

## ON THE BOUNDEDNESS OF A PLANT-HERBIVORE MODEL

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ABSTRACT. In this note, we deal with the boundedness of solutions of the following Plant-Herbivore model

$$\begin{cases} x_{n+1} = \frac{\alpha x_n}{\beta x_n + e^{y_n}} \\ y_{n+1} = \gamma(x_n + 1)y_n \end{cases}, \quad n \in \mathbb{N},$$

where  $\alpha \in (1, \infty)$ ,  $\beta \in (0, \infty)$ ,  $\gamma \in (0, 1)$  with  $\alpha + \beta \geq 1 + \frac{\beta}{\gamma}$ , and the initial values  $x_0, y_0$  are positive real numbers.

### 1. INTRODUCTION

The mathematics of difference equations has always benefited from its involvement with other disciplines which makes this field a very interesting area of research. Difference equations models for interactions between species constitute an essential part in their applications to biology and ecology. Particularly, we deal in this note with a *Plant-Herbivore* model from population dynamics which describes the interaction of the apple twig borer (an insect pest of the grape vine) and grapes in Texas High plains. This last is represented by the following first order system of difference equations

$$\begin{cases} x_{n+1} = \frac{\alpha x_n}{\beta x_n + e^{y_n}} \\ y_{n+1} = \gamma(x_n + 1)y_n \end{cases}, \quad n \in \mathbb{N}, \quad (1)$$

with  $\alpha \in (1, \infty)$ ,  $\beta \in (0, \infty)$ ,  $\gamma \in (0, 1)$  and the initial values  $x_0, y_0$  are positive real numbers. This type of nonlinear systems is usually difficult to solve, but in other ways we can get informations about the dynamics relationship between populations without an analytical solution.

System (1) was developed and studied first by Allen et al. in [1], see also [2], [3] and [6]. It has three equilibrium points:  $P_0 = (0, 0)$  which is unstable,  $P_1 = (\frac{\alpha-1}{\beta}, 0)$ , and under the following hypothesis

$$\alpha + \beta \geq 1 + \frac{\beta}{\gamma} \quad (2)$$

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this system has also another equilibrium point  $P_2 = (\frac{1}{\gamma} - 1, \ln(\alpha + \beta - \frac{\beta}{\gamma}))$ . In their study, those researchers could show that the point  $P_1$  is globally asymptotically stable when  $\alpha + \beta < 1 + \frac{\beta}{\gamma}$ . However, the question about the behavior of this system when (2) is verified has been proposed after as an open problem in [7]. One of the difficult points to deal with in this case is the boundedness of solutions. We note that it is very important for every solution of System (1) to be bounded since the populations of species  $x_n$  and  $y_n$  can not grow infinitely. Recently, this open problem has been studied by Din in [5]. To this end the author supposed (Theorem 2.1) a condition on  $x_n$ , that we think it is hard!, to show that every positive solution of System (1) is bounded. Here, we give another boundedness result, that we believe it is more simple and interesting, and it will be presented in the following section.

## 2. MAIN RESULT

**Theorem 1** Consider the System (1) with

$$\alpha \in (1, \infty), \quad \beta \in (0, \infty), \quad \gamma \in (0, \frac{1}{2}] \quad (3)$$

and the initial values  $x_0, y_0$  are positive real numbers. Assume also that (2) is verified, then every solution of this system is bounded.

**Proof.** Let  $\{(x_n, y_n)\}_{n \geq 0}$  be a solution of System (1). Obviously, we have

$$x_0 = 0$$

if and only if

$$x_n = 0, \quad \text{for all } n \geq 0.$$

Thus, when  $x_0 = 0$  we find

$$0 \leq y_n = \gamma^n y_0 \leq y_0, \quad \text{for all } n \geq 0,$$

which implies that the solution  $\{(x_n, y_n)\}_{n \geq 0}$  is bounded.

Now, we suppose that  $x_0 \neq 0$ . Hence, for all  $n \geq 0$  we get

$$x_{n+1} = \frac{\alpha x_n}{\beta x_n + e^{y_n}} \leq \frac{\alpha x_n}{\beta x_n} \leq \frac{\alpha}{\beta},$$

i.e.,

$$x_n \leq \frac{\alpha}{\beta}, \quad \text{for all } n \geq 1,$$

then the sequence  $\{x_n\}_{n \geq 0}$  is bounded.

We show that  $\{y_n\}_{n \geq 0}$  is bounded also. For this, we distinguish the following cases: when the sequence  $(x_n - 1)$  is positive, negative and when  $(x_n)$  oscillates about 1.

**Case (1):** Suppose that  $x_n \geq 1$ , for all  $n \geq 0$ . We have from (1)

$$x_{n+1}(\beta x_n + e^{y_n}) = \alpha x_n,$$

which can be written as

$$e^{y_n} = \frac{\alpha x_n - \beta x_n x_{n+1}}{x_{n+1}},$$

therefore

$$e^{y_n} \leq \frac{\alpha x_n}{x_{n+1}} \leq \alpha x_n \leq \frac{\alpha^2}{\beta}.$$

From (2) and (3), we have

$$\beta \leq 1 + \beta\left(\frac{1}{\gamma} - 1\right) \leq \alpha \leq \alpha^2,$$

then

$$y_n \leq \ln\left(\frac{\alpha^2}{\beta}\right), \quad \text{for all } n \geq 0,$$

which means that  $\{y_n\}_{n \geq 0}$  is bounded.

**Case (2):** Suppose that  $x_n \leq 1$ , for all  $n \geq 0$ . From (1) we have

$$y_{n+1} = \gamma(x_n + 1)y_n \leq 2\gamma y_n, \quad \text{for all } n \geq 0.$$

Since  $\gamma \in (0, \frac{1}{2}]$ , thus

$$y_{n+1} < y_n, \quad \text{for all } n \geq 0,$$

which means that the sequence  $\{y_n\}_{n \geq 0}$  is decreasing and since  $y_n \geq 0$ , for all  $n \geq 0$ , then  $\{y_n\}_{n \geq 0}$  is bounded.

**Case (3):** Suppose now that  $\{x_n\}_{n \geq 0}$  oscillates about 1. Then, there exists an increasing sequence  $\{n_s\}_{s \geq 1}$  of naturel numbers such that  $x_{n_1} \geq 1$ , for a certain  $n_1 \geq 1$ , and for  $l = 1, 2, \dots$

$$\begin{cases} x_n < 1, & \text{if } n_{2l-1} < n \leq n_{2l}, \\ x_n \geq 1, & \text{if } n_{2l} < n \leq n_{2l+1}. \end{cases}$$

First, we have for all  $l \geq 1$ ,  $x_{n_{2l-1}} \geq 1$ . Thus, proving like in Case (1)

$$e^{y_{n_{2l-1}-1}} = \frac{\alpha x_{n_{2l-1}-1} - \beta x_{n_{2l-1}-1} x_{n_{2l-1}}}{x_{n_{2l-1}}} \leq \frac{\alpha x_{n_{2l-1}-1}}{x_{n_{2l-1}}} \leq \alpha x_{n_{2l-1}-1} \leq \frac{\alpha^2}{\beta},$$

which implies that

$$y_{n_{2l-1}-1} \leq \ln\left(\frac{\alpha^2}{\beta}\right),$$

therefore, we find

$$\begin{aligned} y_{n_{2l-1}} &= \gamma(x_{n_{2l-1}-1} + 1)y_{n_{2l-1}-1} \leq \gamma\left(\frac{\alpha}{\beta} + 1\right)y_{n_{2l-1}-1} \\ &\leq \gamma\left(\frac{\alpha}{\beta} + 1\right)\ln\left(\frac{\alpha^2}{\beta}\right). \end{aligned}$$

Furthermore, for  $n_{2l-1} < n \leq n_{2l}$ , we have

$$y_{n+1} = \gamma(x_n + 1)y_n < 2\gamma y_n < y_n,$$

then, it follows that

$$\begin{aligned} y_{n_{2l}+1} < y_{2l} < y_{n_{2l}-1} < \dots < y_{n_{2l-1}+1} &= \gamma(x_{n_{2l-1}} + 1)y_{n_{2l-1}} \\ &\leq \gamma\left(\frac{\alpha}{\beta} + 1\right)y_{n_{2l-1}} \\ &\leq \gamma^2\left(\frac{\alpha}{\beta} + 1\right)^2 \ln\left(\frac{\alpha^2}{\beta}\right). \end{aligned}$$

Now, for  $n_{2l} < n \leq n_{2l+1}$ , we have  $x_n \geq 1$ . Arguing as in Case (1), we get

$$y_n \leq \ln\left(\frac{\alpha^2}{\beta}\right), \quad \text{for all } n_{2l} \leq n < n_{2l+1},$$

and

$$y_{n_{2l+1}} = \gamma(x_{n_{2l+1}-1} + 1)y_{n_{2l+1}-1} \leq \gamma\left(\frac{\alpha}{\beta} + 1\right)y_{n_{2l+1}-1} \leq \gamma\left(\frac{\alpha}{\beta} + 1\right)\ln\left(\frac{\alpha^2}{\beta}\right).$$

In conclusion, let us take

$$c = \max \left\{ \ln\left(\frac{\alpha^2}{\beta}\right), \gamma\left(\frac{\alpha}{\beta} + 1\right)\ln\left(\frac{\alpha^2}{\beta}\right), \gamma^2\left(\frac{\alpha}{\beta} + 1\right)^2\ln\left(\frac{\alpha^2}{\beta}\right) \right\},$$

hence, for all  $l = 1, 2, \dots$ ,

$$y_n \leq c, \text{ for all } n_{2l-1} \leq n \leq n_{2l+1},$$

i.e.,

$$y_n \leq c, \text{ for all } n \geq n_1.$$

This means that  $\{y_n\}_{n \geq 0}$  is bounded in this case also, and the proof is complete.

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