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GOODMAN-RONNING TYPE CLASS OF HARMONIC ERROR FUNCTION USING SALAGEAN OPERATOR

A. M. GBOLAGADE, I. T. AWOLERE AND A. O. FADARE

Abstract. In this work, the authors wish to establish some results involving coefficient inequality, distortion bounds, extreme points, convolution and convex combinations for a new class of Goodman-Ronning type class $\overline{T}_{H,n}(\alpha, \lambda, \mu, \beta)$ of harmonic univalent functions associated with error function using Salagean operator. Varying some of the parameters involved in the established results, new and various well known results are obtained.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a simply connected complex domain $D \subset C$ if both u and v are real harmonic in D. In any simply connected domain $D, f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $\left|h^{'}(z)\right| > \left|g^{'}(z)\right|, z \in D, [5]$.
Denote $\overline{T}_{H,n}$ the class of function $f = h + \overline{g}$ which are harmonic, univalent in the open unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f^{'}(0) - 1 = 0$. Then, $f = h + \overline{g} \in \overline{T}_{H,n}$. Hence, the analytic function h and g can be written as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1$$

The class $\overline{T}_{H,n}$ reduces to the class S of normalized analytic univalent functions if the co-analytic part of f is zero. Hence, $f(z)$ can be axpressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

For more details and results on harmonic function, refer to $[2, 8, 13]$.
The error function is a special function which occurs in probability, statistics, material science and partial differential equations. The error function was defined

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because of the normal curve and shows up anywhere the normal curve appears. It presents in many disciplines of physics, chemistry, biology, thermo mechanics and mass flow. It also occurs in theoretical aspects of many parts of atmosperic science [1, 12].

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z exp(-t^2)dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)k!} \tag{2}$$

Many authors like Alzer [4], Coman [6] and Elbert et al [9] have studied the properties and inequality of error function with many interesting results. Let A denotes the class of functions of the form (1) which are analytic in the unit disk $U = \{z \in C : |z| < 1\}$ and normalized by $f(0) = f^{'}(0) - 1 = 0$. Let S be the subclass of A consisting of univalent function in U.
A function $f \in A$ is said to be in the class S^* of starlike functions in U if it satisfies the inequality

$$Re\left\{\frac{zf^{'}(z)}{f(z)}\right\} > 0, z \in U \tag{3}$$

A function $f \in A$ is said to be in the class C of convex functions in U if it satisfies the inequality

$$Re\left\{1 + \frac{zf^{''}(z)}{f^{'}(z)}\right\} > 0, z \in U \tag{4}$$

The genegalised Salagean derivative operator introduced by Al-Oboudi [3] is

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k z^k \tag{5}$$

Let $\wp$ be the class of modified error function which was introduced and studied by [11]

$$\wp = A * Erf =$$

$$\left\{F : F(z) = (f * Erf)(z) = z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(2k-1)(k-1)!} a_k z^k, f \in A\right\} \tag{6}$$

where Erf be a normalized analytic function which is given by

$$Erf(z) = \frac{\sqrt{\pi z}}{2} f(\sqrt{z}) = z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(2k-1)(k-1)!} Z^k \tag{7}$$

For $f(z)$ given by (1) and g with the Taylor series $g(z) = z + b_2z^2 + ...$, the Hadamard (or Convolution) denoted by $f * g$ is defined as

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \tag{8}$$

Using (8), this definition is stated.
Definition 1: Let $D_\lambda^n Erf = D_\lambda^n Erh + \overline{D_\lambda^n Erg}$ with h and g be analytic in U,

where

$$D_\lambda^n Erh(z) = z + \sum_{k=2}^{\infty} \frac{[1+\lambda(k-1)]^n(-1)^{k-1}}{(2k-1)(k-1)!} a_k z^k$$

and

$$\overline{D_\lambda^n Erg(z)} = \sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]^n(-1)^{k-1}}{(2k-1)(k-1)!} \overline{b_k}\overline{z^k}, |b_1| < 1$$

Then, $f \in \overline{T}_{H,n}(\alpha, \lambda, \mu, \beta)$ if and only if, $0 \leq \beta < 1, 0 \leq \alpha \leq 1$ and $\mu \in C/0$. Hence,

$$Re\left\{1 + \frac{1-\alpha}{\mu}\left[\frac{zD_\lambda^n Erh'(z) - z\overline{D_\lambda^n Erg'(z)}}{D_\lambda^n Erh(z) + \overline{D_\lambda^n Erg(z)}} - 1\right]\right\} > \beta \tag{9}$$

For the purpose of this work, the following lemmas shall be stated.
Lemma A [10]: Let $f = h + \overline{g}$ with h and g given by (1), then,

$$\sum_{k=2}^{\infty} \frac{k-\beta}{1-\beta} |a_k| + \sum_{k=1}^{\infty} \frac{k+\beta}{1-\beta} |b_k| \leq 1 \tag{10}$$

where $0 \leq \beta < 1$, f is harmonic orientation preserving, univalent in U and $f \in \overline{S_H^*}(\beta)$.
Lemma B [7]: Let $f = h + \overline{g}$ with h and g given by (1), if,

$$\sum_{k=2}^{\infty} k^n \frac{\alpha k(k-1) + k - \beta}{1-\beta} |a_k| + \sum_{k=1}^{\infty} k^n \frac{\alpha k(k+1) + k + \beta}{1-\beta} |b_k| \leq 1 \tag{11}$$

for some $\beta, (0 \leq \beta < 1)$ and $\alpha \geq 0$, then f is harmonic orientation preserving, univalent in U and $f \in \overline{S}_{H,n}^*(\alpha, \beta)$.

2. Main Result

Theorem 1: Let $f \in \overline{T}_{H,n}(\alpha, \lambda, \mu, \beta)$, then,

$$\sum_{k=2}^{\infty} A \frac{[(\alpha-1)(k-1) - (1-\beta)|\mu|]}{(1-\beta)|\mu|} |a_k| + \sum_{k=1}^{\infty} A \frac{[(1-\alpha)(k+1) - (1-\beta)|\mu|]}{(1-\beta)|\mu|} \left|\bar{b}_k\right| \leq 1$$

Proof: From (9)

$$Re\left\{1 + \frac{1-\alpha}{\mu}\left[\frac{zD_\lambda^n Erh'(z) - z\overline{D_\lambda^n Erg'(z)}}{D_\lambda^n Erh(z) + \overline{D_\lambda^n Erg(z)}} - 1\right]\right\} > \beta \tag{12}$$

$$\Rightarrow 1 + \frac{1-\alpha}{\mu}\left[\frac{1 + \sum_{k=2}^{\infty} kAa_k z^{k-1} - \sum_{k=1}^{\infty} kA\bar{b}_k \overline{z}^{k-1}}{1 + \sum_{k=2}^{\infty} Aa_k z^{k-1} + \sum_{k=1}^{\infty} A\bar{b}_k \overline{z}^{k-1}} - 1\right] > \beta \tag{13}$$

where $A = \frac{[1+\lambda(k-1)]^n(-1)^{k-1}}{(2k-1)(k-1)!}$

$$\Rightarrow 1 + \frac{1-\alpha}{\mu}$$

$$\left[\frac{1+\sum_{k=2}^{\infty}kAa_kz^{k-1}-\sum_{k=1}^{\infty}kA\bar{b}_k\bar{z}^{k-1}-1-\sum_{k=2}^{\infty}Aa_kz^{k-1}-\sum_{k=1}^{\infty}A\bar{b}_k\bar{z}^{k-1}}{1+\sum_{k=2}^{\infty}Aa_kz^{k-1}+\sum_{k=1}^{\infty}A\bar{b}_k\bar{z}^{k-1}}\right]>\beta \quad (14)$$

$$\Rightarrow 1-\beta+\left[\frac{\sum_{k=2}^{\infty}(1-\alpha)(k-1)Aa_kz^{k-1}-\sum_{k=1}^{\infty}(1-\alpha)(k+1)A\bar{b}_k\bar{z}^{k-1}}{\mu+\sum_{k=2}^{\infty}\mu Aa_kz^{k-1}+\sum_{k=1}^{\infty}\mu A\bar{b}_k\bar{z}^{k-1}}\right]>0 \quad (15)$$

Further simplification of (15) gives

$$(1-\beta)\mu>\sum_{k=2}^{\infty}\left[(1-\alpha)(k-1)+(1-\beta)\mu\right]Aa_kz^{k-1}+$$

$$\sum_{k=1}^{\infty}\left[(1-\alpha)(k+1)-(1+\beta)\mu\right]A\bar{b}_k\bar{z}^{k-1} \quad (16)$$

Taking bound on (16), we have

$$\sum_{k=2}^{\infty}\left[(1-\alpha)(k-1)+(1-\beta)|\mu|\right]A|a_k|+$$

$$\sum_{k=1}^{\infty}\left[(1-\alpha)(k+1)-(1+\beta)|\mu|\right]A\left|\bar{b}_k\right|\leq(1-\beta)|\mu| \quad (17)$$

which completes the proof.

Corrolary A: Taking $\alpha=0$ in Theorem 1, then $f\in\overline{T}_{H,n}(0,\lambda,\mu,\beta)$

$$\sum_{k=2}^{\infty}\frac{\left[(k-1)+(1-\beta)|\mu|\right]}{(1-\beta)|\mu|}A|a_k|+\sum_{k=1}^{\infty}\frac{\left[(k+1)-(1+\beta)|\mu|\right]}{(1-\beta)|\mu|}A\left|\bar{b}_k\right|\leq 1 \quad (18)$$

where $A=\frac{[1+\lambda(k-1)]^n(-1)^{k-1}}{(2k-1)(k-1)!}$

Theorem 2: Let $f=h+\overline{g}$ with h and g are of the form (1), then $f\in\overline{T}_{H,n}(\alpha,\lambda,\mu,\beta)$ for $|z|=r<1$

$$|f(z)|\leq(1+|b_1|)r-$$

$$\frac{(1+\lambda)^n}{3}\left[\frac{(1-\beta)|\mu|}{(\alpha-1)-(1-\beta)|\mu|}-\frac{2(1-\alpha)-(1-\beta)|\mu|}{(\alpha-1)-(1-\beta)|\mu|}\right]r^2 \quad (19)$$

and

$$|f(z)|\geq(1+|b_1|)r+$$

$$\frac{(1+\lambda)^n}{3}\left[\frac{(1-\beta)|\mu|}{(\alpha-1)-(1-\beta)|\mu|}-\frac{2(1-\alpha)-(1-\beta)|\mu|}{(\alpha-1)-(1-\beta)|\mu|}\right]r^2 \quad (20)$$

Proof: Since $f\in\overline{T}_{H,n}(\alpha,\lambda,\mu,\beta)$, then,

$$|f(z)|\leq(1+|b_1|)r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k\leq(1+|b_1|)r+r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|) \quad (21)$$

$$=(1+|b_1|)r+$$

$$\frac{(1-\beta)\,|\mu|}{[(\alpha-1)-(1-\beta)\,|\mu|]A_2}\sum_{k=2}^{\infty}\frac{[(\alpha-1)-(1-\beta)\,|\mu|]}{(1-\beta)\,|\mu|}A_2(|a_k|+|b_k|)r^2 \tag{22}$$

and so

$$|f(z)| \le (1+|b_1|)r+$$

$$\frac{(1-\beta)\,|\mu|}{[(\alpha-1)-(1-\beta)\,|\mu|]A_2}\sum_{k=2}^{\infty}A[\frac{[(\alpha-1)(k-1)-(1-\beta)\,|\mu|]}{(1-\beta)\,|\mu|}\,|a_k|]r^2$$

$$+[\frac{(1-\alpha)(k+1)-(1-\beta)\,|\mu|}{(1-\beta)\,|\mu|}\,|b_k|]r^2 \tag{23}$$

$$\le (1+|b_1|)r+\frac{(1-\beta)\,|\mu|}{[(\alpha-1)-(1-\beta)\,|\mu|]A_2}\left[1-\frac{2(1-\alpha)-(1-\beta)\,|\mu|}{(1-\beta)\,|\mu|}\,|b_1|\right]r^2 \tag{24}$$

$$=(1+|b_1|)r+\frac{1}{A_2}\left[\frac{(1-\beta)\,|\mu|}{(\alpha-1)-(1-\beta)\,|\mu|}-\frac{2(1-\alpha)-(1-\beta)\,|\mu|}{(\alpha-1)-(1-\beta)\,|\mu|}\right]r^2 \tag{25}$$

where $A_2=-\frac{3}{(1+\lambda)^n}$
The proof of (20) is similar, hence we omit it.
The upper bound given for $f\in\overline{T}_{H,n}(\alpha,\lambda,\mu,\beta)$ is sharp and the equality occurs for the function

$$f(z)=z+|b_1|\,\overline{z}-$$

$$\frac{(1+\lambda)^n}{3}\left[\frac{(1-\beta)\,|\mu|}{(\alpha-1)-(1-\beta)\,|\mu|}-\frac{2(1-\alpha)-(1-\beta)\,|\mu|}{(\alpha-1)-(1-\beta)\,|\mu|}\,|b_1|\right]\overline{z}^2,(z=r^2) \tag{26}$$

$|b_1|\le\frac{(1-\beta)|\mu|}{2(1-\alpha)-(1-\beta)|\mu|}$

Corrolary B: Let $f\in\overline{T}_{H,n}(0,\lambda,\mu,\beta)$, then,

$$f(z)\le(1+|b_1|)r-\frac{(1+\lambda)^n}{3}\left[\frac{(1-\beta)\,|\mu|}{(1-\beta)\,|\mu|}-\frac{2-(1-\beta)\,|\mu|}{(1-\beta)\,|\mu|}\right]r^2 \tag{27}$$

$$\le(1+|b_1|)r-\frac{(1+\lambda)^n}{3}\left[1-\frac{2-(1-\beta)\,|\mu|}{(1-\beta)\,|\mu|}\right]r^2 \tag{28}$$

and

$$f(z)\ge(1+|b_1|)r+\frac{(1+\lambda)^n}{3}\left[1-\frac{2-(1-\beta)\,|\mu|}{(1-\beta)\,|\mu|}\right]r^2 \tag{29}$$

Theorem 3: Let $f\in\overline{T}_{H,n}(\alpha,\lambda,\mu,\beta)$, then,

$$|f'(z)|\le 1+|b_1|-$$

$$2\frac{(1+\lambda)^n}{3}\left[\frac{(1-\beta)\,|\mu|}{(\alpha-1)-(1-\beta)\,|\mu|}-\frac{2(1-\alpha)-(1-\beta)\,|\mu|}{(\alpha-1)-(1-\beta)\,|\mu|}\,|b_1|\right]r \tag{30}$$

and

$$|f'(z)|\ge 1+|b_1|+$$

$$2\frac{(1+\lambda)^n}{3}\left[\frac{(1-\beta)\,|\mu|}{(\alpha-1)-(1-\beta)\,|\mu|}-\frac{2(1-\alpha)-(1-\beta)\,|\mu|}{(\alpha-1)-(1-\beta)\,|\mu|}\,|b_1|\right]r \tag{31}$$

Proof:

$$f'(z) \leq 1 + |b_1| + \sum_{k=2}^{\infty} k(|a_k| + |b_k|)r^{k-1} \leq 1 + |b_1| + r\sum_{k=2}^{\infty} k(|a_k| + |b_k|) \tag{32}$$

$$= 1 + |b_1| +$$

$$\frac{2(1-\beta)|\mu|}{A_2\left[(\alpha-1)-(1-\beta)|\mu|\right]} \sum_{k=2}^{\infty} \frac{\left[(\alpha-1)-(1-\beta)|\mu|\right]}{2(1-\beta)|\mu|} A_2(|a_k| + |b_k|)r \tag{33}$$

and so

$$|f'(z)| \leq 1 + |b_1| + \frac{2(1-\beta)|\mu|}{A_2\left[(\alpha-1)-(1-\beta)|\mu|\right]} \sum_{k=2}^{\infty} A[\frac{(\alpha-1)(k-1)-(1-\beta)|\mu|}{2(1-\beta)|\mu|}|a_k|]+$$

$$[\frac{(1-\alpha)(k+1)-(1-\beta)|\mu|}{2(1-\beta)|\mu|}|b_k]\, r \tag{34}$$

$$= 1 + |b_1| + \frac{2(1-\beta)|\mu|}{A_2\left[(\alpha-1)-(1-\beta)|\mu|\right]}\left[1 - \frac{2(1-\alpha)-(1-\beta)|\mu|}{2(1-\beta)|\mu|}|b_1|\right] r \tag{35}$$

$$\leq 1 + |b_1| + \frac{2}{A_2}\left[\frac{(1-\beta)|\mu|}{(\alpha-1)-(1-\beta)|\mu|} - \frac{2(1-\alpha)-(1-\beta)|\mu|}{(\alpha-1)-(1-\beta)|\mu|}\right] \tag{36}$$

$$= 1 + |b_1| - \frac{2(1+\lambda)^n}{3}\left[\frac{(1-\beta)|\mu|}{(\alpha-1)-(1-\beta)|\mu|} - \frac{2(1-\alpha)-(1-\beta)|\mu|}{(\alpha-1)-(1-\beta)|\mu|}|b_1|\right] r \tag{37}$$

The proof of (31) follows from theorem 3.
Corrolary C: Let $f \in \overline{T}_{H,n}(\alpha, \lambda, \mu, \beta)$, then,

$$|f'(z)| \leq 1 + |b_1| - \frac{2(1+\lambda)^n}{3}\left[1 - \frac{2-(1-\beta)|\mu|}{(1-\beta)|\mu|}|b_1|\right] r \tag{38}$$

and

$$|f'(z)| \geq 1 + |b_1| + \frac{2(1+\lambda)^n}{3}\left[1 - \frac{2-(1-\beta)|\mu|}{(1-\beta)|\mu|}|b_1|\right] r \tag{39}$$

Theorem 4: Let $0 \leq \alpha_1 < \alpha_2$ and $0 \leq \beta < 1$. Then, $\overline{T}_{H,n}(\alpha_2, \beta, \lambda) \subset \overline{T}^*_{H,n}(\alpha_1, \beta, \lambda)$
Proof: From Theorem 1, it follows that

$$\sum_{k=2}^{\infty} \frac{A\left[(\alpha_1-1)(k-1)-(1-\beta)|\mu|\right]}{(1-\beta)|\mu|}|a_k| + \sum_{k=1}^{\infty} \frac{A\left[(1-\alpha_1)(k+1)-(1-\beta)|\mu|\right]}{(1-\beta)|\mu|}|b_k|$$

$$< \sum_{k=2}^{\infty} \frac{A\left[(\alpha_2-1)(k-1)-(1-\beta)|\mu|\right]}{(1-\beta)|\mu|}|a_k| +$$

$$\sum_{k=1}^{\infty} \frac{A\left[(1-\alpha_2)(k+1)-(1-\beta)|\mu|\right]}{(1-\beta)|\mu|}|b_k| \leq 1 \tag{40}$$

For $f \in \overline{T}^*_{H,n}(\alpha_2, \beta, \lambda)$, hence $f \in \overline{T}^*_{H,n}(\alpha_1, \beta, \lambda)$. Supposing $\alpha > 0, 0 \leq \beta < 1$, then, $\overline{T}^*_{H,n+1}(\alpha, \beta, \lambda) \subset \overline{T}^*_{H,n}(\alpha, \beta, \lambda)$.
Theorem 5: Let $f = h + \overline{g}$, where h and g are given by (1), then $f \in clco\overline{T}^*_{H,n}(\alpha, \beta, \lambda)$

if and only if $f(z)=\sum_{k=1}^{\infty}(X_kh_k+Y_kg_k)$
where $h_1(g)=z, h_k(z)=z+\frac{(1-\beta)|\mu|}{A[(\alpha-1)(k-1)-(1-\beta)|\mu|]}z^k, (k=2,3,...)$
$g_k(z)=z+\frac{(1-\beta)|\mu|}{A[(1-\alpha)(k+1)-(1-\beta)|\mu|]}z^k, (k=1,2,...), \sum_{k=1}^{\infty}(X_k+Y_k)=1, X_k\geq 0, Y_k\geq 0$
In particular, the extreme points of the class $\overline{T}^*_{H,n}(\alpha,\beta,\lambda)$ are $\{h_k\}$ and $\{g_k\}$ respectively.
Proof:

$$f(z)=\sum_{k=1}^{\infty}(X_kh_k+Y_kg_k)$$

$$=\sum_{k=1}^{\infty}(X_k+Y_k)(z)+\sum_{k=2}^{\infty}\frac{(1-\beta)\,|\mu|}{A\,[(\alpha-1)(k-1)-(1-\beta)\,|\mu|]}X_kz^k$$

$$+\sum_{k=1}^{\infty}\frac{(1-\beta)\,|\mu|}{A\,[(1-\alpha)(k+1)-(1-\beta)\,|\mu|]}Y_k\overline{z}^k \tag{41}$$

$$=z+\sum_{k=2}^{\infty}\frac{(1-\beta)\,|\mu|}{A\,[(\alpha-1)(k-1)-(1-\beta)\,|\mu|]}X_kz^k+$$

$$\sum_{k=1}^{\infty}\frac{(1-\beta)\,|\mu|}{A\,[(1-\alpha)(k+1)-(1-\beta)\,|\mu|]}Y_k\overline{z}^k \tag{42}$$

But

$$\sum_{k=2}^{\infty}\frac{A\,[(\alpha-1)(k-1)-(1-\beta)\,|\mu|]}{(1-\beta)\,|\mu|}\left[\frac{(1-\beta)\,|\mu|}{A\,[(\alpha-1)(k-1)-(1-\beta)\,|\mu|]}X_k\right]+$$

$$\sum_{k=1}^{\infty}\frac{A\,[(1-\alpha)(k+1)-(1-\beta)\,|\mu|]}{(1-\beta)\,|\mu|}\left[\frac{(1-\beta)\,|\mu|}{A\,[(1-\alpha)(k+1)-(1-\beta)\,|\mu|]}Y_k\right]\leq 1$$

$$\Rightarrow\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1-X_1\leq 1 \tag{43}$$

Thus, $f\in clco\overline{T}_{H,n}(\alpha,\beta,\lambda)$.
Conversely, supposing that $f\in clcoH\overline{T}_{H,n}(\alpha,\beta,\lambda)$, set,

$$X_k=\frac{A\,[(\alpha-1)(k-1)-(1-\beta)\,|\mu|]}{(1-\beta)\,|\mu|}\,|a_k|\,,(k=2,3,...) \tag{44}$$

and

$$Y_k=\frac{A\,[(1-\alpha)(k+1)-(1-\beta)\,|\mu|]}{(1-\beta)\,|\mu|}\,|b_k|\,,(k=1,2,...) \tag{45}$$

Then by the inequality (43), we have $0\leq X_k\leq 1(k=2,3,...)$ and $0\leq Y_k\leq 1(k=1,2,...)$. Define $X_1=1-\sum_{k=2}^{\infty}X_k-\sum_{k=1}^{\infty}Y_k$ and $X_1\geq 0$. Thus, we obtain $f(z)=\sum_{k=1}^{\infty}(X_kh_k+Y_kg_k)$, which completes the proof.

Convolution and Convex Combinations
For two harmonic functions

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k+\sum_{k=1}^{\infty}\overline{b}_k\overline{z}_k \tag{46}$$

and

$$F(z)=z+\sum_{k=2}^{\infty}A_kz^k+\sum_{k=1}^{\infty}\overline{B}_k\overline{z}_k \tag{47}$$

The convolution of (46) and (47) is defined as

$$(f*F)(z)=z+\sum_{k=2}^{\infty}a_kA_kz^k+\sum_{k=1}^{\infty}\overline{b}_k\overline{B}_k\overline{z}^k \tag{48}$$

Using (48), we show that $\overline{T}^{*}_{H,n}(\alpha,\lambda,\mu,\beta)$ is closed under convolution.
Theorem 6: For $0\le\beta<1$ and $\alpha\ge 0$, let $f,F\in\overline{T}_{H,n}(\alpha,\lambda,\mu,\beta)$, then $(f*F)\in\overline{T}_{H,n}(\alpha,\lambda,\mu,\beta)$
Proof: Note that $A_k\le 1$ and $B_k\le 1$. Therefore,

$$(f*F)=\sum_{k=2}^{\infty}\frac{A\left[(\alpha-1)(k-1)-(1-\beta)\left|\mu\right|\right]}{(1-\beta)\left|\mu\right|}\left|a_k\right|+$$

$$\sum_{k=1}^{\infty}\frac{A\left[(1-\alpha)(k+1)-(1-\beta)\left|\mu\right|\right]}{(1-\beta)\left|\mu\right|}\left|\overline{b}_k\right| \tag{49}$$

$$\Rightarrow\sum_{k=2}^{\infty}\frac{A\left[(\alpha-1)(k-1)-(1-\beta)\left|\mu\right|\right]}{(1-\beta)\left|\mu\right|}\left|a_k\right|+$$

$$\sum_{k=1}^{\infty}\frac{A\left[(1-\alpha)(k+1)-(1-\beta)\left|\mu\right|\right]}{(1-\beta)\left|\mu\right|}\left|\overline{b}_k\right|\le 1 \tag{50}$$

Hence, $(f*F)\in\overline{T}_{H,n}(\alpha,\lambda,\mu,\beta)$.
We shall also show that $\overline{T}_{H,n}(\alpha,\lambda,\mu,\beta)$ is closed under convex combination of its members.
Theorem 7: The class $\overline{T}_{H,n}(\alpha,\lambda,\mu,\beta)$ is closed under combination
Proof: For $j=1,2,3...$, let $f_j\in\overline{T}_{H,n}(\alpha,\lambda,\mu,\beta)$, where $f_j(z)$ is given as

$$f_j(z)=z+\sum_{k=2}^{\infty}a_{kj}z^k+\sum_{k=1}^{\infty}\overline{b}_{kj}\overline{z}^k \tag{51}$$

Then by (50)

$$\sum_{k=2}^{\infty}\frac{A\left[(\alpha-1)(k-1)-(1-\beta)\left|\mu\right|\right]}{(1-\beta)\left|\mu\right|}\left|a_{kj}\right|+$$

$$\sum_{k=1}^{\infty}\frac{A\left[(1-\alpha)(k+1)-(1-\beta)\left|\mu\right|\right]}{(1-\beta)\left|\mu\right|}\left|\overline{b}_{kj}\right|\le 1 \tag{52}$$

for $\sum_{j=1}^{\infty} t_j = 1, 0 \le t_j \le 1$, the convex combination of f_j may be written as

$$\sum_{j=1}^{\infty} t_j f_j = z + \sum_{k=2}^{\infty} (\sum_{j=1}^{\infty} t_j a_{kj}) z^k + \sum_{k=1}^{\infty} (\sum_{j=1}^{\infty} t_j \bar{b}_{kj}) \bar{z}^k \tag{53}$$

By convolution,

$$\sum_{k=2}^{\infty} \frac{A\left[(\alpha-1)(k-1)-(1-\beta)\left|\mu\right|\right]}{(1-\beta)\left|\mu\right|} \left|\sum_{j=1}^{\infty} t_j a_{kj}\right| +$$

$$\sum_{k=1}^{\infty} \frac{A\left[(1-\alpha)(k+1)-(1-\beta)\left|\mu\right|\right]}{(1-\beta)\left|\mu\right|} \left|\sum_{j=1}^{\infty} t_j \bar{b}_{kj}\right| \le 1 \tag{54}$$

$$\le \sum_{j=1}^{\infty} t_j (\sum_{k=2}^{\infty} \frac{A\left[(\alpha-1)(k-1)-(1-\beta)\left|\mu\right|\right]}{(1-\beta)\left|\mu\right|} \left|a_{kj}\right| +$$

$$\sum_{k=1}^{\infty} \frac{A\left[(1-\alpha)(k+1)-(1-\beta)\left|\mu\right|\right]}{(1-\beta)\left|\mu\right|} \left|\bar{b}_{kj}\right|)$$

$$\le \sum_{j=1}^{\infty} t_j = 1 \tag{55}$$

Therefore, $\sum_{j=1}^{\infty} t_j f_j \in \overline{T}_{H,n}(\alpha, \lambda, \mu, \beta)$.

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A. M. GBOLAGADE
Department of Mathematics, Emmanuel Alayande College of Education, P. M. B. 1010, Oyo, Oyo State, NIGERIA
E-mail address: gbolagadeam@yahoo.com

I. T. AWOLERE
Department of Mathematics, Emmanuel Alayande College of Education, P. M. B. 1010, Oyo, Oyo State, NIGERIA
E-mail address: awolereibrahim01@gmail.com

A. O. FADARE
Department of Mathematics, Federal College of Education (Special), Oyo, Oyo State, NIGERIA
E-mail address: fadareao2006@yahoo.com