1)(p-1)+r+1]}{\Gamma((n+1)p+r)}\right\}_{n=1}^{\infty} \tag{2.7}$$

is a subordinating factor sequence for the class $\mathcal{K}$.

Proof. From Theorem 2.3, we have

$$\Re\left\{1 + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma[n(p-1)+r+1]}{\Gamma(np+r)} z^{n-1}\right\} > \frac{1}{2}.$$

Multiplying above by 2, changing the summation index to 1 and simplifying, we get

$$\Re\left\{1 + 2\sum_{n=1}^{\infty} \frac{\Gamma(n+2)\Gamma[(n+1)(p-1)+r+1]}{\Gamma((n+1)p+r)} z^{n}\right\} > 0. \tag{2.8}$$

Now using Lemma 1.4, we get the required result. □

Theorem 2.4. *For all $p \geq 2$ and $r \geq \max\{1, 9-2p\}$, then*

$$\Re\{\mathbb{E}'_{p,r}(z)\} > \frac{1}{2} \ (z \in \mathbb{D}).$$

Proof. From (1.10),

$$\mathbb{E}'_{p,r}(z) := 1 + \sum_{n=2}^{\infty} \frac{n\, n!}{\prod_{i=1}^{n-1}(np+r-i)} z^{n-1}. \tag{2.9}$$

We first prove that

$$\{a_n\}_{n=1}^{\infty} = \left\{ \frac{nn!}{\prod_{i=1}^{n-1}(np+r-i)} \right\}_{n=2}^{\infty} ; \ (a_1 = 1)$$

is a non-increasing sequence. Since

$$\begin{aligned} a_n - a_{n+1} &= \frac{n!\left[n\prod_{i=1}^{n}((n+1)p+r-i) - (n+1)^2\prod_{i=1}^{n-1}(np+r-i)\right]}{\prod_{i=1}^{n-1}(np+r-i)\prod_{i=1}^{n}((n+1)p+r-i)} \\ &= \frac{n!\left[n((n+1)p+r-n)\prod_{i=1}^{n-1}((n+1)p+r-i) - (n+1)^2\prod_{i=1}^{n-1}(np+r-i)\right]}{\prod_{i=1}^{n-1}(np+r-i)\prod_{i=1}^{n}((n+1)p+r-i)}. \end{aligned} \tag{2.10}$$

which is greater than equal to zero, as in numerator of (2.10), each term in first finite product is greater than the corresponding term in the second finite product, also

$$\begin{aligned} & n((n+1)p+r-n) \geq (n+1)^2 \\ \iff & n^2(p-2) + n(p+r-2) - 1 \geq 0, \end{aligned}$$

which is true for all $n \geq 1$, $p \geq 2$ and $r \geq 1$. Next we prove that $\{a_n\}_{n=2}^{\infty}$ is a convex decreasing sequence for this we show $a_{n+2} - a_{n+1} \geq a_{n+1} - a_n$ for all $n \geq 2$. That is

$$a_n - 2a_{n+1} + a_{n+2} = n!\left[\frac{n}{\prod_{i=1}^{n-1}(np+r-i)} - \frac{2(n+1)^2}{\prod_{i=1}^{n}((n+1)p+r-i)} + \frac{(n+2)^2(n+1)}{\prod_{i=1}^{n+1}((n+2)p+r-i)}\right].$$

Again the Numerator of the difference of first two term is equal to

$$\begin{aligned} Nr. &= n\prod_{i=1}^{n}((n+1)p+r-i) - 2(n+1)^2\prod_{i=1}^{n-1}(np+r-i) \\ &= n((n+1)p+r-1)\prod_{i=2}^{n}((n+1)p+r-i) - 2(n+1)^2\prod_{i=1}^{n-1}(np+r-i) \end{aligned} \tag{2.11}$$

Again each term in the first finite product is greater than the corresponding term in the second finite product, also

$$\begin{aligned} n((n+1)p+r-1) &\geq 2(n+1)^2 \\ \iff n^2(p-2)+n(p+r-5)-2 &\geq 0, \end{aligned}$$

if the above inequality is true for $n=1$ then it will be true for all n provided $p \geq 2$ and hence we get the condition that $r \geq max\{1, 9-2p\}$. Thus the sequence $\{a_n\}$ is convex decreasing sequence and in view of Lemma 1.3, we have

$$\Re\left\{\sum_{n=1}^{\infty} a_n z^{n-1}\right\} > \frac{1}{2}, \quad (z \in \mathbb{D}).$$

Which is equivalent to

$$\Re\left\{\mathbb{E}'_{p,r}(z)\right\} > \frac{1}{2}, \ (z \in \mathbb{D}).$$

This proves the Theorem 2.4. □

Corollary 2.2. *For all* $p \geq 2$ *and* $r \geq \max\{1, 9-2p\}$*, then*

$$\left\{\frac{(n+1)\,(n+1)!}{\prod\limits_{i=1}^{n}[(n+1)p+r-i]}\right\}_{n=1}^{\infty} \tag{2.12}$$

a subordinating factor sequence for the class $\mathcal{K}$*.*

Proof. From Theorem 2.4, we have

$$\Re\left\{1+\sum_{n=2}^{\infty}\frac{n\,n!}{\prod\limits_{i=1}^{n-1}(np+r-i)}z^{n-1}\right\} > \frac{1}{2}, \ (z \in \mathbb{D}).$$

Multiplying above equation by 2, changing summation index to 1 and simplifying, we get

$$\Re\left\{1+2\sum_{n=1}^{\infty}\frac{(n+1)\,(n+1)!}{\prod\limits_{i=1}^{n}((n+1)p+r-i)}z^{n}\right\} > 0, \ (z \in \mathbb{D}).$$

Now using Lemma 1.4, we get the required result. □

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