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HERMITE-HADAMARD TYPE INEQUALITIES FOR $(h-m)-$CONVEXITY

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Abstract. In this paper, we establish some Hermite-Hadamard type inequalities for $(h-m)-$convex functions of two variables on the co-ordinates. Also some new Hermite-Hadamard type inequalities for product of $(h-m)-$convex functions are given.

1. Introduction

Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be convex if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

holds for all $x, y \in I$ and $\alpha \in [0,1]$. If reverse of above inequality holds, then f is said to be concave function.

If $f : I \to \mathbb{R}$ is a convex function and $a, b \in I$ with $a < b$, then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_b^a f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is known as Hermite-Hadamard inequality. Both inequalities in (1) hold in reversed if f is concave.

In 1984, Toader [1] introduced the following class of functions.

Definition 1 A function $f : [0,b] \to \mathbb{R}$ is said to be $m-$convex, where $m \in [0,1]$, if for every $x, y \in [0,b]$ and $t \in [0,1]$, we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

The following Hermite-Hadamard type inequality for $m-$convex functions is due to Dragomir [15].

Theorem 1 Let $f : [0,\infty) \to \mathbb{R}$ be $m-$convex function, where $m \in (0,1]$ and $0 \leq a < b$. If $f \in L_1[a,b]$, then one has the inequalities:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \left(f(x) + mf\left(\frac{x}{m}\right)\right)dx \leq \frac{1}{8}\left[\mathcal{E}(a) + \mathcal{E}(b)\right],$$

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where

$$\mathcal{E}(t) = f(t) + 2mf\left(\frac{t}{m}\right) + m^2 f\left(\frac{t}{m^2}\right). \tag{2}$$

In [3], Pachpatte proved new inequalities of Hermite-Hadamard type for product of convex functions. They are given in the following theorem.

Theorem 2 Let $f, g : [a, b] \to [0, \infty)$ be convex functions on $[a, b]$. Then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b),$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b),$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

In [4], Bakula *et al.* gave the following Hermite-Hadaramd type inequality for $m-$convex functions.

Theorem 3 Let $f, g : [0, \infty) \to [0, \infty)$ be functions such that $fg \in L_1[a, b]$, where $0 \leq a < b < \infty$. If f is m_1-convex and g is m_2-convex on $[0, b]$ for some fixed $m_1, m_2 \in (0, 1]$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \min\{M_1, M_2\},$$

where

$$M_1 = \frac{1}{3}\left[f(a)g(a) + m_1 m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right] + \frac{1}{6}\left[m_2 f(a) g\left(\frac{b}{m_2}\right) + m_1 f\left(\frac{b}{m_1}\right)g(a)\right],$$

and

$$M_2 = \frac{1}{3}\left[f(b)g(b) + m_1 m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right] + \frac{1}{6}\left[m_2 f(b) g\left(\frac{a}{m_2}\right) + m_1 f\left(\frac{a}{m_1}\right)g(b)\right].$$

The class $P-$function was firstly described in [5] by Dragomir and Toader.

Definition 2 A function $f : I \to \mathbb{R}$ is said to be a $P-$function or belongs to the class $P(I)$. If f is non-negative and

$$f(\alpha x + (1-\alpha)y) \leq f(x) + f(y)$$

for all $x, y \in I$ and $\alpha \in [0, 1]$.

In [5], Dragomir *et al.* proved inequality of Hermite-Hadamard type for class of $P-$functions.

Theorem 4 Let $f \in P(I), a, b \in I$, with $a < b$ and $f \in L_1[a, b]$. Then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)dx \leq 2[f(a) + f(b)].$$

In 1978, Breckner [6] introduced $s-$convex functions as a generalization of convex functions.

Definition 3 Let $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be $s-$convex (in the second sense), if

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y)$$

for all $x, y \in [0, \infty)$ and $\alpha \in [0, 1]$.
In [7], Dragomir and Fitzpatrick established the following Hermite-Hadamard type inequality.
Theorem 5 Let $f : [0, \infty) \to [0, \infty)$ be $s-$convex function in the second sense, where $s \in (0, 1]$, and $f \in L_1[a, b]$, where $0 \le a < b < \infty$. Then one has the inequalities:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}.$$

An analogous result for $s-$convex functions is due to Kirmaci *et al.* given in [8].
Theorem 6 Let $[a, b] \subset [0, \infty)$ and $f, g : [a, b] \to \mathbb{R}$ be functions such that $g \in L_1[a, b]$. If f is convex and non-negative on $[a, b]$, and if g is $s-$convex of second sense on $[a, b]$ for some fixed $s \in (0, 1)$, then

$$2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \int_a^b f(x)g(x)dx \le \frac{1}{(s+1)(s+1)} M(a, b) + \frac{1}{s+2} N(a, b).$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.
In 2006, Varošanec [9] introduced the concept of $h-$convex functions.
Definition 4 Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a non-negative and nonzero function. We say that $f : I \to \mathbb{R}$ is a $h-$convex function or that f is said to be belong to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I, \alpha \in (0, 1)$, we have

$$f(\alpha x + (1-\alpha)y) \le h(\alpha)f(x) + h(1-\alpha)f(y).$$

If the inequality is reversed then f is said to be $h-$concave and and we say that f belongs to the class $SV(h, I)$.
Sarikaya *et al.* in [10], established the following Hermite-Hadamard type inequality for $h-$convex functions.
Theorem 7 Let $f \in SX(h, I), a, b \in I$,with $a < b$ and $f \in L_1[a, b]$. Then

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)dx \le [f(a) + f(b)] \int_0^1 h(\alpha)d\alpha.$$

An analogous result for $h-$convex functions established by Sarikaya *et al.* in [10].
Theorem 8 Let $f \in SX(h_1, I), g \in SX(h_2, I)$, $a, b \in I$, $a < b$, be functions such that $fg \in L_1[a, b]$, and $h_1 h_2 \in L_1[0, 1]$, then

$$\frac{1}{2h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)g(x)dx \le M(a, b) \int_0^1 h_1(t)h_2(t)dt + N(a, b) \int_0^1 h_1(t)h_2(1-t)dt,$$

where M(a,b) = f(a)g(a)+f(b)g(b) and N(a,b) = f(a)g(b)+f(b)g(a).
The concept of $(h-m)-$convexity has been introduce by Özdemir *et al.* in [11].
Definition 5 Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. We say that $f : [0, b] \to \mathbb{R}$ is a $(h-m)-$convex function, if f is non-negative and for all $x, y \in [0, b], m \in [0, 1]$ and $\alpha \in (0, 1)$, we have

$$f(\alpha x + m(1-\alpha)y) \le h(\alpha)f(x) + mh(1-\alpha)f(y).$$

If the inequality is reversed, then f is said to be $(h-m)-$concave function on $[0,b]$.
Remark 1 If we choose $m=1$, then we have $h-$convex function on $[0,b]$. If we choose $h(t)=t$ we have $m-$convex function on [0,b]. If we choose $m=1$ and $h(t)=\{t,1,\frac{1}{t},t^s\}$, then we obtain the following classes of functions, non-negative convex functions, $P-$functions, Godunova-Levin functions and $s-$convex functions on $[0,b]$, respectively.
In [11], Özdemer *et al.* proved the following Hermite-Hadamard type inequalities for $(h-m)-$convex functions as.
Theorem 9 Let $f:[0,\infty)\to\mathbb{R}$ be $(h-m)-$convex function with $m\in(0,1]$, $t\in[0,1]$. If $0\le a<b<\infty$ and $f\in L_1[a,b]$. then the following inequality holds;

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{f(a)\int_0^1 h(t)dt+mf\left(\frac{b}{m}\right)\int_0^1 h(1-t)dt,\right.$$
$$\left. f(b)\int_0^1 h(t)dt+mf\left(\frac{a}{m}\right)\int_0^1 h(1-t)dt\right\}.$$

In [13], Dragomir introduced convex functions on coordinates for rectangle in the plane. Also he gave Hermite-Hadamard type inequality for convex functions on coordinates. Such type of generalization for functions related to convex functions are given by many other mathematicians, e.g. see [12, 13, 14, 16] and references therein. The main purpose of this paper is to define $(h-m)-$convex functions on coordinates and to establish new Hermite-Hadamard type inequalities for $(h-m)-$convex functions of two variables on the coordinates (Section 2). Also some new Hermite-Hadamard type inequalities for product of $(h-m)-$convex functions are established (Section 3).

2. Hermite-Hadamard type inequalities for coordinated $(h-m)-$convex functions

One can give the notion of $(h-m)-$convexity of a function f on a rectangle from the plane $\mathbb{R}^2$ and $(h-m)-$convexity on the coordinates on a rectangle from the plane $\mathbb{R}^2$. For this purpose, we consider bi-dimensional interval $\Delta:=[0,b]\times[0,d]$ in $\mathbb{R}^2$, we will keep this notation for the rest of the paper.
Definition 6 Let $J\subseteq\mathbb{R}$ be an interval containing $(0,1)$ and $h:J\to\mathbb{R}$ be a function. A mapping $f:\Delta\to\mathbb{R}$ is said to be $(h-m)-$convex on Δ if the inequality

$$f(tx+m(1-t)z,ty+m(1-tw)\le h(t)f(x,y)+mh(1-t)f(z,w)$$

holds, for all $(x,y),(z,w)\in\Delta$ and $t\in[0,1],m\in(0,1]$. If the inequality reversed, then f is said to be $(h-m)-$concave on Δ.
As stated in Remark 1, for suitable choices of h and $m=1$, we get different known classes of convex functions.
Definition 7 Let $J\subseteq\mathbb{R}$ be an interval containing $(0,1)$ and $h:J\to\mathbb{R}$ be a function. Also let $f:\Delta\to\mathbb{R}$ be a mapping and define

$$f_x:[0,d]\to\mathbb{R} \text{ by } f_x(v)=f(x,v) \quad\text{and}\quad f_y:[0,b]\to\mathbb{R} \text{ by } f_y(u)=f(u,y). \tag{3}$$

A mapping f is said to be $(h,m)-$convex on the coordinates on Δ if f_x and f_y are $(h-m)-$convex on $[0,d]$ and $[0,b]$ respectively for all $x\in[0,b]$ and $y\in[0,d]$.
Theorem 10 If $f:\Delta\to\mathbb{R}$ is $(h-m)-$ convex function then it is $(h-m)-$convex function on the coordinates, but converse is not true in general.

Proof. Let $f:\Delta\to\mathbb{R}$ is $(h-m)-$convex on Δ. Consider the functions f_x and f_y defined in (3). Then for $t,m\in[0,1]$ and $u_1,u_2\in[0,d]$, we have

$$\begin{aligned}f_x(tu_1+m(1-t)u_2)&=f\left(x,h(t)u_1+mh(1-t)u_2\right)\\&=f(h(t)x+mh(1-t)x,h(t)u_1+mh(1-t)v_2)\\&\le h(t)f(x,u_1)+mh(1-t)f(x,u_2)\\&=h(t)f_x(u_1)+mh(1-t)f_x(u_2).\end{aligned}$$

Therefore, f_x is $(h-m)-$convex on $[0,d]$. The fact that f_y is also $(h-m)-$convex on $[0,b]$ goes likewise. It follows that f is $(h-m)-$convex function on coordinate on Δ.
To prove that converse is not ture in general, we consider a function $f:[0,1]^2\to\mathbb{R}$ defined by $f(x,y)=xy$, then clearly it is $(h-m)-$convex on coordinates. If we take $u,w\in(0,1)$ and $t\in[0,1]$, then

$$f(t(u,0)+(1-t)(0,w))=f(tu,m(1-t)w)=mt(1-t)xw,$$

and

$$h(t)f(u,0)+mh(1-t)f(0,w)=0.$$

This shows that f is not $(h-m)-$convex on $[0,1]^2$.
Now, we establish Hermite-Hadamard type inequality for $(h-m)-$convex functions on the coordinates on rectangle from the plane $\mathbb{R}^2$.
Theorem 11 Let $f:\Delta\to\mathbb{R}$ be an $(h-m)-$convex function on the coordinates on Δ. If $0\le a<b$ and $0\le c<d$, $m\in(0,1]$ with $f\in L_1(\Delta)$ and $h\in L_1[0,1]$. Then one has the inequality

$$\begin{aligned}&\frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \qquad (4)\\&\le\min\left\{\frac{1}{b-a}\int_a^b f(x,c)dx\int_0^1 h(t)dt+\frac{m}{b-a}\int_a^b f\left(x,\frac{d}{m}\right)dx\int_0^1 h(1-t)dt,\right.\\&\left.\frac{1}{b-a}\int_a^b f(x,d)dx\int_0^1 h(t)dt+\frac{m}{b-a}\int_a^b f\left(x,\frac{c}{m}\right)dx\int_0^1 h(1-t)dt,\right\}\\&+\min\left\{\frac{1}{d-c}\int_c^d f(a,y)dx\int_0^1 h(t)dt+\frac{m}{d-c}\int_c^d f\left(\frac{b}{m},y\right)dx\int_0^1 h(1-t)dt,\right.\\&\left.\frac{1}{d-c}\int_c^d f(b,y)dx\int_0^1 h(t)dt+\frac{m}{d-c}\int_c^d f\left(\frac{a}{m},y\right)dx\int_0^1 h(1-t)dt\right\}.\end{aligned}$$

Proof. Since $f:\Delta\to\mathbb{R}$ is $(h-m)-$convex function on the coordinates on Δ it follows that the mapping $f_x:[0,d]\to\mathbb{R}$ defined by $f_x(v)=f(x,v)$ is $(h-m)-$convex on $[0,d]$ for all $x\in[0,b]$, therefore by Theorem 9, one has

$$\begin{aligned}\frac{1}{d-c}\int_c^d f_x(y)dy\le\min&\left\{f_x(c)\int_0^1 h(t)dt+mf_x\left(\frac{d}{m}\right)\int_0^1 h(1-t)dt,\right.\\&\left.f_x(d)\int_0^1 h(t)dt+mf_x\left(\frac{c}{m}\right)\int_0^1 h(1-t)dt\right\},\end{aligned}$$

that is

$$\frac{1}{d-c}\int_c^d f(x,y)dy \le \min\left\{f(x,c)\int_0^1 h(t)dt + mf\left(x,\frac{d}{m}\right)\int_0^1 h(1-t)dt,\right.$$
$$\left. f(x,d)\int_0^1 h(t)dt + mf\left(x,\frac{c}{m}\right)\int_0^1 h(1-t)dt\right\}.$$

Dividing both sides by $\frac{1}{b-a}$ and integrating the inequality on $[a,b]$, we have

$$\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)dy \tag{5}$$
$$\le \min\left\{\frac{1}{b-a}\int_a^b f(x,c)\int_0^1 h(t)dt + \frac{m}{b-a}\int_a^b f\left(x,\frac{d}{m}\right)\int_0^1 h(1-t)dt,\right.$$
$$\left.\frac{1}{b-a}\int_a^b f(x,d)\int_0^1 h(t)dt + \frac{m}{b-a}\int_a^b f\left(x,\frac{c}{m}\right)\int_0^1 h(1-t)dt\right\}.$$

By applying similar arguments to the mapping $f_y : [0,b] \to \mathbb{R}$ defined by $f_y(u) = f(u,y)$, we get

$$\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)dy \tag{6}$$
$$\le \min\left\{\frac{1}{d-c}\int_c^d f(a,y)\int_0^1 h(t)dt + \frac{m}{d-c}\int_c^d f\left(\frac{b}{m},y\right)\int_0^1 h(1-t)dt,\right.$$
$$\left.\frac{1}{d-c}\int_c^d f(b,y)\int_0^1 h(t)dt + \frac{m}{d-c}\int_c^d f\left(\frac{a}{m},y\right)\int_0^1 h(1-t)dt\right\}.$$

We add inequalities (5) and (6), to obtain the required result.
If we take $h(t) = t$ for all $t \in [0,1]$ in above theorem we get [16, Theorem 3.5], is stated in the following corollary.
Corollary 1 If $f : \Delta \to \mathbb{R}$ be $m-$convex on the coordinates on Δ. If $0 \le a < b$ and $0 \le c < d$, $m \in (0,1]$ with $f \in L(\Delta)$. Then one has the inequality

$$\frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx$$
$$\le \min\left\{\frac{1}{b-a}\int_a^b\left(\frac{f(x,c)+mf\left(x,\frac{d}{m}\right)}{2}\right)dx, \frac{1}{b-a}\int_a^b\left(\frac{f(x,d)+mf\left(x,\frac{c}{m}\right)}{2}\right)dx\right\}$$
$$+\min\left\{\frac{1}{b-a}\int_a^b\left(\frac{f(a,y)+mf\left(\frac{b}{m},y\right)}{2}\right)dx, \frac{1}{b-a}\int_a^b\left(\frac{f(b,y)+mf\left(\frac{a}{m},y\right)}{2}\right)dx\right\}$$

Next, we start to state second theorem containing Hermite-Hadamard type inequality.
Theorem 12 Let $f : [0,\infty) \to \mathbb{R}$ be $(h-m)-$convex, where $m \in (0,1]$ and

$0 \leq a < b < \infty$, if $f \in L_1[a,b]$, $h \in L_1[0,1]$. Then one has the inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left(f(x)+mf\left(\frac{x}{m}\right)\right)dx \tag{7}$$
$$\leq \frac{h\left(\frac{1}{2}\right)}{2}\left[\mathcal{E}(a)+\mathcal{E}(b)\right]\int_0^1 h(t)dt,$$

where $\mathcal{E}$ is defined in (2).

Proof. By the $(h-m)-$convexity of f, we have

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right)\left[f(x)+mf\left(\frac{y}{m}\right)\right]. \tag{8}$$

For all $x, y \in [0,\infty)$ If we choose

$$x = ta+(1-t)b, \quad y = (1-t)a+tb,$$

we deduce

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2}\right)\left[f(ta+(1-t)b)+mf\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)\right].$$

Integrating over $t \in [0,1]$, we get

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2}\right)\left[\int_0^1 f(ta+(1-t)b)dt+m\int_0^1 f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)dt\right]. \tag{9}$$

A simple computation gives us that

$$\int_0^1 f(ta+(1-t)b)dt = \frac{1}{b-a}\int_a^b f(x)dx,$$

and

$$\int_0^1 f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)dt = \frac{m}{b-a}\int_{\frac{a}{m}}^{\frac{b}{m}} f(x)dx = \frac{1}{b-a}\int_a^b f\left(\frac{x}{m}\right)dx.$$

Using the above two expression in (9), we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left(f(x)+mf\left(\frac{x}{m}\right)\right)dx. \tag{10}$$

This completes the proof of first inequality in (7). The second inequality in (7) follows from the fact by using the $(h-m)-$convexity of f for all $t \in [0,1]$, we can write

$$h\left(\frac{1}{2}\right)\left[f\left(ta+(1-t)b\right)+mf\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)\right]$$
$$\leq h\left(\frac{1}{2}\right)\left[h(t)f(a)+mh(1-t)f\left(\frac{b}{m}\right)+m^2h(1-t)f\left(\frac{a}{m^2}\right)+h(t)mf\left(\frac{b}{m}\right)\right].$$

Integrating above inequality over $[0,1]$ w.r.t t, we deduce

$$\frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left(f(x)+mf\left(\frac{x}{m}\right)\right)dx \tag{11}$$
$$\leq h\left(\frac{1}{2}\right)\left[f(a)+2mf\left(\frac{b}{m}\right)+m^2f\left(\frac{a}{m^2}\right)\right]\int_0^1 h(t)dt.$$

Now if we choose

$$x = (1-t)a + tb \quad y = ta + (1-t)b$$

we have

$$\frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left(f(x) + mf\left(\frac{x}{m}\right)\right) dx \le h\left(\frac{1}{2}\right)\left[f(b) + 2mf\left(\frac{a}{m}\right) + m^2 f\left(\frac{b}{m^2}\right)\right]\int_0^1 h(t)dt. \tag{12}$$

Adding (11) and (12), we get

$$\frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left(f(x) + mf\left(\frac{x}{m}\right)\right) dx \le \frac{h\left(\frac{1}{2}\right)}{2}\left[\mathcal{E}(a) + \mathcal{E}(b)\right]\int_0^1 h(t)dt,$$

This complete the proof.

Remark 2

1. By putting $h(t) = t$ in above theorem, we get Theorem 1.
2. By putting $m = 1$ and $h(t) = 1$ in above theorem, we get Theorem 4.
3. By putting $m = 1$ and $h(t) = t^s$ in above theorem, we get Theorem 5.
4. By putting $m = 1$ in above theorem, we get Theorem 7.

Theorem 13 Let $f : \Delta \to \mathbb{R}$ be an $(h-m)-$convex function on the coordinates on Δ. If $m \in (0,1]$ with $f \in L_1(\Delta)$ and $h \in L_1[0,1]$. Then we have

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy \\ &\le \frac{h\left(\frac{1}{2}\right)}{(b-a)(d-c)}\int_a^b\int_c^d \left(2f(x,y) + m\left(f\left(x, \frac{y}{m}\right) + f\left(\frac{x}{m}, y\right)\right)\right) dydx \\ &\le \frac{h\left(\frac{1}{2}\right)}{2}\left[\frac{1}{b-a}\int_a^b \left(\mathcal{E}(x,c) + \mathcal{E}(x,d)\right) dx + \frac{1}{d-c}\int_c^d \left(\mathcal{E}(a,y) + \mathcal{E}(b,y)\right) dy\right]\int_0^1 h(t)dt. \end{aligned} \tag{13}$$

where

$$\mathcal{E}(x,t) = f(x,t) + 2mf\left(x, \frac{t}{m}\right) + m^2 f\left(x, \frac{t}{m^2}\right),$$

and

$$\mathcal{E}(t,y) = f(t,y) + 2mf\left(\frac{t}{m}, y\right) + m^2 f\left(\frac{t}{m^2}, y\right).$$

Proof. Since $f : \Delta \to \mathbb{R}$ is $(h-m)-$convex function on the coordinates on Δ it follows that the mapping $f_y : [0,b] \to \mathbb{R}$ defined by $f_y(u) = f(u,y)$ is $(h-m)-$convex on $[0,b]$ for all $x \in [0,d]$, therefore by Theorem 2.4, one has

$$f_y\left(\frac{a+b}{2}\right) \le \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left(f_y(x) + mf_y\left(\frac{x}{m}\right)\right) dx \le \frac{h\left(\frac{1}{2}\right)}{2}\left[\mathcal{E}_y(a) + \mathcal{E}_y(b)\right]\int_0^1 h(t)dt,$$

that is,

$$f\left(\frac{a+b}{2}, y\right) \le \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left(f(x,y) + mf\left(\frac{x}{m}, y\right)\right) dx \le \frac{h\left(\frac{1}{2}\right)}{2}\left[\mathcal{E}(a,y) + \mathcal{E}(b,y)\right]\int_0^1 h(t)dt.$$

Dividing both sides by $\frac{1}{d-c}$ and integrating the inequality on $[c,d]$, we have

$$\begin{aligned} &\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \qquad (14)\\ &\leq \frac{h\left(\frac{1}{2}\right)}{(b-a)(d-c)}\int_c^d\int_a^b \left(f(x,y)+mf\left(\frac{x}{m},y\right)\right)dxdy\\ &\leq \frac{h\left(\frac{1}{2}\right)}{2(d-c)}\int_c^d\left[\mathcal{E}(a,y)+\mathcal{E}(b,y)\right]dy\int_0^1 h(t)dt. \end{aligned}$$

By applying similar arguments to the mapping $f_x : [0,d] \to \mathbb{R}$ defined by $f_x(v) = f(x,v)$, we get

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \qquad (15)\\ &\leq \frac{h\left(\frac{1}{2}\right)}{(b-a)(d-c)}\int_c^d\int_a^b \left(f(x,y)+mf\left(x,\frac{y}{m}\right)\right)dxdy\\ &\leq \frac{h\left(\frac{1}{2}\right)}{2(b-a)}\int_a^b\left[\mathcal{E}(a,y)+\mathcal{E}(b,y)\right]dx\int_0^1 h(t)dt. \end{aligned}$$

By adding (14) and (15) we get the desired result.

Remark 3

(i) If we take $m = 1$ and $h(t) = t^s$ for all $t \in [0,1]$ in above theorem we get partial results of [12, Theorem 2.1].

(ii) If we take $m = 1$ and $h(t) = t$ for all $t \in [0,1]$ in above results of theorem we get [13, Theorem 1].

3. Hermite-Hadamard type inequalities for products of $(h-m)-$convexity

In the following theorem, we proved Hermite-Hadamard type inequality for product of $h-m-$convex functions.

Theorem 14 Let $f,g : [0,\infty) \to \mathbb{R}$ be such that fg is in $L_1[a,b]$, where $0 \leq a < b < \infty$. If f is $(h-m_1)-$convex and g is $(h-m_2)-$convex on $[0,\infty)$ for some fixed $m_1, m_2 \in (0,1]$ and $h \in L_1[0,1]$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \min\{M_1, N_2\}, \qquad (16)$$

where

$$\begin{aligned} M_1 = &\int_0^1 (h(t))^2\,dt\left[f(a)g(a)+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right]\\ &+\int_0^1 h(t)h(1-t)dt\left[m_2f(a)g\left(\frac{b}{m_2}\right)+m_1f\left(\frac{b}{m_1}\right)g(a)\right], \end{aligned}$$

and

$$\begin{aligned} M_2 = &\int_0^1 (h(t))^2\,dt\left[f(b)g(b)+m_1m_2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\\ &+\int_0^1 h(t)h(1-t)dt\left[m_2f(b)g\left(\frac{a}{m_2}\right)+m_1f\left(\frac{a}{m_1}\right)g(b)\right]. \end{aligned}$$

Proof. Since f is $(h-m_1)-$convex and g is $(h-m_2)-$convex on $[0,\infty)$ for some fixed $m_1, m_2 \in (0,1]$ and $t\in[0,1]$, we have

$$\begin{aligned}&f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)\\&\leq (h(t))^2 f(a)g(a)+m_2h(t)h(1-t)f(a)g\left(\frac{b}{m_2}\right)\\&+m_1h(t)h(1-t)f\left(\frac{b}{m_1}\right)g(a)+m_1m_2\left(h(1-t)\right)^2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right).\end{aligned}$$

Integration both sides of the above inequality over $[0,1]$, yield the following

$$\begin{aligned}\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt&=\frac{1}{b-a}\int_a^b f(x)g(x)dx\\&\leq\int_0^1(h(t))^2dt\left(f(a)g(a)+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right)\\&+\int_0^1 h(t)h(1-t)dt\left(m_2f(a)g\left(\frac{b}{m_2}\right)+m_1f\left(\frac{b}{m_1}\right)g(a)\right).\end{aligned}$$

Analogously we obtain

$$\begin{aligned}&=\frac{1}{b-a}\int_a^b f(x)g(x)dx\\&\leq\int_0^1(h(t))^2dt\left(f(b)g(b)+m_1m_2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_1}\right)\right)\\&+\int_0^1 h(t)h(1-t)dt\left(m_2f(b)g\left(\frac{a}{m_2}\right)+m_1f\left(\frac{a}{m_1}\right)g(b)\right).\end{aligned}$$

After a little computation one get inequality (16).

Remark 4 If in above theorem we choose $h(t)=t$, then we obtain Theorem 3.

Theorem 15 Let $f,g:[0,\infty)\to\mathbb{R}$ such that $fg\in L_1[a,b]$, where $0\leq a<b<\infty$. If f is $(h_1-m)-$convex and g is $(h_2-m)-$convex on $[0,\infty)$ for some fixed $m\in(0,1]$, and $h_1,h_2\in L_1[0,1]$, then the following inequalities holds:

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{b-a}\int_a^b\left(f\left(x\right)g\left(x\right)+m^2f\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right)\right)dx \tag{17}$$

$$\begin{aligned}&\leq m\left[f(a)g\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\int_0^1 h_1(t)h_2(1-t)dt\\&+m^2\left[f\left(\frac{b}{m}\right)g\left(\frac{a}{m^2}\right)+f\left(\frac{a}{m^2}\right)g\left(\frac{a}{m}\right)\right]\int_0^1 h_1(1-t)h_2(1-t)dt\\&+\left[f\left(\frac{b}{m}\right)g(a)+f(a)g\left(\frac{b}{m}\right)\right]\int_0^1 h_1(t)h_2(t)dt\\&+m^2\left[f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)+f\left(\frac{a}{m^2}\right)g(a)\right]\int_0^1 h_2(t)h_1(1-t)dt.\end{aligned}$$

Proof. We can write.

$$\frac{a+b}{2}=\frac{at+(1-t)b}{2}+\frac{(1-t)a+tb}{2},$$

so,

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
& = f\left(\frac{at+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) g\left(\frac{at+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) \\
& \leq h_1\left(\frac{1}{2}\right)\left[f\left(at+(1-t)b\right)+mf\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)\right] \times \\
& h_2\left(\frac{1}{2}\right)\left[g\left(at+(1-t)b\right)+mg\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)\right]
\end{aligned}
$$

This gives us

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \leq h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \times \Big\{ & f\left(at+(1-t)b\right) g\left(at+(1-t)b\right) \\
& + m^2 f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right) g\left((1-t)\frac{a}{m}+t\frac{b}{m}\right) \\
& + mf\left(at+(1-t)b\right) g\left((1-t)\frac{a}{m}+t\frac{b}{m}\right) \\
& + mf\left((1-t)\frac{a}{m}+t\frac{b}{m}\right) g\left(at+(1-t)b\right) \Big\}
\end{aligned}
$$

The above expression leads us to

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
& \leq h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \times \{ f\left(at+(1-t)b\right) g\left(at+(1-t)b\right) \\
& + m^2 f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right) g\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)\Big\} \\
& + mh_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \times \left\{\left[h_1(t)f(a)+mh_1(1-t)f\left(\frac{b}{m}\right)\right]\right. \\
& \left[mh_2(1-t)g\left(\frac{a}{m^2}\right)+h_2(t)g\left(\frac{b}{m}\right)\right]+\left[mh_1(1-t)f\left(\frac{a}{m^2}\right)+h_1(t)f\left(\frac{b}{m}\right)\right] \\
& \left.\left[h_2(t)g(a)+mh_2(1-t)g\left(\frac{b}{m}\right)\right]\right\}.
\end{aligned}
$$

$$\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\times\{f\left(at+(1-t)b\right)g\left(at+(1-t)b\right)$$
$$+m^2 f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)g\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)\Big\}$$
$$+mh_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\times\Big\{mh_1(t)h_2(1-t)\left[f(a)g\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]$$
$$+m^2h_1(1-t)h_2(1-t)\left[f\left(\frac{b}{m}\right)g\left(\frac{a}{m^2}\right)+f\left(\frac{a}{m^2}\right)g\left(\frac{a}{m}\right)\right]$$
$$+h_1(t)h_2(t)\left[f\left(\frac{b}{m}\right)g(a)+f(a)g\left(\frac{b}{m}\right)\right]$$
$$+\ mh_2(t)h_1(1-t)\left[f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)+f\left(\frac{a}{m^2}\right)g(a)\right]\Big\}.$$

Integrating over $[0,1]$, we obtain

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{b-a}\int_a^b\left(f\left(x\right)g\left(x\right)+m^2f\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right)\right)dx$$
$$\leq m\left[f(a)g\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\int_0^1 h_1(t)h_2(1-t)dt$$
$$+m^2\left[f\left(\frac{b}{m}\right)g\left(\frac{a}{m^2}\right)+f\left(\frac{a}{m^2}\right)g\left(\frac{a}{m}\right)\right]\int_0^1 h_1(1-t)h_2(1-t)dt$$
$$+\left[f\left(\frac{b}{m}\right)g(a)+f(a)g\left(\frac{b}{m}\right)\right]\int_0^1 h_1(t)h_2(t)dt$$
$$+m^2\left[f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)+f\left(\frac{a}{m^2}\right)g(a)\right]\int_0^1 h_2(t)h_1(1-t)dt.$$

This completes the proof.

Remark 5

(i) If we take $m=1$ and $h_1=h_2=t$ in above theorem we get Theorem 2.
(ii) If we take $m=1$ and $h_1=t, h_2=t^s$ in above theorem we get Theorem 6.
(iii) If we take $m=1$ in above theorem we get Theorem 8.

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