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KÖTHE-TOEPLITZ DUALS AND MATRIX TRANSFORMATIONS OF GENERALIZED DIFFERENCE SEQUENCE SPACES

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Abstract. Orhan [28, 29] introduced the Cesàro difference sequence spaces $C_p, 1 \leq p < \infty$, and C_∞ and determined their generalized Köthe- Toeplitz duals and some of the related matrix transformations. We here propose to derive further properties concerning the space C_∞ (which we denote $C_\infty(\Delta)$ for the sake of notational uniformity) along with the introduction of a new difference sequence space $bv(\Delta)$. It is shown that the non-absolute type sequence spaces $C_\infty(\Delta)$ and $bv(\Delta)$ turn out to be BK spaces, the former of which is inseparable, perfect space and strictly includes the well-known difference sequence spaces $c_0(\Delta), c(\Delta)$ and $\ell_\infty(\Delta)$ of Kizmaz [22], the recently introduced [6, 7] Cesàro summable difference sequence space $C_1(\Delta)$ and the space $bv(\Delta)$, introduced in this paper itself, none of which happens to be perfect. The Köthe-Toeplitz duals of $bv(\Delta)$ are computed and as an application, the matrix classes $(bv(\Delta), \ell_\infty)$, $(bv(\Delta), c)$ and $(bv(\Delta), c_0)$ are also characterized.

1. Notations and terminology

By ω we shall denote the linear space of all complex sequences over $\mathbb{C}$ (the field of complex numbers). Any vector subspace of ω is called a sequence space. $\ell_\infty, c, c_o, \ell_1$ and C_1 denote the spaces of all bounded, convergent, null, absolutely summable and $(C, 1)$ summable sequences $x = (x_k)$ with complex terms, respectively. By cs we denote the space of all convergent series and bv denotes the space of all sequences of bounded variation. Throughout this paper, unless otherwise specified, we write $\sum_k$ for $\sum_{k=1}^{\infty}$ and $\lim_n$ for $\lim_{n\to\infty}$.

The following concepts are of long standing [2, 9, 21, 24].

A complete metric linear space is called a Frèchet space. Let X be a linear subspace of ω such that X is a Frèchet space with continuous coordinate projections. Then we say that X is an FK space. If the metric of an FK space is given by a complete norm then we say that X is a BK space.

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We say that an FK space X has AK, or has the AK property, if (e_k), the sequence of unit vectors, is a Schauder basis for X.

A sequence space X is called
(i) normal (or solid) if $y = (y_k) \in X$ whenever $|y_k| \leq |x_k|$, $k \geq 1$, for some $x = (x_k) \in X$,
(ii) monotone if it contains the canonical preimages of all its stepspaces,
(iii) sequence algebra if $xy = (x_k y_k) \in X$ whenever $x = (x_k), y = (y_k) \in X$,
(iv) convergence free when, if $x = (x_k)$ is in X and if $y_k = 0$ whenever $x_k = 0$, then $y = (y_k)$ is in X,
(v) symmetric if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$ where π is a permutation on $\mathbb{N}$.

The idea of dual sequence spaces was introduced by Köthe and Toeplitz [23] whose main results concerned α-duals; the α-dual of $X \subset \omega$ being defined as

$$X^\alpha = \{a = (a_k) \in \omega : \sum_k |a_k x_k| < \infty \quad for\ all \quad x = (x_k) \in X\}.$$

In the same paper [23], they also introduced another kind of dual, namely, the β-dual (see [8] also, where it is called the g-dual by Chillingworth) defined as

$$X^\beta = \{a = (a_k) \in \omega : \sum_k a_k x_k \quad converges\ for\ all \quad x = (x_k) \in X\}.$$

A still more general notion of a dual was introduced by Garling [21] as

$$X^\gamma = \{a = (a_k) \in \omega : \sup_k |\sum_{i=1}^{k} a_i x_i| < \infty \quad for\ all \quad x = (x_k) \in X\}.$$

Obviously $\phi \subset X^\alpha \subset X^\beta \subset X^\gamma$, where ϕ is the well-known sequence space of finitely non-zero scalar sequences. Also if $X \subset Y$, then $Y^\eta \subset X^\eta$ for $\eta = \alpha$, β or γ. For any sequence space X, we denote $(X^\delta)^\eta$ by $X^{\delta\eta}$ where $\delta, \eta = \alpha, \beta$ or γ. It is clear that $X \subset X^{\eta\eta}$ where $\eta = \alpha, \beta$ or γ.

For a sequence space X, if $X = X^{\alpha\alpha}$ then X is called a Köthe space or a perfect sequence space.

The notion of difference sequence space was introduced by Kizmaz [22] in 1981 as follows:

$$X(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in X\}$$

for $X = \ell_\infty, \quad c, \quad c_0$; where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$ (the set of natural numbers). For a detailed account of difference sequence spaces one may refer to [1-7, 10-20, 25-29] where many more references can be found.

2. Introduction

Soon after the introduction of the notion of difference sequence space, Orhan [28], in the year 1983, applied the same technique of taking differences to the Cesàro spaces $ces_p, 1 \leq p < \infty$ and ces_∞ of Shiue [30] to introduce the Cesàro difference sequence spaces $C_p, 1 \leq p < \infty$, and C_∞, although, surprisingly enough, the reference of the pioneering work of Kizmaz [22] was missing from [28]. Since the initiation of the study of difference sequence space by Kizmaz, a large amount of literature has grown. Keeping aside some exceptions (see, for instance, [1, 3]) in most of these works, the underlying spaces have remained the same, i.e., $\ell_\infty, c,$

or c_0. Quite recently, the Cesàro difference sequence space $C_1(\Delta)$ with underlying space as C_1 has been introduced in [6, 7] as follows :

$$C_1(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in C_1\}.$$

We here propose to derive further properties concerning the space C_∞ (which we denote $C_\infty(\Delta)$ for the sake of notational uniformity) along with the introduction of a new difference sequence space $bv(\Delta)$. Recall [6, 7] that neither $C_1(\Delta) \subset \ell_\infty(\Delta)$ nor $\ell_\infty(\Delta) \subset C_1(\Delta)$ but $c(\Delta) \subset C_1(\Delta) \cap \ell_\infty(\Delta)$ i.e., $C_1(\Delta)$ and $\ell_\infty(\Delta)$ overlap without containing each other. It is worth observing that the difference sequence space $C_\infty(\Delta) = \{x = (x_k) \in \omega : (\frac{1}{k}\sum_{i=1}^{k} \Delta x_i) \in \ell_\infty\}$ of Orhan [28] strictly includes the overlapping spaces $C_1(\Delta)$ and $\ell_\infty(\Delta)$. Before proceeding further, we introduce a new difference sequence space $bv(\Delta)$ as follows:

$$bv(\Delta) = \{(x_k) \in \omega : (\Delta x_k) \in bv\}.$$

As we shall see, the overall picture regarding inclusions (strict, of course) among the already existing spaces $c_0, c, \ell_\infty, \ell_1, bv, C_1, c_0(\Delta), c(\Delta), \ell_\infty(\Delta), \ell_1(\Delta)$, $C_1(\Delta)$, $C_\infty(\Delta)$ and the newly introduced space $bv(\Delta)$ is as shown below:

$$\begin{array}{ccccccccc}
 & & & & C_1 & \subset & C_1(\Delta) \\
 & & & & \cup & & \cup \\
\ell_1 & \subset & bv(\text{ or } c_0) & \subset & c & \subset & \ell_\infty \\
\cap & & \cap & & \cap & & \cap \\
\ell_1(\Delta) & \subset & bv(\Delta)(\text{ or } c_0(\Delta)) & \subset & c(\Delta) & \subset & \ell_\infty(\Delta) \\
 & & & & \cap & & \cap \\
 & & & & C_1(\Delta) & \subset & C_\infty(\Delta)
\end{array}$$

One of the interesting features of the difference sequence space $C_\infty(\Delta)$ is shown to be that it is perfect and strictly includes each of the difference sequence spaces $c_0(\Delta), c(\Delta), \ell_\infty(\Delta), \ell_1(\Delta)$, $C_1(\Delta)$ and $bv(\Delta)$, none of which is itself perfect. In section 3, apart from discussing certain inclusion relations, we examine various topological properties of the spaces $C_\infty(\Delta)$ and $bv(\Delta)$. Section 4 is devoted to the computation of Köthe-Toeplitz and $\gamma-$ duals of these spaces. As an application, the matrix classes $(bv(\Delta), \ell_\infty)$, $(bv(\Delta), c)$ and $(bv(\Delta), c_0)$ are characterized in the concluding section.

3. Inclusion relations and topological structure

We begin with establishing elementary inclusion relations.

Theorem 3.1. *$C_1(\Delta) \subset C_\infty(\Delta)$, the inclusion being strict.*

Proof. Inclusion is obvious. To see that the inclusion is strict, consider the sequence $x = (x_k) = (-1, 2, -3, 4, -5, 6, ...)$.

Theorem 3.2. *$\ell_\infty(\Delta) \subset C_\infty(\Delta)$, the inclusion being strict.*

The proof follows from the fact that the sequence of Cesàro means of a bounded sequence is again bounded. Inclusion is strict in view of the example cited in Theorem 3.1.

Remark 3.3. *In view of Theorem 3.1, Theorem 3.2 and the fact* [6, 7] *that $C_1(\Delta)$ and $\ell_\infty(\Delta)$ overlap , it follows that that the difference sequence space $C_\infty(\Delta)$ strictly includes the overlapping spaces $C_1(\Delta)$ and $\ell_\infty(\Delta)$.*

Theorem 3.4. *$bv \subset bv(\Delta)$, the inclusion being strict.*

Proof. Let $(x_k) \in bv$. Then $(\Delta x_k) \in \ell_1 \subset bv$. For strict inclusion, consider the sequence $x = (x_0, x_1, x_2, \ldots)$ where

$$x_k = \begin{cases} 0, & \text{for } k = 0, \\ k(-1) - (k-1)\frac{1}{2} - (k-2)\frac{1}{2^2} \ldots - \frac{1}{2^{k-1}}, & \text{for } k \geq 1. \end{cases}$$

Theorem 3.5. *$\ell_1(\Delta) \subset bv(\Delta) \subset c(\Delta)$, the inclusions being strict.*

Proof. The result follows from the fact that $\ell_1 \subset bv \subset c$. For strict inclusion $\ell_1(\Delta) \subset bv(\Delta)$, observe that $(k) \in bv(\Delta)$ but $(k) \notin \ell_1(\Delta)$. Inclusion $bv(\Delta) \subset c(\Delta)$ is strict as $(y_k) = (0, -1, -1+\frac{1}{2}, -1+\frac{1}{2}-\frac{1}{3}, \ldots) \in c(\Delta)$ but is missing from $bv(\Delta)$.

Remark 3.6. *In view of Remark 3.3 and Theorem 3.5, we can say that $C_\infty(\Delta)$ is much wider than the difference sequence spaces $c_0(\Delta)$, $c(\Delta)$, $\ell_\infty(\Delta)$ of Kizmaz as well as the spaces $\ell_1(\Delta), bv(\Delta)$ and $C_1(\Delta)$.*

We now propose to study the linear topological structure of the difference sequence spaces $C_\infty(\Delta)$ and $bv(\Delta)$. Note that it was already observed by Orhan [28] that $C_\infty(\Delta)$ is a Banach space with the norm

$$||x||_\infty = |x_1| + \sup_k \frac{1}{k} \mid \sum_{i=1}^{k} \Delta x_i \mid, \; x = (x_k) \in C_\infty(\Delta).$$

We can go ahead and have the following

Theorem 3.7. *$C_\infty(\Delta)$ and $bv(\Delta)$ are BK spaces normed by*

$$||x||_\infty = |x_1| + \sup_k \frac{1}{k} \mid \sum_{i=1}^{k} \Delta x_i \mid, \; x = (x_k) \in C_\infty(\Delta)$$

and

$$||x||_{bv} = |x_1| + |x_2| + \sum_k |\Delta x_k - \Delta x_{k+1}|, \; x = (x_k) \in bv(\Delta), respectively.$$

The proof is a routine verification by using 'standard' techniques and hence is omitted.

Theorem 3.8. *(i) $C_1(\Delta)$ is a closed subspace of $C_\infty(\Delta)$.*
(ii) $C_1(\Delta)$ is a nowhere dense subset of $C_\infty(\Delta)$.

The proof follows from the fact that $C_1(\Delta)$ is a proper and complete subspace of $C_\infty(\Delta)$.

Theorem 3.9. *$C_\infty(\Delta)$ is not separable.*

Proof. Suppose, if possible, that $C_\infty(\Delta)$ is separable, and that $D = \{d_1, d_2, d_3, \ldots\}$ is a countable dense subset, where $d_1 = (d_{1k}) = (d_{11}, d_{12}, \ldots), d_2 = (d_{2k}) = (d_{21}, d_{22}, \ldots), \ldots$. Now define $x = (x_n)$, where for $n \in \mathbb{N}$,

$$x_n = \begin{cases} n + d_{11} + d_{nn} - d_{n1}, \text{ if } |d_{nn} - d_{n1}| \leq n-1; \\ \\ 1 + d_{11}, \text{ if } |d_{nn} - d_{n1}| > n-1. \end{cases}$$

$$
\left|\frac{x_1 - x_{n+1}}{n}\right| = \begin{cases} \left|\frac{1+d_{11}-(\overline{n+1}+d_{11}+d_{(n+1)(n+1)}-d_{(n+1)1})}{n}\right|, \text{ if } |d_{(n+1)(n+1)} - d_{(n+1)1}| \leq n; \\ \\ \left|\frac{1+d_{11}-(1+d_{11})}{n}\right|, \text{ if } |d_{(n+1)(n+1)} - d_{(n+1)1}| > n; \end{cases}
$$

and so $\left|\frac{x_1 - x_{n+1}}{n}\right| \leq 2$ for all $n \in \mathbb{N}$, which in turn implies that $(x_k) \in C_\infty(\Delta)$. Clearly, $\|x - d_1\|_\Delta \geq |x_1 - d_{11}| = 1$ and for $n > 1$

$$
\begin{aligned}
\|x - d_n\|_\Delta &\geq \sup_{k \geq 1} \left|\frac{(x_1 - d_{n1}) - (x_{k+1} - d_{n(k+1)})}{k}\right| \\
&\geq \left|\frac{(x_1 - d_{n1}) - (x_n - d_{nn})}{n-1}\right| \\
&= \begin{cases} \left|\frac{1+d_{11}-d_{n1}-(n+d_{11}-d_{n1})}{n-1}\right|, \text{ if } |d_{nn} - d_{n1}| \leq n-1; \\ \\ \left|\frac{1+d_{11}-d_{n1}-(1+d_{11}-d_{nn})}{n-1}\right|, \text{ if } |d_{nn} - d_{n1}| > n-1; \end{cases} \\
&= \begin{cases} 1, \text{ if } |d_{nn} - d_{n1}| \leq n-1; \\ \\ \left|\frac{d_{nn}-d_{n1}}{n-1}\right|, \text{ if } |d_{nn} - d_{n1}| > n-1; \end{cases}
\end{aligned}
$$

i.e., $\|x - d_n\|_\Delta \geq 1$. Hence $x = (x_k) \in C_\infty(\Delta)$ is such that no $n \in \mathbb{N}$ exists such that $\|x - d_n\|_\Delta < 1$, a contradiction as D is dense in $C_\infty(\Delta)$.

Corollary 3.10. *$C_\infty(\Delta)$ does not have Schauder basis.*

The result follows from the fact that if a normed space has a Schauder basis, then it is separable.

Corollary 3.11. *$C_\infty(\Delta)$ does not have the AK property.*

Theorem 3.12. *$bv(\Delta)$ has Schauder basis namely $\{e, b_1, b_2, b_3, \ldots\}$ where $e = (1, 1, 1, \ldots)$ and for $n \in \mathbb{N}$, $b_n = (b_n^k)_{k \in \mathbb{N}}$ as*

$$
b_n^k = \begin{cases} 0, \ \text{if } k \leq n; \\ k - n \ \text{otherwise}, \ k \in \mathbb{N}; \end{cases}
$$

and any $x = (x_k) \in bv(\Delta)$ has a unique representation of the form $x = x_1 e - \Delta x_1 b_1 + \sum_k (\Delta x_k - \Delta x_{k+1}) b_{k+1}$.

The proof is easy and so omitted.

Corollary 3.13. *$bv(\Delta)$ is a separable space.*

The result follows from the fact that if a normed space has a Schauder basis, then it is separable.

Theorem 3.14. *$bv(\Delta)$ does not have the AK property.*

Proof. Let $x = (x_k) = (1, 2, 3, \ldots) \in bv(\Delta)$. Consider the n^{th} section of the sequence (x_k) as $x^{[n]} = (1, 2, 3, \ldots, n, 0, 0, \ldots)$. Then

$$
\begin{aligned}
\|x - x^{[n]}\|_{bv} &= \|(0, 0, 0, \ldots, n+1, n+2, \ldots)\|_{bv} \\
&= |0| + |0| + |n+1| + |n|
\end{aligned}
$$

which does not tend to 0 as $n \to \infty$.

4. Dual spaces

In the present section, we compute various duals and investigate the perfectness of $C_\infty(\Delta)$ and $bv(\Delta)$.

Theorem 4.1.

$$[C_\infty(\Delta)]^\alpha = [bv(\Delta)]^\alpha = \{a = (a_k) : \sum_k k|a_k| < \infty\} = D_1.$$

The proof is easy and so omitted.

Remark 4.2. *In view of the fact* [6, 7] *that* $[c_0(\Delta)]^\alpha = [c(\Delta)]^\alpha = [\ell_\infty(\Delta)]^\alpha = [C_1(\Delta)]^\alpha = D_1$ *and Theorem 4.1, we conclude that the* α*-duals of the difference sequence spaces* $c_0(\Delta), c(\Delta), \ell_\infty(\Delta), C_1(\Delta)$*,* $bv(\Delta)$ *and* $C_\infty(\Delta)$ *coincide.*

Theorem 4.3.

$$[C_\infty(\Delta)]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^{-1}|a_k| < \infty\} = D_2.$$

The result follows in view of Remark 4.2 and the fact ([6], Theorem 4.3) that $[C_1(\Delta)]^{\alpha\alpha} = D_2$.

Remark 4.4. *It is already known* [6, 7, 20, 21] *that none of the spaces* $c_0(\Delta)$*,* $c(\Delta)$*,* $\ell_\infty(\Delta), \ell_1(\Delta)$*, and* $C_1(\Delta)$ *is perfect. We now show that* $bv(\Delta)$ *is not perfect whereas* $C_\infty(\Delta)$ *is. Thus* $C_\infty(\Delta)$ *is a perfect space which strictly includes the non-perfect spaces* $c_0(\Delta), c(\Delta), \ell_\infty(\Delta), \ell_1(\Delta), C_1(\Delta)$ *and* $bv(\Delta)$*.*

Theorem 4.5. *(i)* $C_\infty(\Delta)$ *is perfect.*
(ii) $bv(\Delta)$ *is not perfect.*

Proof. (i) It is easy to see that $[C_\infty(\Delta)]^{\alpha\alpha} = C_\infty(\Delta)$.
(ii)The proof follows at once when we observe that the sequence $((-1)^k) \in [bv(\Delta)]^{\alpha\alpha}$ but does not belong to $bv(\Delta)$.

Lemma 4.6. [21] *Let* X *be a sequence space. Then we have*
(i) X *is perfect* $\Rightarrow X$ *is normal* $\Rightarrow X$ *is monotone.*
(ii) X *is normal* $\Rightarrow X^\alpha = X^\gamma$*.*
(iii)X *is monotone* $\Rightarrow X^\alpha = X^\beta$*.*

Using Theorem 4.1, Theorem 4.5 and Lemma 4.6, we have

Corollary 4.7. *(i)* $C_\infty(\Delta)$ *is normal as well as monotone.*
(ii) $[C_\infty(\Delta)]^\beta = [C_\infty(\Delta)]^\gamma = [C_\infty(\Delta)]^\alpha = \{a = (a_k) : \sum_k k|a_k| < \infty\}$*.*

Remark 4.8. *The* $\beta-$ *dual of* $C_\infty(\Delta)$ *was originally computed by Orhan* [28]*. Our opproach is indirect. We have benefited from the perfectness of* $C_\infty(\Delta)$*.*

In order to compute the $\beta-$ dual of $bv(\Delta)$, we need the following

Lemma 4.9. [22] $\sum_k ka_k$ *is convergent if and only if* $\sum_k R_k$ *is convergent with* $nR_n = o(1)$*, where* $R_n = \sum_{k=n+1}^\infty a_k$*.*

Theorem 4.10.

$$[bv(\Delta)]^\beta = \{a = (a_k) : \sum_k ka_k \textit{ is convergent}\} = D_3.$$

Proof. The proof follows very closely the lines of proof in [22]. However, for the sake of completeness we are giving the proof.

Let $a = (a_k) \in D_3$. Then $\sum_k ka_k$ is convergent. For $x = (x_k) \in bv(\Delta)$, we have $(\Delta x_k) \in bv \subset c \subset \ell_\infty$ and so there exists $M > 0$ such that $|\Delta x_k| \leq M$, for all $k \in \mathbb{N}$. Abel's summation by parts yields

$$\sum_{k=1}^{n} a_k x_k = -\sum_{j=1}^{n-1} \Delta x_j R_j + R_n \sum_{j=1}^{n-1} \Delta x_j + x_1 \sum_{k=1}^{n} a_k \tag{4.1}$$

where $R_n = \sum_{k=n+1}^{\infty} a_k$ and $n \in \mathbb{N}$. Obviously the last term on the right in (4.1) is convergent. As $\sum_j ja_j$ is convergent, so by Lemma 4.9, $(R_j) \in cs$. Since $(bv)^\beta = cs$ and $(\Delta x_j) \in bv$ so $\sum_j \Delta x_j R_j$ converges, that is, first term on right in (4.1) is convergent. Finally

$$\begin{aligned} |R_n \sum_{j=1}^{n-1} \Delta x_j| &\leq |R_n| \sum_{j=1}^{n-1} |\Delta x_j| \\ &\leq M\ |(n-1)R_n| \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

and so $\sum_k a_k x_k$ converges.
Conversely, let $(a_k) \in [bv(\Delta)]^\beta$. Then $\sum_k a_k x_k$ converges for all $x = (x_k) \in bv(\Delta)$. In particular, taking $x_k = k$, we get $\sum_k ka_k$ is convergent and so $(a_k) \in D_3$.

Theorem 4.11. *$bv(\Delta)$ is not monotone.*

Proof. Take $(x_k) = (1, 1, \ldots) \in bv(\Delta)$ and define $y = (y_k)$ as

$$y_k = \begin{cases} x_k, & \text{if k is odd;} \\ 0, & \text{if k is even,} \end{cases}$$

that is, $(y_k) = (1, 0, 1, 0, \ldots)$. Then $(\Delta y_k) = (1, -1, 1, -1, \ldots)$ and so $(y_k) \notin bv(\Delta)$.

Using Lemma 4.6 and Theorem 4.11, we have

Corollary 4.12. *$bv(\Delta)$ is not normal.*

Theorem 4.13. *None of the spaces $C_\infty(\Delta)$ and $bv(\Delta)$ is convergence free.*

Proof. Let $(x_k) = (1, 0, 3, 0, 5, 0, 7, 0, \ldots) \in C_\infty(\Delta)$.
Take $(y_k) = (1^2, 0, 3^2, 0, 5^2, 0, 7^2, 0, \ldots)$, then $(\Delta y_k) = (1^2, -3^2, 3^2, -5^2, 5^2, -7^2, 7^2, \ldots)$.
Now

$$\frac{1}{k} \sum_{i=1}^{k} \Delta y_i = \begin{cases} \frac{1}{k}, & \text{if k is odd;} \\ \frac{1-(k+1)^2}{k}, & \text{if k is even,} \end{cases}$$

and so $(y_k) \notin C_\infty(\Delta)$. This shows that $C_\infty(\Delta)$ is not convergence free.

Using Corollary 4.12 and the fact [9] that every convergence free space is normal, we see that $bv(\Delta)$ is not a convergence free space.

Next we investigate the symmetry of the spaces $C_\infty(\Delta)$ and $bv(\Delta)$. In checking the symmetric property of the space $C_\infty(\Delta)$, we shall make use of the following

Theorem 4.14. [9] *If X is a perfect symmetric space other than ϕ or ω, then $\ell_1 \subset X \subset \ell_\infty$.*

Theorem 4.15. *None of the spaces $C_\infty(\Delta)$ and $bv(\Delta)$ is symmetric.*

Proof. Let $(x_k) = (1, 2, 3, 4, \dots) \in bv(\Delta)$ and $(y_k) = (2, 1, 4, 3, 6, 5, \dots)$ be a rearrangement of the terms of the sequence (x_k). Here $(\Delta y_k) = (1, -3, 1, -3, 1, -3, \dots) \notin bv$ and so $(y_k) \notin bv(\Delta)$. This shows that $bv(\Delta)$ is not a symmetric space.

In view of Theorem 4.5, Theorem 4.14 and the fact that $C_\infty(\Delta)$ is not contained in ℓ_∞, it follows at once that $C_\infty(\Delta)$ is not a symmetric space.

Theorem 4.16. *None of the spaces $C_\infty(\Delta)$ and $bv(\Delta)$ is a sequence algebra.*

Proof. The sequences $x = y = (k)$ serve the purpose.

5. Matrix maps

Finally, we characterize certain matrix classes. For any complex infinite matrix $A = (a_{nk})$, we shall write $A_n = (a_{nk})_{k\in\mathbb{N}}$ for the sequence in the n^{th} row of A. If X, Y are any two sets of sequences, we denote by (X, Y) the class of all those infinite matrices $A = (a_{nk})$ such that the series $A_n(x) = \sum_k a_{nk}x_k$ converges for all $x = (x_k) \in X$, $(n = 1, 2, \dots)$ and the sequence $Ax = (A_n x)_{n\in\mathbb{N}}$ is in Y for all $x \in X$.

Before proceeding further, we recall the following theorems which will be used in the sequel.

Theorem 5.1. [31] *$A \in (bv, \ell_\infty)$ if and only if*
(i) $\sup_n \sup_j |\sum_{k=j} a_{nk}| < \infty$, *or*
(ii) $\sup_n |\sum_k a_{nk}| < \infty$ *and* $\sup_n \sup_j | \sum_{k=1}^{j} a_{nk}| < \infty$.

Theorem 5.2. [31] *$A = (a_{nk}) \in (bv, c)$ if and only if*
(i) $\lim_n a_{nk}$ *exists for each* $k \in \mathbb{N}$,
(ii) $\lim_n \sum_k a_{nk}$ *exists,*
(iii) $\sup_n \sup_j |\sum_{k=j} a_{nk}| < \infty$ *; or* $\sup_n \sup_j |\sum_{k=1}^{j} a_{nk}| < \infty$.

Theorem 5.3. [31] *$A = (a_{nk}) \in (bv, c_0)$ if and only if*
(i) $\lim_n a_{nk} = 0$ *for each* $k \in \mathbb{N}$,
(ii) $\lim_n \sum_k a_{nk} = 0$,
(iii) $\sup_n \sup_j |\sum_{k=j} a_{nk}| < \infty$ *; or* $\sup_n \sup_j |\sum_{k=1}^{j} a_{nk}| < \infty$.

We are now in a position to characterize the matrix classes $(bv(\Delta), \ell_\infty)$, $(bv(\Delta), c)$ and $(bv(\Delta), c_0)$.

Theorem 5.4. *$A \in (bv(\Delta), \ell_\infty)$ if and only if*
(i) $\sup_n |\sum_k k a_{nk}| < \infty$,
(ii) $\sup_n \sup_j | \sum_{v=j} \sum_{k=v+1}^{\infty} a_{nk}| < \infty$,
(iii) $\sup_n |\sum_k a_{nk}| < \infty$.

Proof. Let the conditions $(i)-(iii)$ hold and suppose that $x = (x_k) \in bv(\Delta)$. It is implicit in (i) that, for each $n \in \mathbb{N}, \sum_k ka_{nk}$ converges and so $(a_{nk})_k \in [bv(\Delta)]^\beta$. This implies that $\sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. As $x = (x_j) \in bv(\Delta)$, so $(\Delta x_j) \in bv$. Further as in in the proof of Theorem 4.10, we have

$$\sum_k a_{nk}x_k = -\sum_j \Delta x_j (\sum_{k=j+1}^{\infty} a_{nk}) + x_1 \sum_k a_{nk}$$
$$= -\sum_j \Delta x_j \; c_{nj} + x_1 \sum_k a_{nk}$$

where $c_{nj} = \sum_{k=j+1}^{\infty} a_{nk}$, for $(n, j = 1, 2, \ldots)$. Making use of (ii), we have $\underset{n}{sup}\ \underset{j}{sup} \mid \sum_{v=j} \; c_{nv}| < \infty$ and so by Theorem 5.1, we have matrix $C = (c_{nv}) \in (bv, \ell_\infty)$. This yields $(\sum_j \Delta x_j \; c_{nj}) \in \ell_\infty$. Thus $A \in (bv(\Delta), \ell_\infty)$.
Conversely, let $A \in (bv(\Delta), \ell_\infty)$. Then $\underset{n}{sup} \mid \sum_k a_{nk}x_k| < \infty$, for all $x = (x_k) \in bv(\Delta)$. As $(k), e = (1, 1, 1, \ldots) \in bv(\Delta)$, so (i) and (iii) hold. Now suppose if possible, $\underset{n}{sup}\ \underset{j}{sup} \mid \sum_{v=j} \; \sum_{k=v+1}^{\infty} a_{nk}| = \infty$. Consider the matrix C defined by

$$c_{nv} = \sum_{k=v+1} a_{nk} \; (n, v = 1, 2, \ldots).$$

Then the matrix $C = (c_{nv}) \notin (bv, \ell_\infty)$. Therefore there exists a sequence $x = (x_k) \in bv$ such that $(\sum_v c_{nv}x_v)_n \notin \ell_\infty$. We define a sequence $y = (y_v)$ as

$$y_v = -\sum_{j=1}^{v-1} x_j + x_1 \; (v = 1, 2, 3, \ldots).$$

Then $(\Delta y_v) = (x_v) \in bv$ and $\sum_v a_{nv}y_v = x_1 \sum_v a_{nv} - \sum_v x_v c_{nv} \neq O(1)$, a contradiction to the fact that $A \in (bv(\Delta), \ell_\infty)$.

Theorem 5.5. $A = (a_{nk}) \in (bv(\Delta), c)$ *if and only if*
(i) $\underset{n}{sup} | \sum_k ka_{nk}| < \infty$,
(ii) $\underset{n}{sup}\ \underset{j}{sup} | \; \sum_{v=j} \; \sum_{k=v+1}^{\infty} a_{nk} \; | < \infty$,
(iii) $\underset{n}{lim} \sum_j \sum_{k=j+1}^{\infty} a_{nk} = \alpha$,
(iv) $\underset{n}{lim} \sum_{k=j+1} a_{nk} = \beta_j$ *for* $j = 0, 1, 2, \ldots$
where $\alpha, \beta_j \in \mathbb{C}$.

Proof. Let the conditions $(i) - (iv)$ hold and $x = (x_k) \in bv(\Delta)$. Using (i) and arguing in the same way as in Theorem 5.4, we have

$$\sum_k a_{nk}x_k = -\sum_j \Delta x_j \; c_{nj} + x_1 \sum_k a_{nk} \; (5.1)$$

where $c_{nj} = \sum_{k=j+1}^{\infty} a_{nk}$, for $(n, j = 1, 2, \ldots)$. Making use of $(ii) - (iv)$ and Theorem 5.2, we have matrix $C = (c_{nk}) \in (bv, c)$. As $(\Delta x_j) \in bv$, so we have $(\sum_j \Delta x_j c_{nj}) \in c$. Therefore it follows from (5.1) that $(\sum_k a_{nk}x_k) \in c$ for all $x = (x_k) \in bv(\Delta)$, i.e., $A \in (bv(\Delta), c)$.
Conversely, it is given that $A \in (bv(\Delta), c)$. So $A \in (bv(\Delta), \ell_\infty)$. Then by Theorem 5.4, (i) and (ii) hold. Also we have $(\sum_k a_{nk}x_k) \in c$ for all $x = (x_k) \in bv(\Delta)$. Taking $x_k = 1$ for all $k \in \mathbb{N}$, we get $(\sum_k a_{nk}) \in c$. It follows from (5.1) that

$\left(\sum_j \ \Delta x_j \ \left(\sum_{k=j+1}^{\infty} a_{nk} \ \right) \right) \in c$, for all $x = (x_k) \in bv(\Delta)$. Now for $(x_j) = (-j) \in bv(\Delta)$ and $(0, 0, 0, \ldots, 0, 1, 1, \ldots) \in bv(\Delta)$, having 0's at first j positions and 1's elsewhere $(j = 1, 2, \ldots)$, we have $\left(\sum_j \left(\sum_{k=j+1}^{\infty} a_{nk} \ \right) \right) \in c$ and $\left(\sum_{k=j+1}^{\infty} a_{nk} \ \right) \in c$.

Using the same technique as in Theorem 5.5 and applying Theorem 5.3, we have

Theorem 5.6. *$A = (a_{nk}) \in (bv(\Delta), c_0)$ if and only if*
(i) $\sup_n |\sum_k k a_{nk}| < \infty$,
(ii) $\sup_n \sup_j | \ \sum_{v=j} \ \sum_{k=v+1}^{\infty} a_{nk} \ | < \infty$,
(iii) $\lim_n \sum_j \sum_{k=j+1}^{\infty} a_{nk} = 0$,
(iv) $\lim_n \sum_{k=j+1} a_{nk} = 0$ *for* $j = 0, 1, 2, \ldots$.

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