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ON A q-ANALOGUE OF HILBERT'S INEQUALITY

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ABSTRACT. In this article an expression relating $\sin_q(\pi_q\alpha)$ and the q-analogue of the beta function $B_q(\alpha, \lambda - \alpha)$ is found. This is used show that the q-analogue of Hilbert's inequality

$$\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x+y} d_qx\, d_qy \leq \frac{2\pi_q}{(2)_q^{1/p'}(2)_q^{1/p}\sin_q(\pi_q/p')}\|f\|_p\|g\|_{p'}$$

holds and a generalization of this inequality is proved.

Hilbert's inequality

$$\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x+y} dxdy \leq \frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_{p'}, \tag{1}$$

where $1 < p < \infty$, $p' = \frac{p}{p-1}$ and $f \in L^p[0, \infty)$ and $g \in L^{p'}[0, \infty)$, has been heavily studied in mathematics. Of particular interest for us is the proof found in [3], where Schur's test is applied to the homogeneous Hilbert kernel

$$k(x, y) = \frac{1}{x+y}$$

on $[0, \infty) \times [0, \infty)$ to prove the result. In this article, we prove the q-analogue of Hilbert's inequality (see Theorem 4),

$$\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x+y} d_qx\, d_qy \leq \frac{2\pi_q}{(2)_q^{\alpha}(2)_q^{1-\alpha}\sin_q(\pi_q/p')}\|f\|_p\|g\|_{p'}, \tag{2}$$

is satisfied. To show this result, a special case of Schur's test is proved in the quantum calculus setup (see Theorem 3).

For $p = p' = 2$, a generalization for the classical Hilbert's inequality is proved in [1]. For those values of p and p' and $\lambda > 0$, they show that given two positive constants a, b and two functions $f, g \in L^2([0, \infty), t^{1-\lambda}dt)$, the inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(ax+by)^\lambda} dxdy < \frac{B(\lambda/2, \lambda/2)}{(ab)^\lambda}\|f\|_2\|g\|_2 \tag{3}$$

is satisfied, with the norms on the right-hand-side are the norms induced by the measure $t^{1-\lambda}dt$. Here $B(\cdot, \cdot)$ is the classical beta function. To obtain a similar, but

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stronger, generalization for the q-analogue, we show that the q-analogue of the beta function satisfies:

$$B_q(x,\lambda-x)=\frac{\pi_q}{\sin_q(\pi_q x)}\prod_{k=1}^{\lambda-1}\left[1+\frac{x}{k}\right]_{q^k}.$$

for $\lambda\in\mathbb{N}$ and $0<x<\lambda$ (See Lemma 1). With this result, we are able to show that under certain conditions:

$$\int_0^\infty\int_0^\infty \frac{|f(x)g(y)|}{(ax+by)_q^\lambda}d_qx\,d_qy\le M_q^{\lambda,\beta}(a,b)\|f\|_p\|g\|_{p'}.$$

is satisfied for some constant $M_q^{\lambda,\beta}(a,b)$ that is explicitly calculated (see Theorem 6).

1. Preliminaries

In this section, we will introduce the basic definitions of q-calculus that will be needed throughout this exposition. For the rest of the article, we will assume a deformation parameter $0<q<1$. The q-analogue of $x\in\mathbb{R}$ will be denoted and defined by

$$[x]=\frac{1-q^x}{1-q}.$$

Later it will be useful to consider the q^k analogue of $x\in\mathbb{R}$ defined by:

$$[x]_{q^k}=\frac{1-q^{kx}}{1-q^k}.$$

Integrals in this article will be understood as the Jackson integrals:

$$\int_0^b f(x)d_qx=(1-q)b\sum_{j=0}^{\infty}q^jf(q^jb),$$
$$\int_0^\infty f(x)d_qx=(1-q)\sum_{j=-\infty}^{\infty}q^jf(q^j).$$

We remark that the Fundamental Theorem of Calculus is satisfied by the Jackson integral (see Theorem 20.1 in [5]) and for later use, we record the q-analogue of the power rule

$$\int x^\alpha d_qx=\frac{x^{\alpha+1}}{[\alpha+1]}\qquad \alpha\ne-1.$$

An important estimate used in this article uses the q-analogues of the gamma (c.f. [4]) and beta functions (c.f. [2]). For $s,t>0$:

$$\Gamma_q(t)=\frac{(1-q)_q^{t-1}}{(1-q)^{t-1}}\tag{4}$$

$$B_q(t,s)=\frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}\tag{5}$$

where,

$$(a+b)_q^n = \prod_{j=0}^{n-1}(a+q^j b) \qquad \text{for } n \in \mathbb{N},$$

$$(1+a)_q^\infty = \prod_{j=0}^{\infty}(1+q^j a) \qquad \text{and}$$

$$(1+a)_q^t = \frac{(1+a)_q^\infty}{(1+aq^t)_q^\infty} \qquad \text{for } t \in \mathbb{C}.$$

These q-binomials satisfy many properties of which we mention the following, for later use, (c.f. [2])

$$(1+x)_q^{s+t} = (1+x)_q^s (1+q^s x)_q^t, \tag{6}$$

for $s, t \in \mathbb{R}$. Integral representations of these functions

$$\Gamma_q(t) = K(A,t)\int_0^{\infty/A(1-q)} x^{t-1} e_q^{-x} d_q x$$

$$B_q(t,s) = K(A,t)\int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_q x, \tag{7}$$

are proved in [2], where

$$K(A,t) = \frac{1}{1+A} A^t \left(1+\frac{1}{A}\right)_q^t (1+A)_q^{1-t} \tag{8}$$

and

$$\int_0^{\infty/A} f(x) d_q x = (1-q) b \sum_{j=0}^{\infty} \frac{q^j}{A} f(q^j/A).$$

In particular, (7) will be especially important to our exposition. One last fact regarding the q-gamma functions will be useful for our purposes. In [6], they obtain:

$$\Gamma_q(t)\Gamma_q(1-t) = \frac{\pi_q}{\sin_q(\pi_q t)}, \tag{9}$$

where

$$\sin_q(x) = \frac{e_q(ix) - e_q(-ix)}{2i}$$

and π_q is the analogue of π defined via the equation

$$\sin_q(\pi_q x) = \pi_q [x] \prod_{k=1}^{\infty} \left[1+\frac{x}{k}\right]_{q^k} \left[1-\frac{x}{k}\right]_{q^k}. \tag{10}$$

In the following lemma, we generalize equation (9).

Lemma 1. *For $\lambda \in \mathbb{N}$ and $0 < x < \lambda$ we have*

$$B_q(x, \lambda - x) = \frac{\pi_q}{\sin_q(\pi_q x)} \prod_{k=1}^{\lambda-1} \left[1+\frac{x}{k}\right]_{q^k}. \tag{11}$$

Proof. It is known that (c.f. Eq. 3, [6]):

$$\Gamma_q(x)^{-1} = [x] \prod_{k=1}^{\infty} \left[1+\frac{x}{k}\right]_{q^k} \left(\left[1+\frac{1}{k}\right]_{q^k}\right)^{-x},$$

we calculate:

$$\begin{aligned} B_q(x,\lambda-x) &= \frac{[\lambda]}{[x][\lambda-x]}\prod_{k=1}^{\infty}\frac{\left[1+\frac{\lambda}{k}\right]_{q^k}}{\left[1+\frac{x}{k}\right]_{q^k}\left[1+\frac{\lambda-x}{k}\right]_{q^k}} \\ &= \frac{[\lambda]}{[x][\lambda-x]}\prod_{k=1}^{\infty}\frac{(1-q^{k+\lambda})(1-q^k)}{(1-q^{k+\lambda-x})(1-q^{k+x})} \\ &= \frac{[\lambda]}{[x][\lambda-x]}\prod_{k=1}^{\lambda}\frac{(1-q^{k-x})}{(1-q^k)}\prod_{k=1}^{\infty}\frac{(1-q^k)(1-q^k)}{(1-q^{k-x})(1-q^{k+x})} \\ &= \frac{[\lambda]}{[x][\lambda-x]}\prod_{k=1}^{\lambda}\frac{(1-q^{k-x})}{(1-q^k)}\prod_{k=1}^{\infty}\frac{1}{\left[1+\frac{x}{k}\right]_{q^k}\left[1-\frac{x}{k}\right]_{q^k}} \\ &= \frac{\pi_q}{\sin_q(\pi_q t)}\prod_{k=1}^{\lambda-1}\frac{(1-q^{k-x})}{(1-q^k)}, \end{aligned}$$

which is the desired result. □

Notice that when $\lambda = 1$, Equation (11) reduces to (9), thus generalizing this result. With all these tools at hand, we are now able to tackle the proof of the q-analogue of Hilbert's inequality.

2. q-Hilbert's Inequality

The first step in our proof will be to obtain the value of the integral of $x^{-\alpha}k(x,y)$ where $k(x,y)$ is the Hilbert kernel $k(x,y) = 1/(x+y)$.

Lemma 2. *For* $0 < \alpha < 1$*,*

$$\int_0^\infty \frac{1}{(x+y)x^\alpha}d_qx = \frac{2\pi_q}{y^\alpha(2)_q^\alpha(2)_q^{1-\alpha}\sin_q(\alpha\pi_q)}. \tag{12}$$

In particular, when $\alpha = 1/p'$*,*

$$\int_0^\infty \frac{1}{(x+y)x^{1/p'}}d_qx = \frac{2\pi_q}{y^{1/p'}(2)_q^{1/p'}(2)_q^{1/p}\sin_q(\pi_q/p')}. \tag{13}$$

Proof. Equation (13) follows from (12). So, using (7) with $t = 1-\alpha$ and $s = \alpha$ we obtain:

$$\int_0^\infty \frac{1}{(1+u)u^\alpha}d_qu = \frac{B_q(1-\alpha,\alpha)}{K(1,1-\alpha)}.$$

Now we use (5) and (8) to obtain:

$$\int_0^\infty \frac{1}{(1+u)u^\alpha}d_qu = \frac{2\Gamma_q(\alpha)\Gamma_q(1-\alpha)}{(2)_q^\alpha(2)_q^{1-\alpha}}.$$

In particular, (9) implies:

$$\int_0^\infty \frac{1}{(1+u)u^\alpha}d_qu = \frac{2\pi_q}{(2)_q^\alpha(2)_q^{1-\alpha}\sin_q(\pi_q\alpha)}.$$

To obtain our result, notice that

$$\int_0^\infty \frac{1}{(x+y)x^\alpha}d_qx = \int_0^\infty \frac{1}{(1+x/y)y^{\alpha+1}(x/y)^\alpha}d_qx,$$

and now the substitution $u = x/y$ gives the result (c.f. Eq. 19.15 [5]). □

To show Hilbert's inequality we will need a q-analogue of Schur's test in the quantum calculus setup. We show this in the following theorem.

Theorem 3 (q-Schur's test)**.** *For $k = k(x, y)$ and $1 < p < \infty$, suppose that there exist functions $s \in L^p[0,\infty)$, $t \in L^{p'}[0,\infty)$ and constants A and B such that such that the integrals exist and satisfy:*

$$\int_0^\infty k(x,y)t(y)^{p'} d_q y \le (As(x))^{p'}$$

for every $x \in [0,\infty)$, and

$$\int_0^\infty k(x,y)s(x)^p d_q x \le (Bt(y))^p$$

for every $y \in [0,\infty)$. Then, if $f \in L^p[0,\infty)$,

$$T(f)(x) = \int_0^\infty k(x,y)f(y)d_q y$$

exists for every $x \in [0,\infty)$, $T(f) \in L^p[0,\infty)$ and

$$\|T(f)\|_p \le AB\|f\|_p.$$

Proof. By the q-Hölder's inequality, it suffices to show that for every non-negative functions $h \in L^{p'}[0,\infty)$ and $g \in L^p[0,\infty)$, the following inequality is satisfied

$$\|h \cdot T(g)\|_1 \le AB\|g\|_p\|h\|_{p'}.$$

Then, by the q-Hölder's inequality and the hypothesis of the theorem we obtain:

$$\begin{aligned}\int_0^\infty k(x,y)g(y)d_q y &= \int_0^\infty \left(t(y)k(x,y)\frac{g(y)}{t(y)}\right) d_q y \qquad (14)\\ &= \int_0^\infty \left(t(y)k(x,y)^{1/p'}\frac{g(y)k(x,y)^{1/p}}{t(y)}\right) d_q y\\ &\le \left(\int_0^\infty t(y)^{p'}k(x,y)d_q y\right)^{1/p'} \left(\int_0^\infty \frac{g(y)^p k(x,y)}{t(y)^p} d_q y\right)^{1/p}\\ &\le As(x)\left(\int_0^\infty \frac{k(x,y)g(y)^p}{t(y)^p} d_q y\right)^{1/p}.\end{aligned}$$

Now, we consider the norm:

$$\begin{aligned}\|h\cdot T(g)\|_1 &\le A\int_0^\infty h(x)s(x)\left(\int_0^\infty \frac{k(x,y)g(y)^p}{t(y)^p}d_q y\right)^{1/p} d_q x \qquad (15)\\ &\le A\|h\|_{p'}\left(\int_0^\infty\int_0^\infty s(x)^p\frac{k(x,y)g(y)^p}{t(y)^p}d_q y\, d_q x\right)^{1/p} \qquad (16)\\ &= A\|h\|_{p'}\left(\int_0^\infty s(x)^p k(x,y)d_q x\int_0^\infty \frac{g(y)^p}{t(y)^p}d_q y\right)^{1/p} \qquad (17)\\ &\le AB\|h\|_{p'}\left(\int_0^\infty g(y)^p d_q y\right)^{1/p}, \qquad (18)\end{aligned}$$

where the (15) follows from (14), (16) follows from the q-Hölder's inequality, (17) follows from Fubini-Tonelli's theorem, and (18) follows from the hypotheses of the theorem. □

We are now able to show the q-analogue of Hilbert's inequality.

Theorem 4 (q-Hilbert's inequality)**.** *Let $1 < p < \infty$ and let $k(x,y) = \dfrac{1}{x+y}$. If $f \in L^p[0,\infty)$ and $g \in L^{p'}[0,\infty)$. Then,*

$$\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x+y} d_qx\, d_qy \le \frac{2\pi_q}{(2)_q^{1/p}(2)_q^{1/p'} \sin_q(\pi_q/p)} \|f\|_p \|g\|_{p'}. \tag{19}$$

Furthermore, the constant $\frac{2\pi_q}{(2)_q^{1/p'}(2)_q^{1/p}\sin_q(\pi_q/p')}$ *is the best possible.*

Proof. Let

$$s(x) = t(x) = \frac{1}{x^{1/(pp')}}.$$

Then, by Lemma 2,

$$\begin{aligned}\int_0^\infty s(x)^p k(x,y) d_qx &= \int_0^\infty \frac{1}{(x+y)x^{1/p'}} d_qx = \frac{2\pi_q}{y^{1/p'}(2)_q^{1/p'}(2)_q^{1/p}\sin_q(\pi_q/p')} \\ &= \frac{2\pi_q}{(2)_q^{1/p'}(2)_q^{1/p}\sin_q(\pi_q/p')} t(y)^p = \frac{2\pi_q}{(2)_q^{1/p'}(2)_q^{1/p}\sin_q(\pi_q/p)} t(y)^p\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty t(y)^p k(x,y) d_qy = \int_0^\infty \frac{1}{(x+y)y^{1/p}} d_qy &= \frac{2\pi_q}{x^{1/p}(2)_q^{1/p'}(2)_q^{1/p}\sin_q(\pi_q/p)} \\ &= \frac{2\pi_q}{(2)_q^{1/p'}(2)_q^{1/p}\sin_q(\pi_q/p)} s(x)^{p'}.\end{aligned}$$

Hence,

$$AB = \left(\frac{2\pi_q}{(2)_q^{1/p'}(2)_q^{1/p}\sin_q(\pi_q/p)}\right)^{1/p+1/p'} = \frac{2\pi_q}{(2)_q^{1/p'}(2)_q^{1/p}\sin_q(\pi_q/p)}$$

in Theorem 3. This completes the proof of the inequality.

Now, we proceed to show that the constant is the best possible. Let $\chi_S : [0,\infty) \to \{0,1\}$ denote the characteristic function of the set S and define

$$f_\lambda(x) = [\lambda - 1]^{1/p} x^{-\lambda/p} \chi_{[1,\infty)}(x)$$

and

$$g_\lambda(y) = [\lambda - 1]^{1/p'} y^{-\lambda/p'} \chi_{[1,\infty)}(y).$$

Then, if $1 < \lambda < p$, we have

$$\begin{aligned}\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x+y} d_qx\, d_qy &= [\lambda - 1] \int_1^\infty \frac{1}{y^{\lambda/p'}} \left(\int_1^\infty \frac{1}{x^{\lambda/p}(x+y)} d_qx \right) d_qy \\ &= [\lambda - 1] \int_1^\infty \frac{1}{y^{\lambda/p'}} \left(\int_0^\infty \frac{1}{x^{\lambda/p}(x+y)} d_qx \right) d_qy \\ &\quad - [\lambda - 1] \int_1^\infty \frac{1}{y^{\lambda/p'}} \left(\int_0^1 \frac{1}{x^{\lambda/p}(x+y)} d_qx \right) d_qy \\ &= \frac{2\pi_q q^{\lambda-1}}{(2)_q^{1-\lambda/p}(2)_q^{\lambda/p} \sin_q(\lambda\pi_q/p)} \\ &\quad - [\lambda - 1] \int_1^\infty \frac{1}{y^\lambda} \left(\int_0^{1/y} \frac{1}{u^{\lambda/p}(1+u)} d_qu \right) d_qy\end{aligned}$$

by (12). For the last integral, notice that $0 \leq x \leq 1$ and $1 \leq y < \infty$. Thus, $1 + u \geq 1$. So, we obtain:

$$\begin{aligned}\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x+y} d_qx\, d_qy \geq &\frac{2\pi_q q^{\lambda-1}}{(2)_q^{1-\lambda/p}(2)_q^{\lambda/p} \sin_q(\lambda\pi_q/p)} \\ &- [\lambda - 1] \int_1^\infty \frac{1}{y^\lambda} \left(\int_0^{1/y} \frac{1}{u^{\lambda/p}} d_qu \right) d_qy.\end{aligned}$$

Observe that

$$\int_0^{1/y} u^{-\lambda/p} d_qu = \frac{y^{(\lambda/p)-1}}{[1 - \lambda/p]}.$$

Then,

$$\int_1^\infty \frac{1}{y^\lambda} \left(\int_0^{1/y} \frac{1}{u^{\lambda/p}} d_qu \right) d_qy = \frac{1}{[1 - \lambda/p]} \int_1^\infty y^{-(\lambda/p')-1} d_qy = -\frac{1}{[1 - \lambda/p][\lambda/p']q^{\lambda/p'}}.$$

Therefore,

$$\begin{aligned}\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x+y} d_qx\, d_qy \geq &\frac{2\pi_q q^{\lambda-1}}{(2)_q^{1-\lambda/p}(2)_q^{\lambda/p} \sin_q(\lambda\pi_q/p)} \\ &+ \frac{[\lambda - 1]}{[1 - \lambda/p][\lambda/p']q^{\lambda/p'}}.\end{aligned}$$

Now, as $\lambda \to 1$,

$$\frac{2\pi_q q^{\lambda-1}}{(2)_q^{1-\lambda/p}(2)_q^{\lambda/p} \sin_q(\lambda\pi_q/p)} + \frac{[\lambda - 1]}{[1 - \lambda/p][\lambda/p']q^{\lambda/p'}} \to \frac{2\pi_q}{(2)_q^{1/p}(2)_q^{1/p} \sin_q(\pi_q/p')}$$

and this implies equality for these particular functions f and g. This completes the proof. □

3. A Generalization of q-Hilbert's Inequality

In this section, we show the q-analogue of (3) for $\lambda \in \mathbb{N}$. We start this section with a generalization of Lemma 2.

Lemma 5. *For $\lambda \in \mathbb{N}$ and $0 < \alpha < 1$,*

$$\int_0^\infty \frac{1}{(ax+by)_q^\lambda y^\alpha} d_q y = b^{\alpha-1}(ax)^{1-\alpha-\lambda} \frac{2\pi_q}{(2)_q^\alpha (2)_q^{1-\alpha} \sin_q(\alpha\pi_q)} \prod_{k=1}^{\lambda-1} \left[1 + \frac{1-\alpha}{k}\right]_{q^k}. \quad (20)$$

and

$$\int_0^\infty \frac{1}{(ax+by)_q^\lambda x^\alpha} d_q x = a^{\alpha-1}(by)^{1-\alpha-\lambda} \frac{2\pi_q}{(2)_q^{\lambda+\alpha-1} (2)_q^{2-\lambda-\alpha} \sin_q(\alpha\pi_q)} \prod_{k=1}^{\lambda-1} \left[1 + \frac{1-\alpha}{k}\right]_{q^k}. \quad (21)$$

Proof. The proof of (20) is similar to the proof of (12) noting that

$$(ax+by)_q^\lambda = \prod_{j=0}^{\lambda-1}(ax + q^j by) = (ax)^\lambda \prod_{j=0}^{\lambda-1}(1 + q^j(by)/(ax)) = (ax)^\lambda (1 + (by)/(ax))_q^\lambda$$

and using (11) instead of (9) at the end. Hence, it is omitted. However, the proof of (21) is inherently different due to the fact that $(a+b)_q^\lambda \neq (b+a)_q^\lambda$ for $\lambda > 1$. For this proof we will use the following property of the q-integrals (c.f [2]):

$$\int_0^{\infty/A} f(x) d_q x = \int_0^{\infty \cdot A} \frac{1}{x^2} f\left(\frac{1}{x}\right) d_q x. \quad (22)$$

We start by rewriting the integral:

$$\int_0^\infty \frac{1}{(ax+by)_q^\lambda x^\alpha} d_q x = \int_0^\infty \frac{1}{x^{\alpha+\lambda} a^\lambda (1 + by/(ax))_q^\lambda} d_q x.$$

Now we let $u = ax/(by)$ to obtain:

$$\int_0^\infty \frac{1}{(ax+by)_q^\lambda x^\alpha} d_q x = a^{\alpha-1}(by)^{1-\alpha-\lambda} \int_0^\infty \frac{1}{u^{\alpha+\lambda}(1+1/u)_q^\lambda} d_q u.$$

To use (22), we rewrite this as:

$$\int_0^\infty \frac{1}{(ax+by)_q^\lambda x^\alpha} d_q x = a^{\alpha-1}(by)^{1-\alpha-\lambda} \int_0^\infty \frac{1}{u^2} \cdot \frac{(1/u)^{\alpha+\lambda-2}}{(1+1/u)_q^\lambda} d_q u.$$

So that, the integral becomes

$$\int_0^\infty \frac{1}{(ax+by)_q^\lambda x^\alpha} d_q x = a^{\alpha-1}(by)^{1-\alpha-\lambda} \int_0^\infty \frac{u^{\alpha+\lambda-2}}{(1+u)_q^\lambda} d_q u.$$

Now, using (7) we obtain

$$\int_0^\infty \frac{1}{(ax+by)_q^\lambda x^\alpha} d_q x = a^{\alpha-1}(by)^{1-\alpha-\lambda} \frac{B_q(\lambda - (1-\alpha), 1-\alpha)}{K(1, \lambda - (1-\alpha))}.$$

Now, (11) and the fact that $\sin_q(\pi_1(1-\alpha)) = \sin_q(\pi_1\alpha)$ give the desired result. □

It must be noted that when $a, b, \lambda = 1$ we recover the result in Lemma 2. To show our generalization of (23), let $0 < \beta < \min\{1/p, 1/p'\}$ and we consider

$$I := \int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{(ax+by)_q^\lambda} d_qx\, d_qy = \int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{(ax)^\lambda (1+(by)/(ax))_q^\lambda} d_qx\, d_qy$$
$$= \int_0^\infty \int_0^\infty \frac{|f(x)|(x/y)^\beta}{(ax)^{\lambda/p}(1+(by)/(ax))_q^{\lambda/p}} \cdot \frac{|g(y)|(y/x)^\beta}{(ax)^{\lambda/p'}(1+q^{\lambda/p}(by)/(ax))_q^{\lambda/p'}} d_qx\, d_qy.$$

By Hölder's inequality, Equation (6), and Lemma 5, we have:

$$I \leq \left(\int_0^\infty \int_0^\infty \frac{|f(x)|^p x^{p\beta}}{(ax+by)_q^\lambda\, y^{p\beta}} d_qy\, d_qx \right)^{1/p} \left(\int_0^\infty \int_0^\infty \frac{|g(y)|^{p'} y^{p'\beta}}{(ax+q^{\lambda/p}by)_q^\lambda\, x^{p'\beta}} d_qx\, d_qy \right)^{1/p'}$$
$$\leq M_q^{\lambda,\beta}(a,b) \left(\int_0^\infty |f(x)|^p x^{1-\lambda} d_qx \right)^{1/p} \left(\int_0^\infty |g(y)|^{p'} y^{1-\lambda} d_qy \right)^{1/p'},$$

where

$$M_q^{\lambda,\beta}(a,b) := \frac{2\pi_q (ab)^{-\lambda}\, q^{\lambda/p(1-p'\beta-\lambda)}}{((2)_q^\beta (2)_q^{1-\beta})^{1/p}((2)_q^{\lambda+\beta-1}(2)_q^{2-\lambda-\beta})^{1/p'} \sin_q(\alpha\pi_q)}$$
$$\cdot \prod_{k=1}^{\lambda-1} \left[1 + \frac{1-p\beta}{k}\right]_{q^k}^{1/p} \left[1 + \frac{1-p'\beta}{k}\right]_{q^k}^{1/p'}.$$

This proves the following theorem:

Theorem 6 (Generalization of the q-Hilbert's inequality)**.** *Let $1 < p < \infty$, $a, b > 0$, and $\lambda \in \mathbb{N}$. Let $\beta \in \mathbb{R}$ such that $0 < \beta < \min\{1/p, 1/p'\}$. Then, if $f \in L^p([0,\infty), t^{1-\lambda} d_qt)$ and $g \in L^{p'}([0,\infty), t^{1-\lambda} d_qt)$, then,*

$$\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{(ax+by)_q^\lambda} d_qx\, d_qy \leq M_q^{\lambda,\beta}(a,b) \|f\|_p \|g\|_{p'}. \tag{23}$$

References

[1] Yang Bicheng. On new generalizations of Hilbert's inequality. *Journal of Mathematical Analysis and Applications*, 248(1):29 – 40, 2000.

[2] A. De Sole and V. Kac. On integral representations of q-gamma and q-beta functions. *ArXiv Mathematics e-prints*, February 2003.

[3] D.J.H. Garling. *Inequalities: A Journey into Linear Analysis*. Cambridge University Press, 2007.

[4] F. H. Jackson. The basic gamma-function and the elliptic functions. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 76(508):127–144, 1905.

[5] V. Kac and P. Cheung. *Quantum Calculus*. Universitext (1979). Springer New York, 2002.

[6] Taekyun Kim Won Sang Chung and Toufik Mansour. The q-deformed Gamma function and q-deformed polygamma function. *B. Korean Math. Soc.*, 51(4):1155, 2014.

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