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HYPERHOLOMORPHICITY OF MULTISPLIT FUNCTIONS

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Abstract. The purpose of this work is to propose a contribution to the study of the multisplit functions. In details we are interested in the study of the hyperholomorphicity of multisplit functions, we propose an extension of the famous Cauchy-Riemann formulas. We show some interesting results regarding continuation of multisplit functions and that of real functions to multisplit algebra. Moreover, we introduce the concept of co-hyperholomorphicity for multisplit function. basing on the generalized Dirac operator.

1. Introduction

The concept of multisplit numbers has been introduced for the first time in the reference [9] as the generalization of hyperbolic or split numbers in higher dimensions. The main key point was to introduce a unit number satisfying $h^n = 1$ and create, inspiring from the concepts of multicomplex and multidual number see for more details [3, 5, 8, 10], the space of multisplit numbers of order n. It is also proved in [9] that the set of multispit numbers forms a $n-$dimensional associative, commutative and unitary generalized Clifford Algebra and posses a matrix representation involving circular matrices. Additionally, many important algebraic properties of multisplit numbers were provided in the already quoted reference.

V. V. Kisil [5] develops a theory of function of split variables and provides the generalization of the concepts of holomorphic functions, Cauchy-Riemann formulas and some other interesting notions.

The main purpose of the present paper is to extend and promote the research on the theory of hyperbolic functions to multisplit functions. Indeed, we begin by generalize the concept of hyperholomorphcity, will-known in Clifford analysis, see [4, 7, 10], to the case of multisplit functions. We also focus on the generalization of the Cauchy-Riemann formulas as well as few properties of hyperholomorphic functions. Moreover, we introduce the conjugate variables and we try to formulate using them the obtained properties. In addition, we prove maximum principle and we focus on the astablishment of some statement concerning the continuation of multisplit functions and that of real functions to multisplit algebra were also obtained. Furthermore, we introduce the concept of co-hyperholomorphicity for

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multisplit functions by the means of Dirac's operator and we show that every co-hyperholomorphic function satisfies Cauchy's condition.

2. Preliminaries

According to the work of F. Messelmi [9], let us state some basic facts of multisplit numbers.

A multisplit number z is an ordered $n-$tuple of real numbers $(x_0, x_1, ..., x_{n-1})$ associated with the real unit 1 and the powers of the multisplit unit h, such that h satisfies $h^n = 1$ where it differs from the real roots of the equation $s^n = 1$. A multisplit number is usually denoted in the form

$$z = \sum_{i=0}^{n-1} x_i h^i. \tag{2.1}$$

for which we admit that $h^0 = 1$.

We denote by $\mathbb{MH}_{n-1}$ the set of multisplit numbers given by

$$\mathbb{MH}_{n-1} = \left\{ z = \sum_{i=0}^{n-1} x_i h^i \mid x_i \in \mathbb{R} \text{ and } h^n = 1 \right\} \tag{2.2}$$

If $n = 1$, $\mathbb{MH}_0 = \mathbb{R}$ and if $n = 2$, $\mathbb{MH}_1$ is the Clifford algebra of hyperbolic numbers or split numbers, see for more details regarding split numbers the references [11]. Moreover, the multisplit numbers form a commutative ring with characteristic 0. Moreover the inherited multiplication gives the multisplit numbers the structure of $n-$dimensional associative, commutative and unitary generalized Clifford Algebra.

It is also shown in the reference [9] that every multisplit number possess a matrix representation, formulated as follows.

Let us denote by $\mathcal{C}_n(\mathbb{R})$ the subset of $\mathcal{M}_n(\mathbb{R})$ constituted of circulant matrices, it means that

$$\begin{aligned} \mathcal{C}_n(\mathbb{R}) &= \left\{ A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R}) \mid A = \begin{bmatrix} a_1 & a_n & a_{n-1} & \cdots & a_2 \\ a_2 & a_1 & a_n & \cdots & a_3 \\ a_3 & a_2 & a_1 & \cdots & a_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{bmatrix} \right. \\ &\qquad \left. \text{where } a_{ij} \in \mathbb{R},\ 1 \le i, j \le n \right\}. \end{aligned}$$

It is well known that $\mathcal{C}_n(\mathbb{R})$ is a subring of $\mathcal{M}_n(\mathbb{R})$ which forms a $n-$dimensional associative, commutative and unitary Algebra, see [6]. Let us introduce the mapping given by

$$\left\{ \begin{array}{c} \mathcal{R} : \mathbb{MH}_{n-1} \longrightarrow \mathcal{C}_n(\mathbb{R}), \\ \mathcal{R}\left(\sum\limits_{i=0}^{n-1} x_i h^i\right) = \begin{bmatrix} x_0 & x_{n-1} & x_{n-2} & \cdots & x_1 \\ x_1 & x_0 & x_{n-1} & \cdots & x_2 \\ x_2 & x_1 & x_0 & \cdots & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_{n-2} & x_{n-3} & \cdots & x_0 \end{bmatrix} \end{array} \right. \tag{2.3}$$

$\mathcal{R}$ is called the matrix representation of multisplit numbers. Notice that the algebras $\mathbb{MH}_{n-1}$ and $\mathcal{C}_n(\mathbb{R})$ are $\mathcal{R}-$isomorphic.

Denoting by $\mathcal{C}_n^0(\mathbb{R})$ the subset of $\mathcal{C}_n(\mathbb{R})$ defined as

$$\mathcal{C}_n^0(\mathbb{R}) = \{A \in \mathcal{C}_n(\mathbb{R}) \mid \det(A) = 0\}.$$

It is clear that $\mathcal{C}_n(\mathbb{R}) - \mathcal{C}_n^0(\mathbb{R})$ is a multiplicative subgroup of $GL(n,\ \mathbb{R})$.

The null part of $\mathbb{MH}_{n-1}$ was defined as the set $\mathcal{D}_{n-1}$ given by

$$\mathcal{D}_{n-1} = \mathcal{R}^{-1}\left(\mathcal{C}_n^0(\mathbb{R})\right), \tag{2.4}$$

consisting of the zero divisors of $\mathbb{MH}_{n-1}$.

As consequence, the subset of $\mathbb{MH}_{n-1}$ given by $\mathbb{MH}_{n-1}^* = \mathbb{MH}_{n-1} - \mathcal{D}_{n-1}$ is a multiplicative group.

Furthermore, The conjugate of a multisplit number $z = \sum\limits_{i=0}^{n-1} x_i h^i$ was defined by the formula

$$z\bar{z} = \det(\mathcal{R}(z)). \tag{2.5}$$

The following formula has been also shown

$$\bar{z} = \frac{1}{n}\left(\frac{\partial \det(\mathcal{R}(z))}{\partial x_0} + \sum_{i=1}^{n-1} \frac{\partial \det(\mathcal{R}(z))}{\partial x_{n-i}} h^i\right). \tag{2.6}$$

The concept of multisplit conjugation was allow us to construct a structure of modulus over the multisplit algebra $\mathbb{MH}_{n-1}$, given by fonctionnelle

$$\begin{cases} \mathcal{P} : \mathbb{MH}_{n-1} \longrightarrow \mathbb{R}_+, \\ \mathcal{P}(z) = \det(\mathcal{R}(z)) = z\bar{z}. \end{cases} \tag{2.7}$$

There is no chance that the modulus $\mathcal{P}$ induces a norm over the algebra $\mathbb{MH}_{n-1}$. However, we can build a seminorm as

$$\|z\|_{\mathbb{MH}_{n-1}} = |z\bar{z}|^{\frac{1}{n}} \tag{2.8}$$

By virtue of the formula (2.8), we can affirm that the map

$$\begin{cases} \left(\mathbb{MH}_{n-1}^*, \times\right) \longrightarrow \left(\mathbb{R}_+^*, \times\right), \\ z \longmapsto \|z\|_{\mathbb{MH}_{n-1}}, \end{cases} \tag{2.9}$$

is a homomorphism of groups where its kernel is given by

$$\ker\left(\|.\|_{\mathbb{MH}_{n-1}}\right) = S_{\mathbb{MH}_{n-1}}(0,1). \tag{2.10}$$

3. Multisplit Function

Let Ω be a subset of $\mathbb{R}^n$, we denote by $\mathcal{L}(\Omega)$ the subset of $\mathbb{MH}_{n-1}$ defined as follows

$$\mathcal{L}(\Omega) = \left\{ z = \sum_{i=0}^{n-1} x_i h^i \mid (x_0, x_1, ..., x_{n-1}) \in \Omega \right\}. \tag{3.1}$$

It results that $\mathcal{L}$ is an isomorphism of vector spaces and the set $\mathbb{MH}_{n-1}$ could be identified with $\mathbb{R}^n$.

From now on we will denote by Ω an open subset of $\mathbb{R}^n$.

Definition 1. A multisplit function is a mapping from a subset $\mathcal{L}(\Omega) \subset \mathbb{MH}_{n-1}$ to $\mathbb{MH}_{n-1}$.

Let $t = \sum\limits_{i=0}^{n-1} y_i h^i \in \mathcal{L}(\Omega)$ and $f : \mathcal{L}(\Omega) \longrightarrow \mathbb{MH}_{n-1}$ a multisplit function.

Definition 2. We say that the multisplit function f is continuous at t if

$$\lim_{z\to t} f(z) = f(t).$$

where the limit is calculated component by component, it means

$$\lim_{z\to t} f(z) = \lim_{x_i\to y_i} \lim_{i=0,\dots,n} f(z), \tag{3.2}$$

where $z = \sum_{i=0}^{n-1} x_i h^i$.

Definition 3. The function f is continuous in $\mathcal{L}(\Omega) \subset \mathbb{MH}_{n-1}$ if it is continuous at every point of $\mathcal{L}(\Omega)$.

Definition 4. The multisplit function f is said to be differentiable in the multisplit sense at $t = \sum_{i=0}^{n} y_i h^i$ if the following limit exists

$$\frac{df}{dz}(t) = \lim_{z\to t} \frac{f(z) - f(t)}{z - t}. \tag{3.3}$$

Notice here that we must impose the condition $z - t \in \mathbb{MH}^*_{n-1}$. The limit $\frac{df}{dz}(t)$ is said to be the derivative of f at the point t.

If f is differentiable for all points in $\mathcal{L}(U)$ where U is a neighbourhood of $\mathcal{L}^{-1}(t)$, then f is called hyperholomorphic at t.

Definition 5. The function f is hyperholomorphic in $\mathcal{L}(\Omega) \subset \mathbb{MH}_{n-1}$ if it is hyperholomorphic at every point of $\mathcal{L}(\Omega)$.

In the following results we generalize the Cauchy-Riemann formulas to multisplit functions.

Theorem 1. Let f be a multisplit function in $\mathcal{L}(\Omega) \subset \mathbb{MH}_{n-1}$, which can be written in terms of its real and multisplit parts as

$$f(z) = \sum_{i=0}^{n-1} f_i(x_0, x_1, \dots, x_{n-1}) h^i. \tag{3.4}$$

Then, f is hyperholomorphic in $\mathcal{L}(\Omega) \subset \mathbb{MH}_{n-1}$ if and only if its partial derivatives satisfy

$$\frac{\partial f}{\partial x_i} = h^i \frac{\partial f}{\partial x_0} \quad \forall i = 1, \dots, n-1. \tag{3.5}$$

Proof. Let us assume that the function f is hyperholomorphic. Since the limit (3.3) has to be always the same for all the paths going to t it has in particular to exist for the n particular paths in which $x_i - y_i$, $i = 0, \dots, n-1$, such that $z = \sum_{i=0}^{n-1} x_i h^i$, $t = \sum_{i=0}^{n-1} y_i h^i$ and $z - t \in \mathbb{MH}^*_{n-1}$. Hence, one has that

$$\begin{aligned} \lim_{z\to t} \frac{f(z) - f(t)}{z - t} &= \lim_{x_i\to y_i} \frac{f(z) - f(t)}{x_i h^i - y_i h^i} \quad i = 1, \dots, n-1 \\ &= \lim_{x_0\to y_0} \frac{f(z) - f(t)}{x_0 - y_0}. \end{aligned}$$

Thus, the relation (3.5) follows.

Conversely, suppose that the formula (3.5) holds. The differential of f at $t = \sum_{i=0}^{n-1} y_i h^i \in \mathcal{L}(\Omega)$ can be computed as

$$\begin{aligned} f(z+t) - f(z) &= \sum_{i=0}^{n-1} \left(f_i(x_0 + y_0, ..., x_{n-1} + y_{n-1}) - f_i(x_0, ..., x_{n-1}) \right) h^i \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left(\frac{\partial f_i}{\partial x_j} y_j + o(y_j) \right) h^i \\ &= \sum_{j=0}^{n-1} y_j \frac{\partial f}{\partial x_j} + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} o(y_j) h^i \end{aligned}$$

Thus, we can infer by virtue of (3.5) that

$$\begin{aligned} f(z+t) - f(z) &= \frac{\partial f}{\partial x_0} \sum_{j=0}^{n-1} y_j h^j + \sum_{i,j=0}^{n-1} o(y_j) h^i \\ &= \frac{\partial f}{\partial x_0} t + \left(\sum_{i=0}^{n-1} h^i \right) o \left(\sum_{i,j=0}^{n-1} y_j \right). \end{aligned}$$

Which permis us to deduce that the function is f is hyperholomorphic.
In addition, we deduce from the above relation that

$$\frac{df}{dz} = \frac{\partial f}{\partial x_0}. \tag{3.6}$$

Theorem 2 (Cauchy-Riemann Formulas). Let $f(z) = \sum_{i=0}^{n-1} f_i(x_0, x_1, ..., x_{n-1}) h^i$ be a multisplit function in $\mathcal{L}(\Omega)$. Then, f is hyperholomorphic in $\mathcal{L}(\Omega)$ if and only if the partial derivatives of the functions f_i, $i = 0, ...n-1$ satisfy

$$\frac{\partial f_i}{\partial x_0} = \begin{cases} \dfrac{\partial f_k}{\partial x_{n+k-i+1}} \text{ for } k = 0, ..., i-1, \\ \\ \dfrac{\partial f_k}{\partial x_{k-i}} \text{ for } k = i, ..., n. \end{cases} \tag{3.7}$$

Proof. We get by employing formula (3.5)

$$\sum_{i=0}^{n-1} \frac{\partial f_i}{\partial x_j} h^i = h^j \sum_{i=0}^{n-1} \frac{\partial f_i}{\partial x_0} h^i.$$

So, for every $j = 1, ..n-1$ we obtain

$$\begin{aligned} \frac{\partial f_0}{\partial x_j} + \frac{\partial f_1}{\partial x_j} h + ... + \frac{\partial f_{n-1}}{\partial x_j} h^{n-1} &= \frac{\partial f_{n-j}}{\partial x_0} + \frac{\partial f_{n-j+1}}{\partial x_0} h + ... + \frac{\partial f_{n-1}}{\partial x_0} h^{j-1} \\ &\quad + \frac{\partial f_0}{\partial x_0} h^j + \frac{\partial f_1}{\partial x_0} h^{j+1} + ... + \frac{\partial f_{n-j-1}}{\partial x_0} h^{n-1}. \end{aligned}$$

Thus, the result will be done.

Proposition 3. Let $f(z) = \sum_{i=0}^{n-1} f_i(x_0, x_1, ..., x_{n-1}) h^i$ be a multisplit function in $\mathcal{L}(\Omega) \subset \mathbb{MH}_{n-1}$. If f is hyperholomorphic in $\mathcal{L}(\Omega)$, then its partial derivatives satisfy the following partial differential system

$$\frac{\partial^n f}{\partial x_j^n} - \frac{\partial^n f}{\partial x_0^n} = 0 \quad \forall j = 1, ..., n-1. \tag{3.8}$$

Further, if there exists two integers k and j such that $kj = n$ then

$$\frac{\partial^k f}{\partial x_j^k} - \frac{\partial^k f}{\partial x_0^k} = 0. \tag{3.9}$$

The proof is a direct consequence of formula (3.5).

We deduce from this proposition that the partial derivatives of the functions f_i solve the partial differential systems

$$\frac{\partial^n f_i}{\partial x_j^n} - \frac{\partial^n f_i}{\partial x_0^n} = 0 \quad \forall i = 0, ..., n-1 \text{ and } \forall j = 1, ..., n-1. \tag{3.10}$$

The following result can be easily established as an immediate consequence of the Cauchy-Riemann formulas.

Proposition 4. Let $f(z) = \sum_{i=0}^{n-1} f_i h^i$ be a multisplit function in $\mathcal{L}(\Omega) \subset \mathbb{MH}_{n-1}$. Suppose that f is hyperholomorphic in $\mathcal{L}(\Omega)$. Then the following formula holds

$$\frac{\partial}{\partial x_j} \sum_{i=0}^{n-1} f_i = \sum_{k=0}^{n-1} \frac{\partial f_m}{\partial x_k} \quad \forall j, m = 0, ..., n-1. \tag{3.11}$$

Define now the concept conjugate variables by the formulas

$$\left\{ \begin{array}{c} \eta_0 = z = x_0 + x_1 h + x_2 h^2 + ... + x_{n-1} h^{n-1}, \\ \eta_1 = -x_0 h^{n-1} + x_1 - x_2 h - ... - x_{n-1} h^{n-2}, \\ \eta_2 = -x_0 h^{n-2} - x_1 h^{n-1} + x_2 - x_3 h - ... - x_{n-1} h^{n-3}, \\ \vdots \\ \eta_{n-1} = -x_0 h - x_1 h^2 - ... - x_{n-2} h^{n-1} + x_{n-1}. \end{array} \right. \tag{3.12}$$

These equations can be also written in matrix form as

$$\begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & h & h^2 & \dots & h^{n-1} \\ -h^{n-1} & 1 & -h & \dots & -h^{n-2} \\ -h^{n-2} & -h^{n-1} & 1 & \dots & -h^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -h & -h^2 & \dots & -h^{n-1} & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}. \tag{3.13}$$

Proposition 5. The following formulas hold

$$dx_0 = \frac{1}{2} \left((3-n)\, d\eta_0 - \sum_{i=1}^{n-1} h^i d\eta_i \right), \tag{3.14}$$

$$dx_i = \frac{1}{2} \left(h^{n-i} d\eta_0 + d\eta_i \right) \text{ for } i = 1, ..., n-1. \tag{3.15}$$

Proof. It is clear according to the definition of conjugate variables that

$$\eta_0 - h\eta_1 = 2x_0 + 2\sum_{i=2}^{n-1} h^i x_i. \tag{3.16}$$

Moreover, one finds for each $i=1,...,n-1$

$$\begin{aligned} x_i &= \frac{1}{2h^i}\left(h^i\eta_i+\eta_0\right) \\ &= \frac{1}{2}\left(\eta_i+h^{n-i}\eta_0\right). \end{aligned}$$

So, formula (3.14) easily follows. To prove the formula (3.15) it is enough to put the two expression (3.15) and (3.16) together and use some simple algebraic manipulations.

We will introduce in the following the notion of the conjugate partial derivatives. To this aim, we check keeping in mind the previous proposition that the total derivative of f can be written in terms of conjugate variables

$$\begin{aligned} df &= \sum_{i=0}^{n-1}\frac{\partial f}{\partial x_i}dx_i \\ &= \frac{1}{2}\frac{\partial f}{\partial x_0}\left((3-n)\,d\eta_0-\sum_{i=1}^{n-1}h^i d\eta_i\right)+\frac{1}{2}\sum_{i=1}^{n-1}\frac{\partial f}{\partial x_i}\left(h^{n-i}d\eta_0+d\eta_i\right) \\ &= \frac{1}{2}\left((3-n)\,\frac{\partial f}{\partial x_0}+\sum_{i=1}^{n-1}h^{n-i}\frac{\partial f}{\partial x_i}\right)d\eta_0+\frac{1}{2}\sum_{i=1}^{n-1}\left(\frac{\partial f}{\partial x_i}-h^i\frac{\partial f}{\partial x_0}\right)d\eta_i. \end{aligned}$$

Hence, we can introduce the concept of conjugate partial derivatives as

$$\frac{\partial}{\partial\eta_0}=\frac{1}{2}\left((3-n)\,\frac{\partial}{\partial x_0}+\sum_{i=1}^{n-1}h^{n-i}\frac{\partial}{\partial x_i}\right), \tag{3.17}$$

$$\frac{\partial}{\partial\eta_i}=\frac{1}{2}\left(\frac{\partial}{\partial x_i}-h^i\frac{\partial}{\partial x_0}\right)\text{ for } i=1,...,n-1. \tag{3.18}$$

The following is a direct consequence of Theorem 1.

Proposition 6. A multisplit function f is hyperholomorphic if and only if it fulfills the relations

$$\frac{\partial f}{\partial\eta_0}=\frac{df}{dz}\text{ and }\frac{\partial f}{\partial\eta_i}=0\text{ for } i=1,...,n-1, \tag{3.19}$$

In the next we show a statment regarding multisplit constant functions.

Proposition 7. Denote by Ω an open and connected set of $\mathbb{R}^n$. Let f be an hyperholomorphic function defined in $\mathcal{L}(\Omega)$.

If $\dfrac{df}{dz}=0$ in $\mathcal{L}(\Omega)$ then $f=$constant.

Proof. If $\dfrac{df}{dz}=0$ then $\dfrac{\partial f}{\partial x_0}=0$. Thus, applying Theorem 1 it comes $\dfrac{\partial f_i}{\partial x_j}=0$ for every $i,j=1,...,n-1$. So, since Ω is connected we can infer that $f=$constant.

Theorem 8 (**Maximum principle).** Denote by Ω an open set of $\mathbb{R}^n$ with piecewise differentiable boundary $\partial\Omega$ such that $\partial\Omega=\Gamma_1\cup\Gamma_2$ where Γ_1 and Γ_2 are two measurable parts satisfying $\Gamma_1\cap\Gamma_2=\emptyset$. Let $f(z)=\sum\limits_{i=0}^{n-1}f_ih^i$ be an hyperholomorphic function in $\mathcal{L}(\Omega)$, verifies $f=0$ on $\mathcal{L}(\Gamma_1)$, then $f=0$ in $\mathcal{L}(\Omega)$.

Proof. Formula (3.11) allows us to find

$$\frac{\partial}{\partial x_j}\sum_{i=0}^{n-1}f_i=\frac{\partial}{\partial x_0}\sum_{i=0}^{n-1}f_i\quad\forall j=1,...,n-1. \tag{3.20}$$

Since $f = 0$ on $\mathcal{L}(\Gamma_1)$, we can infer $\sum_{i=0}^{n-1} f_i = 0$ on $\partial\Omega$. So, under this boundary condition the solution of the linear partial differential system (3.20) will be given by $\sum_{i=0}^{n-1} f_i = 0$. Thus, one can be write the function f as

$$f(z) = \sum_{i=0}^{n-2} f_i h^i - \left(\sum_{i=0}^{n-2} f_i\right) h^{n-1}. \tag{3.21}$$

Furthermore, again by using formula (3.11) one obtains

$$\sum_{j=0}^{n-1} \frac{\partial f_i}{\partial x_j} = \sum_{j=0}^{n-1} \frac{\partial f_0}{\partial x_j} \quad \forall i = 1, ..., n-1. \tag{3.22}$$

In particular, for $i = n-1$ it follows, thanks to (3.21)

$$-\sum_{j=0}^{n-1} \frac{\partial}{\partial x_j} \left(\sum_{k=0}^{n-2} f_k\right) = \sum_{j=0}^{n-1} \frac{\partial f_0}{\partial x_j}. \tag{3.23}$$

Then one can find, puting together equation (3.22) for $i = 1, ..., n-2$ and (3.23)

$$\sum_{j=0}^{n-1} \frac{\partial f_0}{\partial x_j} = 0.$$

This yields, under the boundary condition $f_0 = 0$ on Γ_1,

$$f_0 = 0 \text{ on } \Omega.$$

Thus, by (3.22)

$$\sum_{j=0}^{n-1} \frac{\partial f_i}{\partial x_j} = 0 \quad \forall i = 1, ..., n-1.$$

So, the fact that $f_i = 0$ on Γ_1 permits us to deduce that $f_i = 0$ on Ω. Which allows to conclude the proof.

Theorem 9 (Multisplit continuation principle). Denote by Ω an open and connected set of $\mathbb{R}^n$ with piecewise differentiable boundary and by Ω_0 an open subset of Ω possessing a piecewise differentiable boundary strictly contained in Ω. Denote by f and g two hyperholomorphic functions in $\mathcal{L}(\Omega)$ such that

$$f = g \text{ in } \mathcal{L}(\Omega_0). \tag{3.24}$$

Then

$$f = g \text{ in } \mathcal{L}(\Omega). \tag{3.25}$$

Proof. The fact that f and g are hyperholomorphic in $\mathcal{L}(\Omega)$ and $f = g$ in $\mathcal{L}(\Omega_0)$ leads to $f - g = 0$ in $\mathcal{L}(\partial\Omega_0)$. So, using the previous maximum principle we can infer that $f - g = 0$ in $\mathcal{L}\left(\Omega \setminus \overline{\Omega_0}\right)$. So, the proof can be deduced.

Theorem 10 (Continuation of real functions). Let g be a $\mathcal{C}^1-$real function in an open connected set U of $\mathbb{R}$. Then, there exists an open and connected set Ω of $\mathbb{R}^n$ containing U and an hyperholomorphic function $\widetilde{g}$ in $\widetilde{U} = \mathcal{L}(\Omega)$ such that

$$\widetilde{g} = g \text{ in } U. \tag{3.26}$$

$\widetilde{U}$ is called the hyperholomorphicity domain.

Proof. It is straightforward to check that the solution of the linear partial differential system (3.20), characterizing hyperholomorphic functions, fulfills the relation

$$\sum_{i=0}^{n-1} f_i = \varphi\left(\sum_{i=0}^{n-1} x_i\right), \tag{3.27}$$

where φ is a $\mathcal{C}^1-$real function.

Suppose now that $f(x_0) \in \mathbb{R}$. This allows to get

$$\begin{aligned} f_0(x_0, 0, ...,) &= f(x_0) = \varphi(x_0), \\ f_i(x_0, 0, ...,) &= 0 \ \ \forall i = 1, ..., n-1. \end{aligned}$$

So, to conclude the proof it is enough to choose $\varphi = g$.

We remark in this proof that if $z \in \mathcal{L}(\Omega)$ we have necessarily $\sum\limits_{i=0}^{n-1} x_i \in U$.

Proposition 11. Denoting by Ω an open connected set of $\mathbb{R}^n$. Let $f = \sum\limits_{i=0}^{n-1} f_i h^i$ be an hyperholomorphic function in $\mathcal{L}(\Omega)$. Then,

$$f(z) - f\left(\sum_{i=0}^{n-1} x_i\right) \in \mathcal{D}_{n-1}. \tag{3.28}$$

Proof. Taking into account (3.27), f can be rewritten

$$f(z) = \sum_{i=0}^{n-2} f_i h^i + \left[f\left(\sum_{i=0}^{n-1} x_i\right) - \sum_{i=0}^{n-2} f_i\right] h^{n-1}. \tag{3.29}$$

Thus,

$$f(z) - f\left(\sum_{i=0}^{n-1} x_i\right) h^{n-1} = \sum_{i=0}^{n-2} f_i \left(1 - h^{n-i-1}\right) h^i.$$

This yields, by multiplying the two side members by $\sum\limits_{i=0}^{n-1} h^i$

$$\left[f(z) - f\left(\sum_{i=0}^{n-1} x_i\right) h^{n-1}\right] \sum_{i=0}^{n-1} h^i = \left[f(z) - f\left(\sum_{i=0}^{n-1} x_i\right)\right] \sum_{i=0}^{n-1} h^i = 0.$$

Which eventualy gives (3.28).

We deduce in particular that $f\left(\sum\limits_{i=0}^{n-1} x_i\right)$ is an eigenvalue of the matrix $\mathcal{R}(f(z))$.

Proposition 12. Let g be a $\mathcal{C}^1-$real function in an open connected set U of $\mathbb{R}$. If $n = 2$ then

$$\widetilde{U} = \mathcal{L}\left(\left\{(x_0, x_1) \in \mathbb{R}^2 \mid x_0 + x_1 \in U \text{ and } x_0 - x_1 \in U\right\}\right),$$

and

$$\widetilde{g}(x_0 + x_1 h) = \frac{1}{2}\left(g(x_0 + x_1) + g(x_0 - x_1)\right) + \frac{1}{2}\left(g(x_0 + x_1) - g(x_0 - x_1)\right) h.$$

Proof. Formula (3.29) leads to

$$\widetilde{g}(x_0 + x_1 h) = \widetilde{g_0}(x_0, x_1) + \left(g(x_0 + x_1) - \widetilde{g_0}(x_0, x_1)\right) h^n.$$

So, we can infer making use (3.7)

$$\frac{\partial \widetilde{g_0}}{\partial x_0} + \frac{\partial \widetilde{g_0}}{\partial x_1} = g'(x_0 + x_1).$$

Thus, it comes

$$\widetilde{g_0}(x_0, x_1) = \varphi(x_0 - x_1) + \frac{1}{2} g(x_0 + x_1).$$

Then since $\widetilde{g_0}(x_0, 0) = g(x_0)$ the statement immediately results.

Proposition 13.

1. Let w be the real-valued $1-$form given by

$$w = \left(\sum_{i=0}^{n-1} f_i\right)\left(\sum_{i=0}^{n-1} dx_i\right). \tag{3.30}$$

If f is hyperholomorphic then w is closed.

2. Let w_i be the real-valued $1-$forms given by

$$f(z)\, dz = \sum_{i=0}^{n-1} h^i w_i. \tag{3.31}$$

If f is hyperholomorphic then w_i are closed.

Proof. 1. Denoting by φ the real function

$$\varphi = \sum_{i=0}^{n-1} f_i.$$

Then

$$\begin{aligned} dw &= d\varphi \wedge \left(\sum_{i=0}^{n-1} dx_i\right) \\ &= \left(\sum_{j=0}^{n-1} \frac{\partial \varphi}{\partial x_j} dx_j\right) \wedge \left(\sum_{i=0}^{n-1} dx_i\right) \end{aligned}$$

So, we obtain by applying Proposition 4 that

$$dw = \frac{\partial \varphi}{\partial x_0}\left(\sum_{j=0}^{n-1} dx_j\right) \wedge \left(\sum_{i=0}^{n-1} dx_i\right) = 0.$$

Which achievs the proof of the first assertion.

2. Relation (3.31) can be written making use the conjugate variables

$$f(\eta_0)\, d\eta_0 = \sum_{i=0}^{n-1} h^i w_i.$$

Thus, it becomes

$$-\sum_{j=1}^{n-1} \frac{\partial f}{\partial \eta_i} d\eta_0 \wedge d\eta_i = \sum_{i=0}^{n-1} h^i dw_i.$$

Consequently, we deduce that if f is hyperholomorphic then w_i are necessarily closed.

Defining now the Dirac operator for multisplit function as

$$D = \sum_{i=0}^{n-1} h^i \frac{\partial}{\partial x_i}. \tag{3.32}$$

Let now Ω be a bounded and orientable volume of $\mathbb{R}^n$ with piecewise differentiable boundary and denoting by $d\mu$ the surface measure on $\partial\Omega$ given by

$$d\mu = \sum_{i=0}^{n-1} (-1)^i h^i dx_0 \wedge ... \wedge dx_{i-1} \wedge \widehat{dx_i} \wedge dx_{i+1} \wedge ... \wedge dx_n. \tag{3.33}$$

Definition 6. Let f be a multisplit function definied in $\mathcal{L}(\Omega)$.
1. The function f is called co-hyperholomorphic if

$$Df = 0. \tag{3.34}$$

2. We say that the function f satisfies the Cauchy condition if

$$\int_{\partial\Omega} f d\mu = 0. \tag{3.35}$$

We can finally state the following Theorem.
Theorem 14.
1. The function f is co-hyperholomorphic in $\mathcal{L}(\Omega)$ if and only if it satisfies the Cauchy condition.
2. If f is hyperholomorphic then $Df \in \mathcal{D}_{n-1}$
Proof. 1. By Green's formula one has that

$$\begin{aligned}
\int_{\partial\Omega} f d\mu &= \int_{\Omega} d(f d\mu) \\
&= \int_{\Omega} d\left(f \sum_{i=0}^{n-1} (-1)^i h^i dx_0 \wedge ... \wedge dx_{i-1} \wedge \widehat{dx_i} \wedge dx_{i+1} \wedge ... \wedge dx_n \right) \\
&= \sum_{i=0}^{n-1} (-1)^i h^i \int_{\Omega} \frac{\partial f}{\partial x_i} dx_i \wedge dx_0 \wedge ... \wedge dx_{i-1} \wedge \widehat{dx_i} \wedge dx_{i+1} \wedge ... \wedge dx_n \\
&= \sum_{i=0}^{n-1} h^i \int_{\Omega} \frac{\partial f}{\partial x_i} dx_0 \wedge ... \wedge dx_n = \int_{\Omega} Df dx_0 \wedge ... \wedge dx_n.
\end{aligned}$$

This permits us to conclude the proof.
2. Obviously, if f is hyperholomorphic, Theorem 1 allows us to get

$$Df = \left(\sum_{i=1}^{n-1} h^i \right) \frac{\partial f}{\partial x_0}.$$

Hence, since $\det \mathcal{R} \left(\sum_{i=1}^{n-1} h^i \right) = 0$, the claim follows.

Proposition 15. Let $f(z) = \sum_{i=0}^{n-1} f_i h^i$ be a multisplit function in $\mathcal{L}(\Omega) \subset \mathbb{MH}_{n-1}$. If f is co-hyperholomorphic in $\mathcal{L}(\Omega)$. Then

$$\sum_{i,j=0}^{n-1} \frac{\partial f_i}{\partial x_j} = 0. \tag{3.36}$$

Proof. It follows using some algebraic manipulations

$$\begin{aligned} Df &= \sum_{i=0}^{n-1} h^i \frac{\partial f}{\partial x_i} \\ &= \sum_{i=0}^{n-1} \left(\sum_{j=0}^{i} \frac{\partial f_{i-j}}{\partial x_j} + \sum_{j=i+1}^{n-1} \frac{\partial f_{n+i-j}}{\partial x_j} \right) h^i. \end{aligned}$$

So, if f is co-hyperholomorphic, we get

$$\begin{cases} \frac{\partial f_0}{\partial x_0} + \frac{\partial f_{n-1}}{\partial x_1} + \frac{\partial f_{n-2}}{\partial x_2} + ... + \frac{\partial f_1}{\partial x_{n-1}} = 0, \\ \frac{\partial f_1}{\partial x_0} + \frac{\partial f_0}{\partial x_1} + \frac{\partial f_{n-1}}{\partial x_2} + ... + \frac{\partial f_2}{\partial x_{n-1}} = 0, \\ \frac{\partial f_2}{\partial x_0} + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_0}{\partial x_2} + \frac{\partial f_{n-1}}{\partial x_3} + ... + \frac{\partial f_3}{\partial x_{n-1}} = 0, \\ \vdots \\ \frac{\partial f_{n-1}}{\partial x_0} + \frac{\partial f_{n-2}}{\partial x_1} + \frac{\partial f_{n-3}}{\partial x_2} + ... + \frac{\partial f_0}{\partial x_{n-1}} = 0. \end{cases}$$

Thus, the formula can be easily deduced.

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