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YU'S RESULT - A FURTHER EXTENSION

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Abstract. Taking Yu's [23] result into background, we employ the notion of weighted sharing to investigate the uniqueness of rational function of a meromorphic function sharing a small function with its generalized differential polynomial. Our results will improve a number of results specially those of Banerjee-Dhar [5] and Li-Yang-Liu [18]. A number of examples have been exhibited in the paper to justify our certain claims.

1. Introduction

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities).

Throughout the paper the standard notations of Nevanlinna's value distribution theory of entire and meromorphic functions which are discussed in [12] have been adopted.

A meromorphic function a is said to be a small function of f provided that $T(r,a) = S(r,f)$, that is $T(r,a) = o(T(r,f))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure. Also we use I to denote any set of infinite linear measure of $0 < r < \infty$.
We also recall that if $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a;f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N(r,a;f)}{T(r,f)}$$

is called Nevanlinna deficiency of the value a and by ramification index we mean

$$\Theta(a;f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

We begin our discussion recalling the following famous result of R. Brück [7].

Theorem A. [7] *Let f be a non-constant entire function. If f and f' share the value 1 CM and if $N(r,0;f') = S(r,f)$ then $\dfrac{f'-1}{f-1}$ is a nonzero constant.*

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In [7], R. Brück posed the following conjecture concerning a non-constant entire function.

Conjecture: *Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ of f is not a positive integer or infinite. If f and f' share a finite value a CM, then*

$$\frac{f'-a}{f-a}=c,$$

where c is a non zero constant.

Many authors like Zhang [24], Yang [21], Gundersen-Yang [11] tried to solve the above conjecture and naturally obtained different aspects of it. Next we demonstrate the following definition known as weighted sharing of values which has a remarkable influence on the results of Brück conjecture.

Definition 1.1. [13, 14] *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f)=E_k(a;g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k), then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

If a is a small function we define that f and g share a IM or a CM or with weight l according as $f-a$ and $g-a$ share $(0,0)$ or $(0,\infty)$ or $(0,l)$ respectively.

We now explain some definitions and notations which are used in the paper.

Definition 1.2. [17]*Let p be a positive integer and $a \in \mathbb{C}\cup\{\infty\}$.*

- (i) *$N(r,a;f \mid\geq p)$ $(\overline{N}(r,a;f \mid\geq p))$denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.*
- (ii) *$N(r,a;f \mid\leq p)$ $(\overline{N}(r,a;f \mid\leq p))$denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.*

Definition 1.3. [1] *Let f and g be two non-constant meromorphic functions such that f and g share the value a IM. Let z_0 be a a-point of f with multiplicity p, a a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p > q$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r,a;f)$ the counting function of those a-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. Similarly, one can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$.*

Definition 1.4. [13, 14] *Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

In 2003 Yu [23] obtained the following results in the direction of the above conjecture.

Theorem B. [23] *Let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share 0 CM and $\delta(0; f) > \frac{3}{4}$ then $f \equiv f^{(k)}$.*

Theorem C. [23] *Let f be a non-constant non-entire meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If*

i) *f and a have no common poles.*
ii) *$f - a$ and $f^{(k)} - a$ share the value 0 CM.*
iii) $4\delta(0; f) + 2(8 + k)\Theta(\infty; f) > 19 + 2k$

then $f \equiv f^{(k)}$ where k is a positive integer.

Using weighted sharing of values Lahiri-Sarkar [17] improved the results of Yu [23]. In 2005, Zhang [25] further extended the result of Lahiri-Sarkar.

In 2010, Chen-Wang and Zhang [8] generalized the results of Yu [23] by considering the problem of uniqueness of f and $(f^n)^{(k)}$. In 2014, Banerjee-Majumder [6], rectify some gaps in the main results of [8] and presented its correct form considering the uniqueness of f^m and $(f^n)^{(k)}$.

To proceed further, we recall the following well known definition.

Definition 1.5. [3] *Let $n_{0j}, n_{1j}, \ldots, n_{kj}$ be non-negative integers.*
• *The expression $\mathcal{M}_j[f] = (f)^{n_{0j}}(f')^{n_{1j}} \ldots (f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $d(\mathcal{M}_j) = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{\mathcal{M}_j} = \sum_{i=0}^{k} (1 + i)n_{ij}$.*
• *The sum $\mathcal{P}[f] = \sum_{j=1}^{t} b_j \mathcal{M}_j[f]$ is called a differential polynomial generated by f of degree $\overline{d}(\mathcal{P}) = \max\{d(\mathcal{M}_j) : 1 \leq j \leq t\}$ and weight $\Gamma_{\mathcal{P}} = \max\{\Gamma_{\mathcal{M}_j} : 1 \leq j \leq t\}$, where $T(r, b_j) = S(r, g)$ for $j = 1, 2, \ldots, t$.*
• *The numbers $\underline{d}(\mathcal{P}) = \min\{d(\mathcal{M}_j) : 1 \leq j \leq t\}$ and k the highest order of the derivative of f in $\mathcal{P}[f]$ are called respectively the lower degree and order of $\mathcal{P}[f]$.*
• *$\mathcal{P}[f]$ is called* homogeneous *if $\overline{d}(\mathcal{P}) = \underline{d}(\mathcal{P}) = d(say)$.*
• *$\mathcal{P}[f]$ is called a* linear differential polynomial *generated by f if $\overline{d}(\mathcal{P}) = 1$. Otherwise $\mathcal{P}[f]$ is called* non-linear differential polynomial. *We denote by $Q = \max\{\Gamma_{\mathcal{M}_j} - d(\mathcal{M}_j) : 1 \leq j \leq t\}$.*

Definition 1.6. *For any two positive integers n, and $r \leq 3$,*

$$\mu_r(n) = \min\{r, n\} \quad \text{and} \quad \mu_r^*(n) = (r + 1) - \mu_r(n).$$

Definition 1.7. [22] *For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots + \overline{N}(r, a; f \mid\geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.*

Definition 1.8. [25] *For a positive integer p and $a \in \mathbb{C} \cup \{\infty\}$ we put*

$$\delta_p(a; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N_p(r, a; f)}{T(r, f)}$$

It is clear that

$$0 \leq \delta(a; f) \leq \delta_p(a; f) \leq \delta_{p-1}(a; f) \leq \ldots \leq \delta_2(a; f) \leq \delta_1(a; f) = \Theta(a; f).$$

Extending all the above mentioned results, in 2015, Banerjee - Dhar [5] obtained the following.

Theorem D. *Let f be a non-constant meromorphic function and $n(\geq 1)$, $p(\geq 0)$ be integers. Let $a \equiv a(z)(\not\equiv 0, \infty)$ be a small function. Suppose further that $\mathcal{P}[f]$ be a differential polynomial generated by f such that $\mathcal{P}[f]$ contains at least one derivative and $f^n - a$ and $\mathcal{P}[f] - a$ share $(0, p)$. If $p = \infty$ and*

$$3\Theta(\infty; f) + \underline{d}(\mathcal{P})\delta(0; f) + \mu_2(n)\delta_{\mu_2^*(n)}(0; f) > \mu_2(n) + 3 \tag{1.1}$$

or, $2 \leq p < \infty$ and

$$3\Theta(\infty; f) + \underline{d}(\mathcal{P})\delta(0; f) + \mu_3(n)\delta_{\mu_3^*(n)}(0; f) > \mu_3(n) + 3 \tag{1.2}$$

or, $p = 1$ and

$$4\Theta(\infty; f) + \underline{d}(\mathcal{P})\delta(0; f) + \Theta(0; f) + \mu_2(n)\delta_{\mu_2^*(n)}(0; f) > \mu_2(n) + 5 \tag{1.3}$$

or, $p = 0$ and

$$\begin{aligned} & (2Q + 6)\Theta(\infty; f) + 3\underline{d}(\mathcal{P})\delta(0; f) + \Theta(0; f) + \mu_2(n)\delta_{\mu_2^*(n)}(0; f) \\ > \; & 2Q + 2\overline{d}(\mathcal{P}) + \mu_2(n) + 7 \end{aligned} \tag{1.4}$$

then $f^n \equiv \mathcal{P}[f]$.

Recently, in this direction, for homogeneous differential polynomials, Li - Yang - Liu [18] obtained the following result .

Theorem E. *Let f be a non-constant meromorphic function and $\mathcal{P}[f]$ be a non-constant homogeneous differential polynomial of degree d and weight Γ satisfying $\Gamma \geq (k+2)d - 2$. Let $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $f - a$ and $\mathcal{P}[f] - a$ share $(0, p)$. If $p \geq 2$ and*

$$3\Theta(\infty; f) + d\, \delta_{2+\Gamma-d}(0, f^d) + \delta_2(0, f) + \delta(a, f) > 4 \tag{1.5}$$

or, $p = 1$ and

$$\begin{aligned} & \frac{7 + \Gamma - d}{2}\Theta(\infty; f) + \frac{d}{2}\, \delta_{1+\Gamma-d}(0; f^d) + d\, \delta_{2+\Gamma-d}(0; f^d) + \delta_2(0; f) \\ & + \delta(a; f) > \frac{\Gamma + 9}{2} \end{aligned} \tag{1.6}$$

or, $p = 0$ and

$$\begin{aligned} & [2(\Gamma - d) + 6]\Theta(\infty; f) + d\, \delta_{1+\Gamma-d}(0, f^d) + d\, \delta_{2+\Gamma-d}(0; f^d) + \delta_2(0; f) \\ & + \Theta(0; f) + \delta(a; f) > 2\Gamma + 8, \end{aligned} \tag{1.7}$$

then $\dfrac{\mathcal{P}[f] - a}{f - a} = C$, where C is a non-zero constant. Specially when $p = 0$ i.e., when f and $\mathcal{P}[f]$ share $(a, 0)$, then $f \equiv \mathcal{P}[f]$.

We note that $f^{(k)}$ and $(f^{(k)})^m$ both are differential monomials and $(f^n)^{(k)}$ a differential polynomial generated by f. But both of them can be accommodated if one considers a differential monomial of a power of f. So it will be interesting to investigate whether *Theorems D - E* can be extended up to generalized differential polynomial generated by some power of f and at the same time the first setting of function in *Theorems D - E* can also be extended up to a rational function in f. This is the main motivation of writing this paper.

Henceforth we denote by $\mathcal{R}(f)$ as defined in Lemma 2.3, p_i $(1 \leq i \leq u)$ and q_j $(1 \leq j \leq l)$ are positive integers. Let $P_n(f) = a_n \prod_{i=1}^{u}(f - d_i)^{p_i}$, $1 \leq u \leq n$ and $P_m(f) = b_m \prod_{j=1}^{l}(f - c_j)^{q_j}$, $1 \leq l \leq m$ respectively, where u and l are two positive integers. Let $c_0 \neq c_j (j = 1, \ldots, l)$ be a complex constant.

Throughout this paper, we denote by, $k^* = \begin{cases} \frac{k}{2} + 1, & \text{if } k \text{ is even}, \\ \left[\frac{k}{2}\right] + 2, & \text{if } k \text{ is odd}. \end{cases}$ and

$\chi_m = \begin{cases} 0, & \text{if } m = 0, \\ 1, & \text{if } m \geq 1. \end{cases}$

Let us define $u^* = \begin{cases} u, & \text{if none of } d_i \text{ is zero}, \\ u-1, & \text{if one among the } d_i \text{ is zero}. \end{cases}$ and $l^* = l\,\chi_m$.

We define

$$\Theta_{(t}(a; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, a; f \mid \geq t)}{T(r, f)}.$$

In this paper, we have obtained a combined result improving and extending all the Theorems A - E. Actually our aim is two fold. In one direction we will put the improved version of all the above theorems under a single umbrella and at the same time we will devote to weaken the conditions also.

The following results are the main results of this paper.

Theorem 1.1. *Let f be a non-constant meromorphic function and $\mathcal{P}[f^q]$ be a differential polynomial containing atleast one derivative and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Let $n > m(\geq 0)$, $u(\geq 1)$, $k(\geq 1)$, $q(\geq 1)$ and $p(\geq 0)$ are integers such that $q \geq k^*$ where k is the highest order derivative in $\mathcal{P}[f^q]$. Suppose that $\mathcal{R}(f) - a$ and $\mathcal{P}[f^q] - a$ share $(0, p)$ with (i) $\chi_m q\, \underline{d}(\mathcal{P}) \leq \sum_{i=1}^{u} \Theta(d_i; f)$ and (ii) $q\, \underline{d}(\mathcal{P})\delta(0; f) \leq n$. If $p \geq 2$ and*

$$3\Theta(\infty; f) + q\, \underline{d}(\mathcal{P})\delta(0; f) + \sum_{j=0}^{l^*} \chi_j \left\{\Theta(c_j; f) + \Theta_{(2}(c_j; f)\right\} \tag{1.8}$$

$$+ \sum_{\substack{i=1 \\ d_1, d_2, \ldots, d_{u^*} \neq 0}}^{u^*} \Theta_{(2}(d_i; f) + \sum_{i=1}^{u} \Theta(d_i; f) > 3 + 2l^* + u + u^*,$$

or, if $p = 1$ and

$$4\Theta(\infty; f) + q\,\underline{d}(\mathcal{P})\delta(0; f) + \sum_{j=0}^{l^*} \chi_{_j} \left\{ \Theta_{(2}(c_{_j}; f) + 2\Theta(c_{_j}; f) \right\} \tag{1.9}$$

$$+ \sum_{\substack{i=1 \\ d_1, d_2, \ldots, d_{u^*} \neq 0}}^{u^*} \Theta_{(2}(d_{_i}; f) + 2\sum_{i=1}^{u} \Theta(d_{_i}; f) > 4 + 3l^* + u^* + 2u,$$

or, if $p = 0$ and

$$(6 + 2Q)\Theta(\infty; f) + 3q\,\underline{d}(\mathcal{P})\delta(0; f) + \sum_{j=0}^{l^*} \chi_{_j} \left\{ \Theta_{(2}(c_{_j}; f) + 2\Theta(c_{_j}; f) \right\}$$

$$+ \sum_{\substack{i=1 \\ d_1, d_2, \ldots, d_{u^*} \neq 0}}^{u^*} \Theta_{_{(2}}(d_{_i}; f) + 2\sum_{i=1}^{u} \mu_2(p_i)\delta_{\mu_2^*(p_i)}(d_i; f) > 6 + 2Q + 2q\overline{d}(\mathcal{P}) + 3l^*$$

$$+2\sum_{i=1}^{u} \mu_2(p_i) + u^*, \tag{1.10}$$

then $\mathcal{R}(f) \equiv \mathcal{P}[f^q]$. In particular, if $P_n(0) = 0$, then the condition (ii) is no longer required.

Theorem 1.2. *Let f be a non-constant meromorphic function and $\mathcal{P}[f^q]$ be a differential polynomial containing atleast one derivative and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Let $n > m(\geq 0)$, $u(\geq 1)$, $k(\geq 1)$, $q(\geq 1)$ and $p(\geq 0)$ are integers such that $q < k^*$ where k is the highest order derivative in $\mathcal{P}[f^q]$. Suppose that $\mathcal{R}(f) - a$ and $\mathcal{P}[f^q] - a$ share $(0, p)$ with (i) $\chi_{_m} q\,\underline{d}(\mathcal{P}) \leq \sum_{i=1}^{u} \Theta(d_{_i}; f)$ and (ii) $q\,\underline{d}(\mathcal{P})\delta(0; f) \leq n$.*
If $p = \infty$ and

$$3\Theta(\infty; f) + q\,\underline{d}(\mathcal{P})\delta(0; f) + \sum_{j=0}^{l^*} \chi_{_j} \left\{ \Theta_{_{(2}}(c_{_j}; f) + \Theta(c_{_j}; f) \right\} \tag{1.11}$$

$$+ \sum_{i=1}^{u} \mu_2(p_i)\delta_{\mu_2^*(p_i)}(d_i; f) > 3 + 2l^* + \sum_{i=1}^{u} \mu_2(p_i),$$

or, if $2 \leq p < \infty$ and

$$3\Theta(\infty; f) + q\,\underline{d}(\mathcal{P})\delta(0; f) + \sum_{j=0}^{l^*} \chi_{_j} \left\{ \Theta_{(2}(c_{_j}; f) + \Theta(c_{_j}; f) \right\} \tag{1.12}$$

$$+ \sum_{i=1}^{u} \mu_3(p_i)\delta_{\mu_3^*(p_i)}(d_i; f) > 3 + 2l^* + \sum_{i=1}^{u} \mu_3(p_i),$$

or, if $p=1$ *and*

$$4\Theta(\infty;f)+q\,\underline{d}(\mathcal{P})\delta(0;f)+\sum_{j=0}^{l^*}\chi_{j}\left\{\Theta_{(2}(c_{j};f)+2\Theta(c_{j};f)\right\} \tag{1.13}$$
$$+\sum_{i=1}^{u}\mu_2(p_i)\delta_{\mu_2^*(p_i)}(d_{i};f)+\sum_{i=1}^{u}\Theta(d_{i};f)>4+3l^*+\sum_{i=1}^{u}\mu_2(p_i)+u,$$

or, if $p=0$ *and*

$$(6+2Q)\Theta(\infty;f)+3q\,\underline{d}(\mathcal{P})\delta(0;f)+\sum_{j=0}^{l^*}\chi_{j}\left\{\Theta_{(2}(c_{j};f)+2\Theta(c_{j};f)\right\}$$
$$+\sum_{i=1}^{u}\mu_2(p_i)\delta_{\mu_2^*(p_i)}(d_{i};f)+\sum_{i=1}^{u}\Theta(d_{i};f)>6+2Q+3l^*+2q\overline{d}(\mathcal{P}) \tag{1.14}$$
$$+\sum_{i=1}^{u}\mu_2(p_i)+u,$$

then $\mathcal{R}(f)\equiv\mathcal{P}[f^q]$. *In particular, if* $P_n(0)=0$, *then the condition (ii) is no longer required.*

From Theorem 1.2 we can immediately deduce the following corollaries.

Corollary 1.1. *Let* f *be a non-constant meromorphic function and* $n(\geq 1)$, $u(\geq 1)$, $k(\geq 1)$ *and* $p(\geq 0)$ *are integers such that and* $a\equiv a(z)(\not\equiv 0,\infty)$ *be a meromorphic small function of* f. *Let* $\mathcal{P}[f]$ *be a differential polynomial containing at least one derivative. Suppose that* $P_n(f)-a$ *and* $\mathcal{P}[f]-a$ *share* $(0,p)$ *with* $\underline{d}(\mathcal{P})\delta(0;f)\leq n$. *If* $p=\infty$ *and*

$$3\Theta(\infty;f)+\underline{d}(\mathcal{P})\delta(0;f)+\sum_{i=1}^{u}\mu_2(p_i)\delta_{\mu_2{}^*(p_i)}(d_{i};f) \tag{1.15}$$
$$>3+\sum_{i=1}^{u}\mu_2(p_i),$$

or, if $2\leq p<\infty$ *and*

$$3\Theta(\infty;f)+\underline{d}(\mathcal{P})\delta(0;f)+\sum_{i=1}^{u}\mu_3(p_i)\delta_{\mu_3^*(p_i)}(d_{i};f) \tag{1.16}$$
$$>3+\sum_{i=1}^{u}\mu_3(p_i),$$

or, if $p=1$ *and*

$$4\Theta(\infty;f)+\underline{d}(\mathcal{P})\delta(0;f)+\sum_{i=1}^{u}\mu_2(p_i)\delta_{\mu_2^*(p_i)}(d_{i};f)+\sum_{i=1}^{u}\Theta(d_{i};f) \tag{1.17}$$
$$>4+\sum_{i=1}^{u}\mu_2(p_i)+u,$$

or, if $p = 0$ *and*

$$\begin{aligned}&(6+2Q)\Theta(\infty;f)+\underline{d}(\mathcal{P})\delta(0;f)+\sum_{i=1}^{u}\mu_2(p_i)\delta_{\mu_2^*(p_i)}(d_i;f)+\sum_{i=1}^{u}\Theta(d_i;f)\\ > \;& 6+2Q+2\overline{d}(\mathcal{P})+\sum_{i=1}^{u}\mu_2(p_i)+u, \qquad (1.18)\end{aligned}$$

then $P_n(f) \equiv \mathcal{P}[f]$. *In particular, if* $P_n(0) = 0$*, then the condition* $\underline{d}(\mathcal{P})\delta(0;f) \le n$ *is no longer required.*

Let $\mathcal{P}[f]$ be a homogeneous differential polynomial. Then we know from Definition 1.5 that $\underline{d}(\mathcal{P}) = \overline{d}(\mathcal{P}) = d$ and hence $Q = \Gamma - d$ and if $P_n(f) = f$, $\mu_2(1) = 1 = \mu_3(1)$, $\mu_2^*(1) = 2$, $\mu_3^*(1) = 3$. Keeping this in mind we obtain the following corollary for homogeneous differential polynomials.

Corollary 1.2. *Let* f *be a non-constant meromorphic function and* $\mathcal{P}[f]$ *be a homogeneous differential polynomial. Let* $a(z)(\not\equiv 0,\infty)$ *be a small meromorphic function. Suppose* $f - a$ *and* $\mathcal{P}[f] - a$ *share* $(0,p)$. *If* $p = \infty$ *and*

$$3\Theta(\infty;f)+d\,\delta(0;f)+\delta_2(0;f)>4, \qquad (1.19)$$

or, $2 \le p < \infty$ *and*

$$3\Theta(\infty;f)+d\,\delta(0;f)+\delta_3(0;f)>4, \qquad (1.20)$$

or, $p = 1$ *and*

$$4\Theta(\infty;f)+d\,\delta(0;f)+\Theta(0;f)+\delta_2(0;f)>6, \qquad (1.21)$$

or, $p = 0$ *and*

$$[2(\Gamma-d)+6]\Theta(\infty;f)+3d\,\delta(0;f)+\Theta(0;f)+\delta_2(0;f)>2\Gamma+8, \qquad (1.22)$$

then $f \equiv \mathcal{P}[f]$.

Remark 1.1. *It is clear that Corollary 1.1 is a direct extension and improvement of Theorem F. Now if we compare the conditions of Corollary 1.2 and Theorem E, we see that*

$$\begin{aligned}&3\Theta(\infty;f)+d\delta_{2+\Gamma-d}(0,f^d)+\delta_2(0,f)+\delta(a,f)\\ > \;& 3\Theta(\infty;f)+d\,\delta(0;f)+\delta_3(0;f).\end{aligned}$$

and

$$\begin{aligned}&\frac{7+\Gamma-d}{2}\Theta(\infty;f)+\frac{d}{2}\delta_{1+\Gamma-d}(0;f^d)+d\delta_{2+\Gamma-d}(0;f^d)+\delta_2(0;f)+\delta(a;f)\\ > \;& 4\Theta(\infty;f)+d\,\delta(0;f)+\Theta(0;f)+\delta_2(0;f).\end{aligned}$$

and

$$\begin{aligned}&[2(\Gamma-d)+6]\Theta(\infty;f)+d\delta_{1+\Gamma-d}(0,f^d)+d\delta_{2+\Gamma-d}(0;f^d)+\delta_2(0;f)+\Theta(0;f)\\ &+\delta(a;f)>[2(\Gamma-d)+6]\Theta(\infty;f)+3d\,\delta(0;f)+\Theta(0;f)+\delta_2(0;f).\end{aligned}$$

Hence we see that Corollary 1.2 is a direct improvement of Theorem E.

The following examples show that $a \not\equiv 0$ is necessary in Theorem 1.1 and Theorem 1.2.

Example 1.1. *Let* $\mathcal{R}(f) = f^7$ *and* $\mathcal{P}[f^6] = \frac{1}{6}\left(f^6\right)'$, *where* $f = e^z$. *Clearly* $q \geq k^*$ *as* $q = 6$, $k = 1$ *and* $\mathcal{R}(f) = e^{7z}$ *and* $\mathcal{P}[f^6] = e^{6z}$ *share* $(0, \infty)$. *Here* $\underline{d}(\mathcal{P}) = 1 = \overline{d}(\mathcal{P})$, $d_1 = 0$, $\Theta(\infty; f) = \delta(0; f) = \Theta(0; f) = 1$. *All the conditions* (1.8) - (1.10) *in Theorem 1.1 are satisfied but* $\mathcal{R}(f) \not\equiv \mathcal{P}[f^6]$.

Example 1.2. *Let* $\mathcal{R}(f) = \dfrac{f^n}{f - \alpha}$, *where* $n \geq 2$ *be an integer,* $\alpha \neq 0$ *be any complex number and* $\mathcal{P}[f] = \frac{1}{3}f + \frac{2}{3}f'$, *where* $f = e^z$. *Clearly* $q < k^*$ *as* $q = 1 = k$ *and* $\mathcal{R}(f)$ *and* $\mathcal{P}[f]$ *share* $(0, \infty)$. *Here* $\underline{d}(\mathcal{P}) = 1 = \overline{d}(\mathcal{P})$, $d_1 = 0$, $c_1 = \alpha$, $\Theta(\infty; f) = \Theta_{(2}(c_1; f) = \delta(0; f) = 1$, $\Theta(c_1; f) = 0$. *We see that all the conditions* (1.11) - (1.14) *in Theorem 1.2 are satisfied but* $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

The next example shows that the deficiency conditions stated in Theorem 1.2 are not necessary.

Example 1.3. [4] *Let* $f(z) = \mathcal{A}\cos z + \mathcal{B}\sin z$, $\mathcal{AB} \neq 0$. *Then* $\overline{N}(r, f) = S(r, f)$ *and*

$$\overline{N}(r, 0; f) = \overline{N}\left(r, \frac{\mathcal{A} + i\mathcal{B}}{\mathcal{A} - i\mathcal{B}}; e^{2iz}\right) \sim T(r, f).$$

Here $\Theta(\infty, f) = 1$ *and* $\delta(0, f) = 0$. *Let* $\mathcal{R}(f) = f$.

Therefore it is clear that $\mathcal{P}[f] = f^{(4k)}$, *for* $k \in \mathbb{N}$ *and* $\mathcal{R}(f)$ *share* (a, ∞) *and the deficiency conditions* (1.11) - (1.14) *in Theorem 1.2 are not satisfied, but* $\mathcal{R}(f) \equiv \mathcal{P}[f]$.

The following examples show that the conditions in Theorem 1.1 and Theorem 1.2 can not be removed when $m = 0$ under different sharing environment.

Example 1.4. *Let* $\mathcal{R}(f) = f^3 + 1$ *and* $\mathcal{P}[f^2] = -f^2 + \frac{1}{2}(f^2)'$, *where* $f = \dfrac{e^z}{e^z - 1}$. *We see that* $q \geq k^*$, *where* $q = 2$, $k = 1$ *and* $\mathcal{R}(f) - \frac{1}{2}$ *and* $\mathcal{P}[f^2] - \frac{1}{2}$ *share* $(0, \infty)$. *Here* $q\underline{d}(\mathcal{P})\delta(0; f) \leq n$, *as* $n = 3$, $\underline{d}(\mathcal{P}) = 1 = \overline{d}(\mathcal{P})$, $\Theta(\infty; f) = \Theta(d_i; f) = \delta_s(d_i; f) = 0$, $\forall s \in \mathbb{N}$ *and* $\delta(0; f) = 1$. *It is clear that none of the condition* (1.8)-(1.10) *in Theorem 1.1 is satisfied and* $\mathcal{R}(f) \not\equiv \mathcal{P}[f^2]$.

Example 1.5. *Let* $\mathcal{R}(f) = 2f^2 - 1$ *and* $\mathcal{P}[f^2] = \frac{1}{4}(f^2)' + \frac{1}{2}f^2$, *where* $f = e^z$. *Now it is clear that* $q \geq k^*$ *and* $\mathcal{R}(f) - 1 = 2(e^{2z} - 1)$ *and* $\mathcal{P}[f^2] - 1 = e^{2z} - 1$ *share* $(0, \infty)$. *Here* $\Theta(\infty; f) = 1 = \delta(0; f)$, $q = 2$, $\underline{d}(\mathcal{P}) = 1 = \overline{d}(\mathcal{P})$, $d_i = \pm\frac{1}{\sqrt{2}}$, $\Theta(d_i; f) = 0$, $\Theta_{(2}(d_i; f) = 1$, $u = 2$. *We see that none of the conditions* (1.8)-(1.10) *in Theorem 1.1 is satisfied and* $\mathcal{R}(f) \not\equiv \mathcal{P}[f^2]$.

Example 1.6. *Let* $f(z) = e^{Nz}$, *where* N *is a non-zero integer. For* $n \geq 2$ *let*

$$\mathcal{R}(f) = -N^{2n} \sum_{r=0}^{2n-1} (-1)^r \binom{2n}{r} f^{2n-r} \quad \text{and} \quad \mathcal{P}[f] = f^{(2n)}.$$

Then it is clear that

$$\mathcal{R}(f) - N^{2n} = -N^{2n}(e^{Nz} - 1)^{2n} \quad \text{and} \quad \mathcal{P}[f] - N^{2n} = N^{2n}(e^{Nz} - 1).$$

Thus we see that $\mathcal{R}(f)$ *and* $\mathcal{P}[f]$ *share* $(N^{2n}, 0)$. *Here* $\Theta(\infty; f) = 1$ *and* $\delta_q(0; f) = 1, \forall q \in \mathbb{N}$.

We see that the condition (1.14) *in Theorem 1.2 is not satisfied and* $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

Example 1.7. *Let* $f(z) = -\sin(\alpha z) + a - \frac{a}{\alpha^{4k}}$, $k \in \mathbb{N}$; *where* $\alpha \neq 0, \alpha^{4k} \neq 1$ *and* $a \in \mathbb{C} - \{0\}$. *Let* $\mathcal{R}(f) = f$ *and* $\mathcal{P}[f] = f^{(4k)}$. *Then* $\mathcal{P}[f] = -\alpha^{4k}\sin(\alpha z)$. *Here* $\Theta(\infty; f) = 1$ *and since*

$$\overline{N}(r, 0; f) = \overline{N}\left(r, a - \frac{a}{\alpha^{4k}}; \sin(\alpha z)\right) \sim T(r, f),$$

so $\delta(0; f) = 0$. *Also it is clear that* $\mathcal{R}(f)$ *and* $\mathcal{P}[f]$ *share* (a, ∞) *but none of the inequalities* (1.11) - (1.14) *of Theorem 1.2 is satisfied and* $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

Example 1.8. *Let* $f(z) = e^{\beta z} + a - \frac{a}{\beta^2}$; *where* $a \neq 0, \infty$ *and* $\beta \neq 0, \pm 1$. *Let* $\mathcal{R}(f) = f$. *Again let* $\mathcal{P}[f] = f^{(2)}$. *Then* $\mathcal{P}[f] = \beta^2 e^{\beta z}$. *Here* $\Theta(\infty; f) = 1$ *and since*

$$\overline{N}(r, 0; f) = \overline{N}\left(r, \frac{a}{\beta^2} - a; e^{\beta z}\right) \sim T(r, f),$$

so $\delta_q(0; f) = 0, \forall q \in \mathbb{N}$. *Also it is clear that* $\mathcal{R}(f)$ *and* $\mathcal{P}[f]$ *share* (a, ∞) *but none of the inequalities* (1.11) - (1.14) *in Theorem 1.2 is satisfied and* $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

Example 1.9. *Let* $\mathcal{R}(f) = f^2 + 1$ *and* $\mathcal{P}[f] = f' - f$, *where* $f = \frac{e^z}{e^z - 1}$. *We see that* $q < k^*$, *where* $q = 1 = k$ *and* $\mathcal{R}(f) - \frac{1}{2}$ *and* $\mathcal{P}[f] - \frac{1}{2}$ *share* $(0, \infty)$. *Here* $q\underline{d}(\mathcal{P})\delta(0; f) \leq n$, *as* $n = 2$, $\underline{d}(\mathcal{P}) = 1 = \overline{d}(\mathcal{P})$, $\Theta(\infty; f) = \Theta(d_i; f) = \delta_s(d_i; f) = 0$, $\forall s \in \mathbb{N}$ *and* $\delta(0; f) = 1$. *It is clear that none of the conditions* (1.11)-(1.14) *in Theorem 1.2 is satisfied and* $\mathcal{R}(f) \not\equiv \mathcal{P}[f]$.

2. Some Lemmas

In this section we present some lemmas which will be needed in the sequel. Let $\mathcal{F}$, $\mathcal{G}$ be two non-constant meromorphic functions. Henceforth we shall denote by $\mathcal{H}$ the following function.

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F} - 1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G} - 1}\right). \tag{2.1}$$

Lemma 2.1. [25] *Let* f *be a non-constant meromorphic function and* k *be a positive integer, then*

$$N_p(r, 0; f^{(k)}) \leq N_{p+k}(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.2. [16] *If* $N(r, 0; f^{(k)} \mid f \neq 0)$ *denotes the counting function of those zeros of* $f^{(k)}$ *which are not the zeros of* f, *where a zero of* $f^{(k)}$ *is counted according to its multiplicity then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f \mid < k) + k\overline{N}(r, 0; f \mid \geq k) + S(r, f).$$

Lemma 2.3. [19] *Let* f *be a non-constant meromorphic function and let*

$$\mathcal{R}(f) = \frac{\sum\limits_{i=0}^{n} a_i f^i}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_i\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, \mathcal{R}(f)) = d\, T(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 2.4. *Let f be a meromorphic function and $\mathcal{P}[f]$ be a differential polynomial. Then*

$$m\left(r, \frac{\mathcal{P}[f]}{f^{q\,\overline{d}(\mathcal{P})}}\right) \leq (\overline{d}(\mathcal{P}) - \underline{d}(\mathcal{P}))m\left(r, \frac{1}{f^q}\right) + S(r, f).$$

Proof. The proof can be conducted along the same lines as the proof of *Lemma 2.4* in [5]. □

Lemma 2.5. *Let f be a non-constant meromorphic function and $\mathcal{P}[f^q]$ be a differential polynomial. Then*

$$\begin{aligned} & N(r, 0; \mathcal{P}[f^q]) \\ \leq \;& T(r, \mathcal{P}[f^q]) - q\, \underline{d}(\mathcal{P})T(r, f) + q\, \underline{d}(\mathcal{P})N(r, 0; f) + S(r, f). \end{aligned}$$

Proof. The proof can be conducted along the same lines as the proof of *Lemma 2.5* in [5]. □

Lemma 2.6. [10] *Let $\mathcal{P}[f^q]$ be a differential polynomial generated by f. Then*

$$m(r, \mathcal{P}[f^q]) \leq \overline{d}(\mathcal{P})m(r, f^q) + S(r, f).$$

Proof. The proof can be conducted along the same lines as the proof of *Lemma 2.6* in [5]. □

Lemma 2.7. *Let f be a non-constant meromorphic function and $\mathcal{P}[f]$ be a differential polynomial. Then $S(r, \mathcal{P}[f^q])$ can be replaced by $S(r, f)$.*

Proof. From *Lemma 2.6* it is clear that $T(r, \mathcal{P}[f^q]) = O(T(r, f))$ and so the lemma follows. □

Lemma 2.8. *Let $\mathcal{P}[f^q]$ be a differential polynomial generated by f. Then*

$$\begin{aligned} & T(r, \mathcal{P}[f^q]) \\ \leq \;& q\, \overline{d}(\mathcal{P})T(r, f) + Q\overline{N}(r, \infty; f) + S(r, f). \end{aligned}$$

Proof. The proof can be conducted along the same lines as the proof of *Lemma 2.8* in [5]. □

3. Proofs of the theorems

Proof of Theorem 1.1. Let $\mathcal{F} = \dfrac{\mathcal{R}(f)}{a}$ and $\mathcal{G} = \dfrac{\mathcal{P}[f^q]}{a}$. Then $\mathcal{F} - 1 = \dfrac{\mathcal{R}(f) - a}{a}$ and $\mathcal{G} - 1 = \dfrac{\mathcal{P}[f^q] - a}{a}$. Since $\mathcal{R}(f) - a$ and $\mathcal{P}[f^q] - a$ share $(0, p)$ it follows that $\mathcal{F}$, $\mathcal{G}$ share $(1, p)$ except the zeros and poles of $a(z)$. Now we consider the following cases.

Case 1. Let $\mathcal{H} \not\equiv 0$.

From (2.1) it can be easily calculated that the possible poles of $\mathcal{H}$ occur at (i) multiple zeros of $\mathcal{F}$ and $\mathcal{G}$, (ii) those 1 points of $\mathcal{F}$ and $\mathcal{G}$ whose multiplicities are different related to $\mathcal{F}$ and $\mathcal{G}$, (iii) those common poles of $\mathcal{F}$ and $\mathcal{G}$ with different multiplicities, (iv) multiple poles of $\mathcal{F}$ and $\mathcal{G}$, (v) zeros of $\mathcal{F}^{'}$ ($\mathcal{G}^{'}$) which are not the zeros of $\mathcal{F}(\mathcal{F}-1)$ ($\mathcal{G}(\mathcal{G}-1)$).

Let z_0, a d_i, $i=1,2,\ldots,u$ point of f with multiplicity $r\geq 2$ such that $a(z_0)\neq 0,\infty$. If $d_i=0$, then since $\mathcal{G}$ contains at least one derivative, z_0 would be a zero of $\mathcal{G}$ with multiplicity at least $2q-k$. Since $q\geq k^{*}$, it follows that z_0 will be a multiple zero of $\mathcal{G}$ too. Since $\mathcal{H}$ has only simple poles we get from (2.1)

$$\begin{aligned}
& N(r,\infty;\mathcal{H}) && (3.1)\\
\leq\; & \overline{N}(r,\infty;f)+\sum_{j=0}^{l^{*}}\chi_{j}\overline{N}(r,c_{j};f\mid\geq 2)+\sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^{*}}\overline{N}(r,d_{i};f\mid\geq 2)\\
+\; & \overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})+\overline{N}(r,0;\mathcal{G}\mid\geq 2)+\overline{N}_{0}(r,0;\mathcal{F}^{'})+\overline{N}_{0}(r,0;\mathcal{G}^{'})+\overline{N}(r,0;a)\\
+\; & \overline{N}(r,\infty;a). && (3.2)
\end{aligned}$$

where $\overline{N}_0(r,0;\mathcal{F}')$ is the reduced counting function of those zeros of $\mathcal{F}'$ which are not the zeros of $\mathcal{F}(\mathcal{F}-1)$ and $\overline{N}_0(r,0;\mathcal{G}^{'})$ is similarly defined. Let z_0 be a simple zero of $\mathcal{F}-1$. Then by a simple calculation we see that z_0 is a zero of $\mathcal{H}$ and hence

$$N_{E}^{1)}(r,1;\mathcal{F})=N(r,1;\mathcal{F}\mid=1)\leq N(r,0;\mathcal{H})\leq N(r,\infty;\mathcal{H})+S(r,\mathcal{F}) \qquad (3.3)$$

By using the Second Fundamental Theorem, *Lemma 2.7*, (3.1) and noting that $\overline{N}(r,\infty;\mathcal{G})=\overline{N}(r,\infty;f)+S(r,f)$, we get

$$\begin{aligned}
& T(r,\mathcal{G}) && (3.4)\\
\leq\; & \overline{N}(r,\infty;\mathcal{G})+\overline{N}(r,0;\mathcal{G})+\overline{N}(r,1;\mathcal{G})-N_{0}(r,0;\mathcal{G}^{'})+S(r,\mathcal{G})\\
\leq\; & 2\overline{N}(r,\infty;f)+\overline{N}(r,0;\mathcal{G})+\overline{N}(r,0;\mathcal{G}\mid\geq 2)+\sum_{j=0}^{l^{*}}\chi_{j}\overline{N}(r,c_{j};f\mid\geq 2)\\
+\; & \sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^{*}}\overline{N}(r,d_{i};f\mid\geq 2)+\Big\{\overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})+\overline{N}(r,1;F\mid\geq 2)\\
+\; & \overline{N}_{0}(r,0;\mathcal{F}')\Big\}+S(r,f).
\end{aligned}$$

Subcase 1.1. While $p\geq 2$. Then

$$\begin{aligned}
& \overline{N}_{*}(r,1;\mathcal{F},\mathcal{G})+\overline{N}(r,1;\mathcal{F}\mid\geq 2)+\overline{N}_{0}(r,0;\mathcal{F}') && (3.5)\\
\leq\; & \overline{N}(r,0;\mathcal{F}'\mid\mathcal{F}\neq 0).
\end{aligned}$$

So,

$$\begin{aligned}
& T(r,\mathcal{G})\\
\leq\; & 2\,\overline{N}(r,\infty;f)+N_{2}(r,0;\mathcal{G})+\sum_{j=0}^{l^{*}}\chi_{j}\overline{N}(r,c_{j};f)+\sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^{*}}\overline{N}(r,d_{i};f\mid\geq 2)\\
+\; & \overline{N}(r,0;\mathcal{F}^{'}\mid\mathcal{F}\neq 0)+S(r,f).
\end{aligned}$$

By *Lemma 2.2*, we get

$$\begin{aligned} & N(r,0;\mathcal{F}' \mid \mathcal{F}\neq 0) \\ \leq\ & \overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_{j}\overline{N}(r,c_{j};f)+\sum_{i=1}^{u}\overline{N}(r,d_{i};f)+S(r,f). \end{aligned}$$

Using *Lemma 2.5*, we get from above

$$\begin{aligned} & T\left(r,\mathcal{P}[f^q]\right) \\ \leq\ & 3\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_{j}\overline{N}(r,c_{j};f\mid\geq 2)+\sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^*}\overline{N}(r,d_{i};f\mid\geq 2) \\ +\ & \sum_{j=0}^{l^*}\chi_{j}\overline{N}(r,c_{j};f)+T(r,\mathcal{P}[f^q])-q\ \underline{d}(\mathcal{P})T(r,f)+q\ \underline{d}(\mathcal{P})N(r,0;f) \\ +\ & \sum_{i=1}^{u}\overline{N}(r,d_{i};f)+S(r,f). \end{aligned}$$

i.e., for any $\epsilon>0$,

$$\begin{aligned} & \left\{3\Theta(\infty;f)+q\ \underline{d}(\mathcal{P})\delta(0;f)+\sum_{j=0}^{l^*}\chi_{j}\left\{\Theta_{(2}(c_{j};f)+\Theta(c_{j};f)\right\}\right. \\ +\ & \left.\sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^*}\Theta_{(2}(d_{i};f)+\sum_{i=1}^{u}\Theta(d_{i};f)\right\}T(r,f) \\ \leq\ & \left\{3+2l^*+u+u^*+\epsilon\right\}T(r,f)+S(r,f), \end{aligned}$$

which contradicts (1.8)

Subcase 1.2. While $p=1$, (3.5) becomes

$$\begin{aligned} & \overline{N}(r,1;\mathcal{F}\mid\geq p+1)+\overline{N}(r,1;\mathcal{F}\mid\geq 2)+\overline{N}_0(r,0;\mathcal{F}^{'}) \\ \leq\ & N(r,0;\mathcal{F}^{'}\mid\mathcal{F}\neq 0\mid\geq 1)+N(r,0;\mathcal{F}^{'}\mid\mathcal{F}\neq 0) \\ \leq\ & 2\overline{N}(r,0;\mathcal{F}^{'}\mid\mathcal{F}\neq 0). \end{aligned} \tag{3.6}$$

Proceeding same way as in Subcase 1.1, we get

$$\begin{aligned} & T(r,\mathcal{G}) \\ \leq\ & 4\,\overline{N}(r,\infty;f)+N(r,0;\mathcal{G})+\sum_{j=0}^{l^*}\chi_{j}\overline{N}(r,c_{j};f\mid\geq 2)+2\sum_{j=0}^{l^*}\chi_{j}\overline{N}(r,c_{j};f) \\ +\ & \sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^*}\overline{N}(r,d_{i};f\mid\geq 2)+2\sum_{i=1}^{u}\overline{N}(r,d_{i};f)+S(r,f). \end{aligned}$$

Again using *Lemmas 2.5* and *2.2*, as above we get that

$$\begin{aligned}
& T(r,P[f^q]) \\
\leq\ & 4\,\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_{j}\left\{\overline{N}(r,c_{j};f\mid\geq 2)+2\overline{N}(r,c_{j};f)\right\} \\
+\ & \sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^*}\overline{N}(r,d_{i};f\mid\geq 2)+2\sum_{i=1}^{u}\overline{N}(r,d_{i};f)+T(r,\mathcal{P}[f^q])-q\ \underline{d}(\mathcal{P})T(r,f^q) \\
+\ & q\ \underline{d}(\mathcal{P})N(r,0;f)+S(r,f).
\end{aligned}$$

i.e., for any $\epsilon>0$,

$$\begin{aligned}
& \Bigg\{4\Theta(\infty;f)+q\ \underline{d}(\mathcal{P})\delta(0;f)+\sum_{j=0}^{l^*}\chi_{j}\left\{\Theta_{(2}(c_{j};f)+2\Theta(c_{j};f)\right\} \\
+\ & \sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^*}\Theta_{(2}(d_{i};f)+2\sum_{i=1}^{u}\Theta(d_{i};f)\Bigg\}T(r,f) \\
\leq\ & \left\{4+3l^*+2u+u^*+\epsilon\right\}T(r,f)+S(r,f),
\end{aligned}$$

which contradicts (1.9).

Subcase 1.3. While $p=0$.

In this case $\mathcal{F}$ and $\mathcal{G}$ share $(1,0)$ except the zeros and poles of $a(z)$. Here, proceeding same way as in [3, Subcase 1.2, Proof of Theorem 1.1], we obtain by applying *Lemma 2.2* and *Lemma 2.5*,

$$\begin{aligned}
& T(r,\mathcal{G}) \\
\leq\ & 4\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_{j}\overline{N}(r,c_{j};f\mid\geq 2)+\sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^*}\overline{N}(r,d_{i};f\mid\geq 2) \\
+\ & 2T(r,\mathcal{G})+T(r,\mathcal{P}[f^q])-3q\ \underline{d}(\mathcal{P})T(r,f)+3q\ \underline{d}(\mathcal{P})N(r,0;f) \\
+\ & 2\left\{N_2(r,0;\mathcal{F})+\overline{N}(r,\infty;\mathcal{F})\right\}+S(r,f).
\end{aligned}$$

Applying *Lemmas 2.8*, we get from above

$$\begin{aligned}
& 3q\ \underline{d}(\mathcal{P})T(r,f) \\
\leq\ & (2Q+6)\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_{j}\left\{\overline{N}(r,c_{j};f\mid\geq 2)+2\overline{N}(r,c_{j};f)\right\} \\
+\ & \sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^*}\overline{N}(r,d_{i};f\mid\geq 2)+2q\ \overline{d}(\mathcal{P})T(r,f)+3q\ \underline{d}(\mathcal{P})N(r,0;f) \\
+\ & 2\sum_{i=1}^{u}\mu_2(p_i)N_{\mu_2^*(p_i)}(r,d_{i};f)+S(r,f),
\end{aligned}$$

i.e., for any $\epsilon > 0$,

$$\begin{aligned}&\left\{(6+2Q)\Theta(\infty;f)+\sum_{j=0}^{l^*}\chi_{_j}\left\{\Theta_{(2}(c_{_j};f)+2\Theta(c_{_j};f)\right\}+\sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^*}\Theta_{_{(2}}(d_{_i};f)\right.\\ +&\left.\quad 3q\ \underline{d}(\mathcal{P})\delta(0;f)+2\sum_{i=0}^{u}\mu_2(p_i)\delta_{\mu_2^*(p_i)}(d_i;f)\right\}T(r,f)\\ \leq&\quad\left\{6+2Q+2q\overline{d}(\mathcal{P})+3l^*+2\sum_{i=1}^{u}\mu_2(p_i)+u^*+\epsilon\right\}T(r,f)+S(r,f),\end{aligned}$$

which contradicts (1.10).

Case 2. Let $\mathcal{H} \equiv 0$.

On integration we get from (2.1)

$$\frac{1}{\mathcal{F}-1}\equiv\frac{\mathcal{C}}{\mathcal{G}-1}+\mathcal{D}, \tag{3.7}$$

where $\mathcal{C}(\neq 0)$, $\mathcal{D}$ are constants. We claim that $\mathcal{D} = 0$. Suppose that there exist a pole z_0 of f with multiplicity p which is neither a pole nor a zero of $a(z)$. As $n > m$, z_0 will be a pole of $\mathcal{F}$ with multiplicity $(n-m)p$ and a pole of $\mathcal{G}$ with multiplicity M (say). We assume that $(n-m)p \neq M$, since otherwise we know from (3.7) that $\mathcal{D} = 0$ and we are done.

Subcase 2.1. Suppose $\mathcal{D} \neq 0$.

Since $(n-m)p \neq M$, we get a contradiction from (3.10). So,

$$N(r,\infty;f)\leq N(r,0;a)+N(r,\infty;a)=S(r,f),$$

and hence $\Theta(\infty; f) = 1$. Also it is clear that $\overline{N}(r,\infty;\mathcal{G}) = \overline{N}(r,\infty;f) = S(r,f)$.

$$\begin{aligned}&q\ \underline{d}(\mathcal{P})\delta(0;f)+\sum_{j=0}^{l^*}\chi_{_j}\left\{\Theta(c_{_j};f)+\Theta_{_{(2}}(c_{_j};f)\right\}+\sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^*}\Theta_{_{(2}}(d_{_i};f)\\ +&\quad\sum_{i=1}^{u}\Theta(d_{_i};f)>2l^*+u+u^*,\end{aligned}\tag{3.8}$$

$$\begin{aligned}&q\ \underline{d}(\mathcal{P})\delta(0;f)+\sum_{j=0}^{l^*}\chi_{_j}\left\{\Theta_{_{(2}}(c_{_j};f)+2\Theta(c_{_j};f)\right\}+\sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^*}\Theta_{_{(2}}(d_{_i};f)\\ +&\quad 2\sum_{i=1}^{u}\Theta(d_{_i};f)>3l^*+u^*+2u,\end{aligned}\tag{3.9}$$

$$\begin{aligned}&3q\ \underline{d}(\mathcal{P})\delta(0;f)+\sum_{j=0}^{l^*}\chi_{_j}\left\{\Theta_{_{(2}}(c_{_j};f)+2\Theta(c_{_j};f)\right\}+\sum_{\substack{i=1\\ d_1,d_2,\ldots,d_{u^*}\neq 0}}^{u^*}\Theta_{_{(2}}(d_{_i};f)\\ +&\quad 2\sum_{i=1}^{u}\mu_2(p_i)\delta_{\mu_2{}^*(p_i)}(d_{_i};f)>2q\overline{d}(\mathcal{P})+3l^*+2\sum_{i=1}^{u}\mu_2(p_i)+u^*,\end{aligned}\tag{3.10}$$

Since $D \neq 0$, from (3.7) we get

$$-\frac{\mathcal{D}\left(\mathcal{F}-1-\frac{1}{\mathcal{D}}\right)}{\mathcal{F}-1} \equiv \frac{\mathcal{C}}{\mathcal{G}-1}.$$

So

$$\overline{N}\left(r,1+\frac{1}{\mathcal{D}};\mathcal{F}\right)=\overline{N}(r,\infty;\mathcal{G})=S(r,f).$$

Subcase 2.1.1. Let $\mathcal{D} \neq -1$. First suppose $m \neq 0$.
Using the second fundamental theorem for $\mathcal{F}$ we get

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,\infty;\mathcal{F})+\overline{N}(r,0;\mathcal{F})+\overline{N}\left(r,1+\frac{1}{\mathcal{D}};\mathcal{F}\right) \\ &\leq \sum_{j=0}^{l^*}\chi_{_j}\overline{N}(r,c_{_j};f)+\sum_{i=1}^{u}\overline{N}(r,d_{_i};f)+S(r,f). \end{aligned}$$

i.e.,

$$\sum_{j=0}^{l^*}\chi_{_j}\Theta(c_{_j};f)+\sum_{i=1}^{u}\Theta(d_{_i};f)\leq l^*+u-n\leq l^*, \tag{3.11}$$

which contradicts (1.8) - (1.10).

Next let $m=0$. Then (3.11) reduces to $\sum_{i=1}^{u}\Theta(d_{_i};f)\leq u-n$.

If $u<n$, then we get a contradiction. So we must have $u=n$. So we have $\Theta(d_i;f)=0$ for each i. Then from (3.8) we get $n<q\underline{d}(\mathcal{P})\delta(0;f)$, which is not possible. In particular, if $P_n(0)=0$, then one among the $d_{_i}$ is zero and so $\Theta(0;f)=0$, which implies $\delta(0;f)=0$ and so (ii) is no longer required.

Subcase 2.1.2. Let $\mathcal{D}=-1$.

Then

$$\frac{\mathcal{F}}{\mathcal{F}-1}\equiv\mathcal{C}\frac{1}{\mathcal{G}-1}. \tag{3.12}$$

If $\mathcal{C}\neq -1$ we know from (3.12) that $\overline{N}(r,1+\mathcal{C};\mathcal{G})=\overline{N}(r,\infty;\mathcal{F})$. So from *Lemmas 2.1* and *2.5* and by the second fundamental theorem we get

$$\begin{aligned} & q\ \underline{d}(\mathcal{P})T(r,f) \\ \leq\ & \overline{N}(r,\infty;\mathcal{G})+q\ \underline{d}(\mathcal{P})N(r,0;f)+\overline{N}(r,1+\mathcal{C};\mathcal{G})+S(r,f) \\ \leq\ & q\ \underline{d}(\mathcal{P})N(r,0;f)+\sum_{j=0}^{l^*}\chi_{_j}\overline{N}(r,c_{_j};f)+S(r,f). \end{aligned}$$

i.e.,

$$q\ \underline{d}(P)\delta(0;f)+\sum_{j=0}^{l^*}\chi_{_j}\Theta(c_{_j};f)\leq l^*,$$

which contradicts (3.8)-(3.10).

So $\mathcal{C}=-1$ and we get from (3.12) that $\mathcal{F}\mathcal{G}\equiv 1$, which gives $\mathcal{R}(f)\mathcal{P}[f^q]\equiv a^2$.

From above we have $N(r,0;f)=S(r,f)$ and $N(r,\infty;f)=S(r,f)$.

In view of the first fundamental theorem *Lemma 2.4*, we get from above

$$\begin{aligned}
& (n+q\,\overline{d}(\mathcal{P}))T(r,f) \\
= \;& T\left(r,\frac{a^2}{\mathcal{R}(f)f^{q\,\overline{d}(P)}}\right)+S(r,f) \\
\leq \;& T\left(r,\frac{\mathcal{P}[f^q]}{f^{q\,\overline{d}(\mathcal{P})}}\right)+S(r,f) \\
= \;& m\left(r,\frac{\mathcal{P}[f^q]}{f^{q\,\overline{d}(\mathcal{P})}}\right)+N\left(r,\infty;\frac{\mathcal{P}[f^q]}{f^{q\,\overline{d}(\mathcal{P})}}\right)+S(r,f) \\
\leq \;& (\overline{d}(\mathcal{P})-\underline{d}(\mathcal{P}))m\left(r,\frac{1}{f^q}\right)+N\left(r,\infty;\mathcal{P}[f^q]\right)+q\,\overline{d}(\mathcal{P})N(r,0;f)+S(r,f) \\
= \;& q\,(\overline{d}(\mathcal{P})-\underline{d}(\mathcal{P}))(T(r,f)-N(r,0;f))+S(r,f).
\end{aligned}$$

i.e.,

$$(n+q\,\underline{d}(\mathcal{P}))T(r,f)\leq S(r,f),$$

which is impossible.

Subcase 2.2. Let $\mathcal{D}=0$ and so from (3.10) we get

$$\mathcal{G}-1\equiv\mathcal{C}\,(\mathcal{F}-1).$$

If $\mathcal{C}\neq 1$, then

$$\mathcal{F}\equiv\frac{\mathcal{G}-1+\mathcal{C}}{\mathcal{C}}$$

and

$$\overline{N}(r,0;\mathcal{F})=\overline{N}(r,1-\mathcal{C};\mathcal{G}).$$

By the second fundamental theorem and using *Lemmas 2.1*, *2.5* and *2.7*, we have

$$\begin{aligned}
& T(r,\mathcal{G}) \\
\leq \;& \overline{N}(r,\infty;\mathcal{G})+\overline{N}(r,0;\mathcal{G})+\overline{N}(r,1-\mathcal{C};\mathcal{G})+S(r,\mathcal{G}) \\
\leq \;& \overline{N}(r,\infty;f)+\overline{N}(r,0;\mathcal{F})+T(r,P[f^q])-q\,\underline{d}(\mathcal{P})T(r,f)+q\,\underline{d}(\mathcal{P})N(r,0;f) \\
+ \;& S(r,f).
\end{aligned}$$

i.e.,

$$q\,\underline{d}(P)T(r,f)\leq\overline{N}(r,\infty;f)+q\,\underline{d}(P)N(r,0;f)+\sum_{i=1}^{u}\overline{N}(r,d_i;f)+S(r,f),$$

which implies

$$\Theta(\infty;f)+q\,\underline{d}(\mathcal{P})\delta(0;f)+\sum_{i=1}^{u}\Theta(d_i;f)\leq 1+u. \tag{3.13}$$

Now with the help of (3.13), we get contradiction to (1.8) - (1.10) respectively. Hence $\mathcal{C}=1$ and so $\mathcal{F}\equiv\mathcal{G}$, i.e., $\mathcal{R}(f)\equiv\mathcal{P}[f^q]$. □

Proof of Theorem 1.2. Let $\mathcal{F}$ and $\mathcal{G}$ be given as in the proof of *Theorem 1.1.* When $\mathcal{H} \not\equiv 0$ we observe that (3.1) can be changed into

$$\begin{aligned} & N(r,\infty;\mathcal{H}) \qquad (3.14) \\ \leq\ & \overline{N}(r,\infty;f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r,c_j;f \mid\geq 2) + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + \overline{N}(r,0;\mathcal{F} \mid\geq 2) \\ +\ & \overline{N}(r,0;\mathcal{G} \mid\geq 2)\overline{N}_0(r,0;\mathcal{F}^{'}) + \overline{N}_0(r,0;\mathcal{G}^{'}) + \overline{N}(r,0;a) + \overline{N}(r,\infty;a). \end{aligned}$$

Now for the next cases we follow [3, Subcase 1.1 and Subcase 1.2, Proof of Theorem 1.1] and apply *Lemmas 2.1* and *2.2*.

Finally we omit the rest of the proof as that is similar to the proof of *Theorem 1.1.* □

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