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APPROXIMATE SOLUTIONS FOR A CUBIC AUTOCATALYTIC REACTION

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ABSTRACT. We aim to present an algorithm (presumably new) by using Adomian Decomposition Method (ADM) and Variational Iteration Method (VIM) to solve Cubic Isothermal Autocatalytic Chemical System (CIACS). This paper studies the approximate analytical solution of the isothermal chemical reaction $U + 2V \to 3V$ involving two chemical species, a reactant U and an autocatalyst V, whose diffusion coefficients, ε_U and ε_V. In this paper , we have assumed $\varepsilon_U = \varepsilon_V$ for species U and V in region I, and region II for simplicity. The ADM and VIM solutions are compared with numerical solutions evaluated by symbolic computation program Mathematica and very good agreement is obtained. We also show the behaviour of the ADM and VIM solutions.

1. Introduction

Recently, Merkin *et al.* in [1] considered the following reaction-diffusion traveling waves system in region I as follows: for quadratic autocatalytic reaction

$$U + V \to 2V(rate\ \kappa_1 uv), \tag{1}$$

together with a linear decay step

$$V \to W(rate\ \kappa_2 v), \tag{2}$$

for cubic autocatalytic reaction

$$U + 2V \to 3V(rate\ \kappa_3 uv^2) \tag{3}$$

together with a linear decay step

$$V \to W(rate\ \kappa_4 v), \tag{4}$$

where u and v are concentrations of reactant U and auto-catalyst V, $\kappa_i(i = 1, 4)$ are the rate constants and W is some inert product of reaction. On the region II we assume that only the (1) and (3) are taking place for quadratic autocatalytic reaction and cubic autocatalytic reaction respectively. noindent Here, we consider the following system for the dimensionless concentrations (α_1, β_1) and (α_2, β_2) in region I and II of species U and V, respectively with $\zeta > 0$ and $\eta > 0$:

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$$\frac{\partial \alpha_1}{\partial \eta} = \frac{\partial^2 \alpha_1}{\partial \zeta^2} - \alpha_1 \beta_1^2, \tag{5}$$

$$\frac{\partial \beta_1}{\partial \eta} = \frac{\partial^2 \beta_1}{\partial \zeta^2} + \alpha_1 \beta_1^2 - k\beta_1 + \gamma(\beta_2 - \beta_1), \tag{6}$$

$$\frac{\partial \alpha_2}{\partial \eta} = \frac{\partial^2 \alpha_2}{\partial \zeta^2} - \alpha_2 \beta_2^2, \tag{7}$$

$$\frac{\partial \beta_2}{\partial \eta} = \frac{\partial^2 \beta_2}{\partial \zeta^2} + \alpha_2 \beta_2^2 + \gamma(\beta_1 - \beta_2), \tag{8}$$

with the boundary conditions

$$\alpha_i(0,\eta) = \alpha_i(L,\eta) = 1, \ \ \beta_i(0,\eta) = \beta_i(L,\eta) = 0. \tag{9}$$

and the initial conditions

$$\alpha_1(\zeta,0) = 1 - \sum_{n=1}^{\infty} a_1 \sin\left(\frac{\pi n}{2}\right) \cos\left(0.5\mu_n(L-2)\right), \tag{10}$$

$$\beta_1(\zeta,0) = \sum_{n=1}^{\infty} b_1 \sin\left(\frac{\pi n}{2}\right) \cos\left(0.5\mu_n(L-2)\right), \tag{11}$$

$$\alpha_2(\zeta,0) = 1 - \sum_{n=1}^{\infty} a_2 \sin\left(\frac{\pi n}{2}\right) \cos\left(0.5\mu_n(L-2)\right), \tag{12}$$

$$\beta_2(\zeta,0) = \sum_{n=1}^{\infty} b_2 \sin\left(\frac{\pi n}{2}\right) \cos\left(0.5\mu_n(L-2)\right), \tag{13}$$

where $\mu_n = \frac{n\pi}{L}$. The dimensionless constants k and γ represent the strength of the autocatalyst decay and the coupling between the two regions respectively. Also Metcalf *et al.* have studied reaction-diffusion waves in coupled isothermal autocatalytic chemical systems in detail [2]. The cubic reaction relation has been documented in the literature and appeared in various chemical reactions fields [3–16].

Nonlinear differential equations play a major role in the mathematical description of the problems of the real world. It is therefore very important to have an accurate solution to these equations. Since most of these equations do not have an exact solution, numerical and analytical methods are required to study these types of problems. Therefore the motivation for studying this problem is to find the approximate analytical solution and compare it with numerical solution that found it by Mathematica program. Also, we compare two new methods that give us an accurate and effective solution. For more details about the approximate and the numerical methods see [17–31].

Adomian derived a new techniques called ADM for computing the solutions of linear and nonlinear equations [32, 33]. Various authors have studied the convergence of Adomain's method [34–37]. It has recently been proven that it is a very effective method and can be applied successfully to many problems such as systems of ordinary and partial differential equations as well as integral equations [38–46].

The essentials of the VIM and its applicability for several kinds of differential equation are given in [41,47–52]. The comparison between ADM and VIM has been studied in [53, 54]. The aim of this paper is to obtain the approximate analytic solutions of the CIACS by ADM and VIM, and to determine the accuracy of these methods in solving CIACS. We will make some comparisons between these methods through finding the approximate solutions.

The present paper is organized as follows. The second and third sections are devoted the basic idea of the standard ADM and VIM respectively. The fourth and fifth sections are devoted to the applied the ADM and VIM on CIACS respectively. Section six is devoted to the numerical results. Conclusions are presented in section seven.

2. Description of ADM and VIM

In this section, we introduce the basic ideas of the VIM and ADM respectively.

2.1. **Basic Idea of ADM.** In this subsection, we present the basic idea of the ADM [55] by considering the following nonlinear partial differential equation

$$L(\psi(\zeta,\eta)) + R(\psi(\zeta,\eta)) + N(\psi(\zeta,\eta)) = 0, \tag{2.1}$$

$$\psi(\zeta,0) = \phi(\zeta), \tag{2.2}$$

where L is the highest order derivative which is assumed to be invertible, R is the remaining linear operator, N represent a nonlinear operator. Now, applying the inverse operator L^{-1} to both the sides of (2.1), we get

$$\psi(\zeta,\eta) = \phi(\zeta) - L^{-1}(R(\psi(\zeta,\eta)) + N(\psi(\zeta,\eta)), \tag{2.3}$$

Let

$$\psi(\zeta,\eta) = \sum_{m=0}^{\infty} \psi_m(\zeta,\eta), \tag{2.4}$$

and

$$N(\psi) = \sum_{m=0}^{\infty} \chi_m, \tag{2.5}$$

where χ_m are Adomin polynomials which depend upon ψ. In view of Equations (2.4)–(2.5), (2.3) takes the form

$$\sum_{m=0}^{\infty} \psi_m(\zeta,\eta) = \phi() - L^{-1}(R(\psi(\zeta,\eta)) + \sum_{m=0}^{\infty} \chi_m(\psi(\zeta,\eta)). \tag{2.6}$$

We set

$$\psi_0(\zeta,\eta) = \phi(); \tag{2.7}$$

$$\psi_{m+1}(\zeta,\eta) = -L^{-1}(R(\psi(\zeta,\eta)) + \sum_{m=0}^{\infty} \chi_m(\psi(\zeta,\eta)), \quad m = 0,1,\cdots \tag{2.8}$$

where

$$\chi_m(\psi(\zeta,\eta)) = \left[\frac{1}{m!}\frac{d^m}{d\lambda^m} N(\sum_{m=0}^{\infty} \psi_m(\zeta,\eta)\lambda^m)\right]_{\lambda=0}. \tag{2.9}$$

Hence, (2.7)–(2.8) and (2.9) lead to the following recurrence relations

$$\psi_0(\zeta,0) = \phi(), \quad \psi_{m+1}(\zeta,\eta) = -L^{-1}\left(R(\psi(\zeta,\eta)) + A_m(\psi(\zeta,\eta))\right) \tag{2.10}$$

The solution $\psi(\zeta,\eta)$ can be approximated by the truncated series

$$\varphi_k(\zeta,\eta) = \sum_{m=0}^{k-1} \psi_m(\zeta,\eta), \quad lim_{k\to\infty}\varphi_k = \psi(\zeta,\eta).$$

2.2. **Basic Idea of the VIM.** In order to introduce the VIM, let us consider the differential equation

$$D\psi(\zeta,\eta) + L\psi(,\eta) = \phi(\zeta,\eta), \tag{2.1}$$

where D, L, and $\phi(\zeta,\eta)$ are a linear operator, a nonlinear operator, a source term, respectively. According to the VIM, we construct the correction functional in the η-direction as

$$\psi_{n+1}(\zeta,\eta) = \psi_n(\zeta,\eta) + \int_0^\eta \lambda \left(L\psi_n(,t) + N\tilde{\psi}_n(\zeta,t) - \phi(\zeta,t) \right) dt \tag{2.2}$$

where λ is a general Lagrangian multiplier [47–49], which can be determined optimally through the variational theory. The subscript n indicates the nth order approximation. $\tilde{\psi}_n(\zeta,\eta)$ is considered as a restricted variation [47–49], i.e. $\delta\tilde{\psi}_n(\zeta,\eta) = 0$.

3. Derivation of approximate solution of CIACS via ADM

In this subsection, we apply the ADM to evaluate the approximate solutions of (5)–(8). If we operate L_η^{-1} on both sides of (5)–(8), we obtain

$$\alpha_1 = L_\eta^{-1} \left(\frac{\partial^2 \alpha_1}{\partial^2} - \alpha_1 \beta_1^2 \right), \tag{3.3}$$

$$\beta_1 = L_\eta^{-1} \left(\frac{\partial^2 \beta_1}{\partial x^2} + \alpha_1 \beta_1^2 - k\beta_1 + \gamma(\beta_2 - \beta_1) \right), \tag{3.4}$$

$$\alpha_2 = L_\eta^{-1} \left(\frac{\partial^2 \alpha_2}{\partial x^2} - \alpha_2 \beta_2^2 \right), \tag{3.5}$$

$$\beta_2 = L_\eta^{-1} \left(\frac{\partial^2 \beta_2}{\partial x^2} + \alpha_2 \beta_2^2 + \gamma(\beta_1 - \beta_2) \right), \tag{3.6}$$

where

$$L_\eta^{-1} := \int_0^\eta (.) \tag{3.7}$$

Now the ADM solutions and the nonlinear functions $N_1(\alpha_1, \beta_1)$ and $N_2(\alpha_2, \beta_2)$ can be presented as an infinite series

$$\alpha_1(\zeta,\eta) = \alpha_{1,0}(,\eta) + \sum_{m=1}^{\infty} \alpha_{1,m}(\zeta,\eta), \tag{3.8}$$

$$\beta_1(\zeta,\eta) = \beta_{1,0}(,\eta) + \sum_{m=1}^{\infty} \beta_{1,m}(\zeta,\eta), \tag{3.9}$$

$$\alpha_2(\zeta,\eta) = \alpha_{2,0}(,\eta) + \sum_{m=1}^{\infty} \alpha_{2,m}(\zeta,\eta), \tag{3.10}$$

$$\beta_2(\zeta,\eta) = \beta_{2,0}(,\eta) + \sum_{m=1}^{\infty} \beta_{2,m}(\zeta,\eta), \tag{3.11}$$

and

$$N_1(\alpha_1, \beta_1) = \alpha_1 \beta_1^2 = \sum_{m=0}^{\infty} \chi_m, \tag{3.12}$$

$$N_2(\alpha_2,\beta_2)=\alpha_2\beta_2^2=\sum_{m=0}^{\infty}\xi_m, \tag{3.13}$$

where

$$\chi_m=\frac{1}{m!}\left[\frac{d^m}{d\lambda^m}N_1(\alpha_1,\beta_1)\right]_{\lambda=0}, \tag{3.14}$$

$$\xi_m=\frac{1}{m!}\left[\frac{d^m}{d\lambda^m}N_2(\alpha_2,\beta_2)\right]_{\lambda=0}, \tag{3.15}$$

where χ_m are called the Adomian polynomials. $\alpha_{1,m}(\zeta,\eta)$ and $\beta_{1,m}(\zeta,\eta)$ are the components of the solutions $\alpha_1(\zeta,\eta)$ and $\beta_1(\zeta,\eta)$ will be determined by the following recurrence relations

$$\alpha_{1,0}=\alpha_1(\zeta,0),\quad \alpha_{1,m+1}=L_\eta^{-1}\left(\frac{\partial^2\alpha_{1,m}}{\partial^2}-\chi_m\right), \tag{3.16}$$

$$\beta_{1,0}=\beta_1(\zeta,0),\quad \beta_{1,m+1}=L_\eta^{-1}\left(\frac{\partial^2\beta_{1,m}}{\partial^2}-k\beta_{1,m}+\gamma(\beta_{2,m}-\beta_{1,m})-\chi_m\right), \tag{3.17}$$

where ξ_m are called the Adomian polynomials. $\alpha_{2,m}(\zeta,\eta)$ and $\beta_{2,m}(\zeta,\eta)$ are the components of the solutions $\alpha_2(\zeta,\eta)$ and $\beta_2(\zeta,\eta)$ will be determined by the following recurrence relations

$$\alpha_{2,0}=\alpha_2(\zeta,0),\quad \alpha_{2,m+1}=L_\eta^{-1}\left(\frac{\partial^2\alpha_{2,m}}{\partial^2}-\xi_m\right), \tag{3.18}$$

$$\beta_{2,0}=\beta_2(\zeta,0),\quad \beta_{2,m+1}=L_\eta^{-1}\left(\frac{\partial^2\beta_{2,m}}{\partial^2}+\gamma(\beta_{1,m}-\beta_{2,m})-\xi_m\right), \tag{3.19}$$

In view of (2.9) and using Mathematica software, we evaluate the Adomian polynomials χ_n and ξ_n as follows:

$$\begin{aligned}
\chi_0&=\alpha_{1,0}\beta_{1,0}^2,\\
\chi_1&=\alpha_{1,1}\beta_{1,0}^2+2\alpha_{1,0}\beta_{1,0}\beta_{1,1},\\
\chi_2&=\alpha_{1,2}\beta_{1,0}^2+2\alpha_{1,1}\beta_{1,0}\beta_{1,1}+\frac{1}{2}\alpha_{1,0}(2\beta_{1,1}^2+4\beta_{1,0}\beta_{1,2},
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
\xi_0&=\alpha_{2,0}\beta_{2,0}^2,\\
\xi_1&=\alpha_{2,1}\beta_{2,0}^2+2\alpha_{2,0}\beta_{2,0}\beta_{2,1},\\
\xi_2&=\alpha_{2,2}\beta_{2,0}^2+2\alpha_{2,1}\beta_{2,0}\beta_{2,1}+\frac{1}{2}\alpha_{2,0}(2\beta_{2,1}^2+4\beta_{2,0}\beta_{2,2}.
\end{aligned} \tag{3.21}$$

In the first iteration we have

$$\alpha_{1,1}=L_\eta^{-1}\left(\frac{\partial^2\alpha_{1,0}}{\partial^2}-\chi_0\right), \tag{3.22}$$

$$\beta_{1,1}=L_\eta^{-1}\left(\frac{\partial^2\beta_{1,0}}{\partial^2}-k\beta_{1,0}+\gamma(\beta_{2,m}-\beta_{1,0})+\chi_0\right), \tag{3.23}$$

$$\alpha_{2,1}=L_\eta^{-1}\left(\frac{\partial^2\alpha_{2,0}}{\partial^2}-\xi_0\right), \tag{3.24}$$

$$\beta_{2,1}=L_\eta^{-1}\left(\frac{\partial^2\beta_{2,0}}{\partial^2}+\gamma(\beta_{1,0}-\beta_{2,0})-\xi_0\right). \tag{3.25}$$

The components $\alpha_{1,2}, ..., \beta_{1,2}, ..., \alpha_{2,2}, ..., \beta_{2,2}, ...$ were also determined and will be used, but for brevity are not listed. The general form of the approximations $\alpha_1, \beta_1, \alpha_2, \beta_2$are given by (3.8)–(3.15), i. e.

$$\alpha_1 = \alpha_{1,0} + \alpha_{1,1} + \alpha_{1,2} + \cdots, \tag{3.26}$$

$$\beta_1 = \beta_{1,0} + \beta_{1,1} + \beta_{1,2} + \cdots, \tag{3.27}$$

$$\alpha_2 = \alpha_{2,0} + \alpha_{2,1} + \alpha_{2,2} + \cdots, \tag{3.28}$$

$$\beta_2 = \beta_{2,0} + \beta_{2,1} + \beta_{2,2} + \cdots, \tag{3.29}$$

4. Derivation of approximate solution of CIACS via VIM

In this section, we apply the VIM to evaluate the approximate solutions of (5)–(8). We can approximate the correction formula of (5)–(8) as follows:

$$\begin{aligned} \alpha_{1,n+1}(\zeta,\eta) &= \alpha_{1,n}(\zeta,\eta) + \int_0^\eta \mu_1(t)\left(\frac{\partial}{\partial t}\alpha_{1,n}(\zeta,t)\right. \\ &\quad \left. -\frac{\partial^2}{\partial^2}\tilde{\alpha}_{1,n}(\zeta,t) + \tilde{\alpha}_{1,n}(\zeta,t)\tilde{\beta}^2_{1,n}(\zeta,t)\right)dt, \end{aligned} \tag{4.30}$$

$$\begin{aligned} \beta_{1,n+1}(\zeta,\eta) &= \beta_{1,n}(\zeta,\eta) + \int_0^\eta \mu_2(t)\left(\frac{\partial}{\partial t}\beta_{1,n}(\zeta,t)\right. \\ &\quad -\frac{\partial^2}{\partial^2}\tilde{\beta}_{1,n}(\zeta,t) - \tilde{\alpha}_{1,n}(\zeta,t)\tilde{\beta}^2_{1,n}(\zeta,t)dt \\ &\quad \left. +k\tilde{\beta}_{1,n}(\zeta,t) + \gamma(\tilde{\beta}_{1,n}(\zeta,t) - \tilde{\beta}_{2,n}(\zeta,t))\right)dt, \end{aligned} \tag{4.31}$$

$$\begin{aligned} \alpha_{2,n+1}(\zeta,\eta) &= \alpha_{2,n}(\zeta,\eta) + \int_0^\eta \mu_3(t)\left(\frac{\partial}{\partial t}\alpha_{2,n}(\zeta,t)\right. \\ &\quad \left. -\frac{\partial^2}{\partial^2}\tilde{\alpha}_{2,n}(\zeta,t) + \tilde{\alpha}_{2,n}(\zeta,t)\tilde{\beta}^2_{2,n}(\zeta,t)\right)dt, \end{aligned} \tag{4.32}$$

$$\begin{aligned} \beta_{2,n+1}(\zeta,\eta) &= \beta_{2,n}(\zeta,\eta) + \int_0^\eta \mu_4(t)\left(\frac{\partial}{\partial t}\beta_{2,n}(\zeta,t) - \frac{\partial^2}{\partial^2}\tilde{\beta}_{2,n}(\zeta,t)\right. \\ &\quad \left. -\tilde{\alpha}_{2,n}(\zeta,t)\tilde{\beta}^2_{2,n}(\zeta,t) + \gamma(\tilde{\beta}_{2,n}(\zeta,t) - \tilde{\beta}_{1,n}(\zeta,t))\right)dt. \end{aligned} \tag{4.33}$$

where $\tilde{\alpha}_{1,n}(\zeta,\eta), \tilde{\beta}_{1,n}(\zeta,\eta), \tilde{\alpha}_{2,n}(\zeta,\eta), \tilde{\beta}_{2,n}(\zeta,\varsigma)$, denote the restrictive variation, that is,

$$\delta\tilde{\alpha}_{1,n}(\zeta,\eta) = 0,\ \delta\tilde{\beta}_{1,n}(\zeta,\eta) = 0,\ \delta\tilde{\alpha}_{2,n}(\zeta,\eta) = 0,\ \delta\tilde{\beta}_{2,n}(\zeta,\eta) = 0.$$

Thus, we have

$$\begin{aligned} \delta\alpha_{1,n+1}(\zeta,\eta) &= \delta\alpha_{1,n}(\zeta,\eta) + \int_0^\eta \delta\mu_1(t)\left(\frac{\partial}{\partial t}\alpha_{1,n}(\zeta,t)\right. \\ &\quad \left. -\frac{\partial^2}{\partial^2}\tilde{\alpha}_{1,n}(\zeta,t) + \tilde{\alpha}_{1,n}(\zeta,t)\tilde{\beta}^2_{1,n}(\zeta,t)\right)dt \\ &= \delta\alpha_{1,n}(\zeta,\eta + \int_0^\eta \delta\mu_1(t)\left(\frac{\partial}{\partial t}\alpha_{1,n}(\zeta,t)\right)dt, \end{aligned} \tag{4.34}$$

$$\begin{aligned}\delta\beta_{1,n+1}(\zeta,\eta) &= \delta\beta_{1,n}(\zeta,\eta)+\int_0^\eta \delta\mu_2(t)\left(\frac{\partial}{\partial\eta}\beta_{1,n}(\zeta,t)\right.\\ &\quad -\frac{\partial^2}{\partial^2}\tilde{\beta}_{1,n}(,t)-\tilde{\alpha}_{1,n}(,t)\tilde{\beta}^2_{1,n}(,t)d\eta\\ &\quad \left.+k\tilde{\beta}_{1,n}(,t)+\gamma(\tilde{\beta}_{1,n}(,t)-\tilde{\beta}_{2,n}(,t))\right)dt\\ &= \delta\beta_{1,n}(,\eta)+\int_0^\eta \delta\mu_2(t)\left(\frac{\partial}{\partial t}\beta_{1,n}(,t)\right)dt, \end{aligned}\tag{4.35}$$

$$\begin{aligned}\delta\alpha_{2,n+1}(\zeta,\eta) &= \delta\alpha_{2,n}(\zeta,\eta)+\int_0^\eta \delta\mu_3(t)\left(\frac{\partial}{\partial t}\alpha_{2,n}(\zeta,t)\right.\\ &\quad \left.-\frac{\partial^2}{\partial^2}\tilde{\alpha}_{2,n}(\zeta,t)+\tilde{\alpha}_{2,n}(\zeta,t)\tilde{\beta}^2_{2,n}(\zeta,t)\right)dt\\ &= \delta\alpha_{2,n}(\zeta,\eta)+\int_0^\eta \delta\mu_3(t)\frac{\partial}{\partial t}\alpha_{2,n}(\zeta,t)dt \end{aligned}\tag{4.36}$$

$$\begin{aligned}\delta\beta_{2,n+1}(\zeta,\eta) &= \delta\beta_{2,n}(\zeta,\eta)+\int_0^\eta \delta\mu_4(t)\left(\frac{\partial}{\partial t}\beta_{2,n}(\zeta,t)-\frac{\partial^2}{\partial^2}\tilde{\beta}_{2,n}(,t)\right.\\ &\quad \left.-\tilde{\alpha}_{2,n}(\zeta,t)\tilde{\beta}^2_{2,n}(\zeta,t)+\gamma(\tilde{\beta}_{2,n}(\zeta,t)-\tilde{\beta}_{1,n}(\zeta,t))\right)dt\\ &= \delta\beta_{2,n}(\zeta,\eta)+\int_0^\eta \delta\mu_4(t)\frac{\partial}{\partial t}\beta_{2,n}(\zeta,t)dt, \end{aligned}\tag{4.37}$$

where $\tilde{\alpha}_{1,n},\tilde{\beta}_{1,n},\tilde{\alpha}_{2,n}$ and $\tilde{\beta}_{2,n}$ are considered as restricted variations, i.e. $\delta\tilde{\alpha}_{1,n}=0,\delta\tilde{\beta}_{1,n}=0,\delta\tilde{\alpha}_{2,n}=0$ and $\delta\tilde{\beta}_{2,n}=0$. We have

$$\delta\alpha_{1,n}(\zeta,\eta)+\int_0^\eta \mu_1(t)\left(\frac{\partial}{\partial t}\delta\alpha_{1,n}(\zeta,t)\right)dt=0,\tag{4.38}$$

$$\delta\beta_{1,n}(\zeta,\eta)+\int_0^\eta \mu_2(t)\left(\frac{\partial}{\partial t}\delta\beta_{1,n}(\zeta,t)\right)dt=0,\tag{4.39}$$

$$\delta\alpha_{2,n}(\zeta,\eta)+\int_0^\eta \mu_1(t)\left(\frac{\partial}{\partial t}\delta\alpha_{2,n}(\zeta,t)\right)dt=0,\tag{4.40}$$

$$\delta\beta_{2,n}(\zeta,\eta)+\int_0^\eta \mu_2(t)\left(\frac{\partial}{\partial t}\delta\beta_{2,n}(\zeta,t)\right)dt=0.\tag{4.41}$$

By integrating by parts we obtain obtain the stationary conditions as follows:

$$\mu_i^{'}(t)=0,\quad 1+\mu_i(t)|_{t=\eta}=0.\tag{4.42}$$

Now, it can be determined the Lagrange multiplier $\mu_1(t)=\mu_2(t)=\mu_3(t)=\mu_4(t)=-1$. As a consequence, we obtain the following iterations formula:

$$\begin{aligned}\alpha_{1,n+1}(\zeta,\eta) &= \alpha_{1,n}(\zeta,\eta)-\int_0^\eta\left(\frac{\partial}{\partial t}\alpha_{1,n}(\zeta,t)\right.\\ &\quad \left.-\frac{\partial^2}{\partial^2}\alpha_{1,n}(\zeta,t)+\alpha_{1,n}(\zeta,t)\beta^2_{1,n}(\zeta,t)\right)dt, \end{aligned}\tag{4.43}$$

$$\begin{aligned}\beta_{1,n+1}(\zeta,\eta) &= \beta_{1,n}(\zeta,\eta)-\int_0^\eta\Big(\frac{\partial}{\partial t}\beta_{1,n}(\zeta,t)\\ &\quad -\frac{\partial^2}{\partial^2}\beta_{1,n}(\zeta,t)-\alpha_{1,n}(\zeta,t)\beta_{1,n}^2(\zeta,t)\\ &\quad +k\beta_{1,n}(\zeta,t)+\gamma(\beta_{1,n}(\zeta,t)-\beta_{2,n}(\zeta,t))\big)\,dt, \qquad (4.44)\end{aligned}$$

$$\begin{aligned}\alpha_{2,n+1}(,\eta) &= \alpha_{2,n}(\eta)-\int_0^\eta\Big(\frac{\partial}{\partial t}\alpha_{2,n}(\zeta,t)\\ &\quad -\frac{\partial^2}{\partial^2}\alpha_{2,n}(\zeta,t)+\alpha_{2,n}(\zeta,t)\beta_{2,n}^2(\zeta,\eta)\Big)\,dt, \qquad (4.45)\end{aligned}$$

$$\begin{aligned}\beta_{2,n+1}(\zeta,\eta) &= \beta_{2,n}(\zeta,\eta)-\int_0^\eta\Big(\frac{\partial}{\partial t}\beta_{2,n}(\zeta,t)-\frac{\partial^2}{\partial^2}\beta_{2,n}(\zeta,t)\\ &\quad -\alpha_{2,n}(\zeta,t)\beta_{2,n}^2(\zeta,t)+\gamma(\beta_{2,n}(\zeta,t)-\beta_{1,n}(\zeta,t))\big)\,dt. \qquad (4.46)\end{aligned}$$

5. Numerical Results

In this section, we apply ADM and VIM to evaluate the approximate solutions of (5)–(8). First applying the recurrence relations (3.16)–(3.19) and the initial conditions (10)–(13), we obtain the following ADM successive approximations

$$\alpha_{1,0}(\zeta,\eta)=\alpha_1(,0), \qquad (5.1)$$

$$\begin{aligned}\alpha_{1,1}(\zeta,\eta) &= \eta\sum_{n=1}^{\infty}a_1\mu_n^2\cos\Big[(L-2)\frac{\mu_n}{2}\Big]\sin(\frac{n\pi}{2})\\ &\quad -\eta\sum_{n=1}^{\infty}a_1\cos\Big[(L-2)\frac{\mu_n}{2}\Big]\sin(\frac{n\pi}{2})\left(\sum_{m=1}^{\infty}b_1\cos\Big[(L-2x)\frac{\mu_m}{2}\Big]\sin(\frac{m\pi}{2})\right)^2, \qquad (5.2)\end{aligned}$$

$$\beta_{1,0}(\zeta,\eta)=\beta_1(,0), \qquad (5.3)$$

$$\begin{aligned}\beta_{1,1}(\zeta,\eta) &= -\eta\sum_{n=1}^{\infty}b_1\mu_n^2\cos\Big[(L-2)\frac{\mu_n}{2}\Big]\sin(\frac{n\pi}{2})-k\sum_{n=1}^{\infty}b_1\cos\Big[(L-2)\frac{\mu_n}{2}\Big]\sin(\frac{n\pi}{2})\\ &\quad +\eta\sum_{n=1}^{\infty}a_1\cos\Big[(L-2)\frac{\mu_n}{2}\Big]\sin(\frac{n\pi}{2})\left(\sum_{m=1}^{\infty}b_1\cos\Big[(L-2)\frac{\mu_m}{2}\Big]\sin(\frac{m\pi}{2})\right)^2. \qquad (5.4)\\ &\quad +\gamma\sum_{n=1}^{\infty}(b_2-b_1)\cos\Big[(L-2)\frac{\mu_n}{2}\Big]\sin(\frac{n\pi}{2}),\end{aligned}$$

$$\alpha_{2,0}(,\eta)=\alpha_2(,0), \qquad (5.5)$$

$$\begin{aligned}\alpha_{2,1}(\zeta,\eta) &= \eta\sum_{n=1}^{\infty}a_2\mu_n^2\cos\Big[(L-2x)\frac{\mu_n}{2}\Big]\sin(\frac{n\pi}{2})\\ &\quad -\eta\sum_{n=1}^{\infty}a_2\cos\Big[(L-2)\frac{\mu_n}{2}\Big]\sin(\frac{n\pi}{2})\left(\sum_{m=1}^{\infty}b_2\cos\Big[(L-2)\frac{\mu_m}{2}\Big]\sin(\frac{m\pi}{2})\right)^2, \qquad (5.6)\end{aligned}$$

$$\beta_{2,0}(,\eta)=\beta_2(,0), \qquad (5.7)$$

$$\beta_{2,1}(\zeta,\eta) = -\eta\sum_{n=1}^{\infty} b_2\mu_n^2 \cos\left[(L-2)\frac{\mu_n}{2}\right]\sin(\frac{n\pi}{2})$$
$$+\,\eta\sum_{n=1}^{\infty} a_2 \cos\left[(L-2x)\frac{\mu_n}{2}\right]\sin(\frac{n\pi}{2})\left(\sum_{m=1}^{\infty} b_2\cos\left[(L-2)\frac{\mu_m}{2}\right]\sin(\frac{m\pi}{2})\right)^2 \quad (5.8)$$
$$+\,\gamma\sum_{n=1}^{\infty}(b_1-b_2)\cos\left[(L-2)\frac{\mu_n}{2}\right]\sin(\frac{n\pi}{2}).$$

Now, we apply the VIM to solve (5)–(8). By taking the same initial values as for ADM we obtain the successive approximations as follow:

$$\begin{aligned}\alpha_{1,1}(\zeta,\eta) &= \alpha_{1,0}(,\eta) - \int_0^{\eta}\left(\frac{\partial}{\partial t}\alpha_{1,0}(\zeta,t)\right.\\ &\quad \left. -\frac{\partial^2}{\partial^2}\alpha_{1,0}(\zeta,t) + \alpha_{1,0}(\zeta,t)\beta_{1,0}^2(\zeta,t)\right)dt\\ &= \alpha_{1,0}(\zeta,\eta) + \eta\frac{\partial^2}{\partial^2}\alpha_{1,0}(\zeta,\eta) - \alpha_{1,0}(\zeta,\eta)\beta_{1,0}^2(\zeta,\eta) \end{aligned} \quad (5.9)$$

$$\begin{aligned}\beta_{1,1}(\zeta,\eta) &= \beta_{1,0}(\zeta,\eta) - \int_0^{\eta}\left(\frac{\partial}{\partial t}\beta_{1,0}(\zeta,t)\right.\\ &\quad -\frac{\partial^2}{\partial x^2}\beta_{1,0}(\zeta,t) - \alpha_{1,0}(\zeta,t)\beta_{1,0}^2(\zeta,t)\\ &\quad +k\beta_{1,0}(\zeta,t) + \gamma(\beta_{1,0}(\zeta,t) - \beta_{2,0}(\zeta,t)))\,dt\\ &= \beta_{1,0}(\zeta,\eta) + \eta\;\frac{\partial^2}{\partial^2}\beta_{1,0}(\zeta,\eta) + \eta\;\alpha_{1,0}(\zeta,\eta)\beta_{1,0}^2(\zeta,\eta)\\ &\quad -k\;\eta\beta_{1,0}(\zeta,\eta) + \eta\;\gamma(\beta_{2,0}(\zeta,\eta) - \beta_{1,0}(\zeta,\eta)))\,, \end{aligned} \quad (5.10)$$

$$\begin{aligned}\alpha_{2,1}(\zeta,\eta) &= \alpha_{2,0}(\zeta,\eta) - \int_0^{\eta}\left(\frac{\partial}{\partial t}\alpha_{2,0}(\zeta,t)\right.\\ &\quad \left. -\frac{\partial^2}{\partial^2}\alpha_{2,0}(\zeta,t) + \alpha_{2,0}(\zeta,t)\beta_{2,0}^2(\zeta,t)\right)d\eta\\ &= \alpha_{2,0}(,\eta) + \eta\;\frac{\partial^2}{\partial^2}\alpha_{2,0}(\zeta,\eta) - \eta\;\alpha_{2,0}(\zeta,\eta)\beta_{2,0}^2(\zeta,\eta), \end{aligned} \quad (5.11)$$

$$\begin{aligned}\beta_{2,1}(\zeta,\eta) &= \beta_{2,0}(\zeta,\eta) - \int_0^{\eta}\left(\frac{\partial}{\partial t}\beta_{2,0}(\zeta,t) - \frac{\partial^2}{\partial^2}\beta_{2,0}(\zeta,t)\right.\\ &\quad \left. -\alpha_{2,0}(\zeta,t)\beta_{2,0}^2(\zeta,t) + \gamma(\beta_{2,0}(\zeta,t) - \beta_{1,0}(\zeta,t))\right)d\eta\\ &= \beta_{2,0}(\zeta,\eta) + \eta\;\frac{\partial^2}{\partial^2}\beta_{2,0}(\zeta,\eta) + \eta\;\alpha_{2,0}(\zeta,\eta)\beta_{2,0}^2(\zeta,\eta)\\ &\quad +\eta\;\gamma(\beta_{1,0}(\zeta,\eta) - \beta_{2,0}(\zeta,\eta)). \end{aligned} \quad (5.12)$$

After substituting the initial values for $\alpha_{1,0}(\zeta,\eta), \beta_{1,0}(\zeta,\eta), \alpha_{2,0}(\zeta,\eta)$ and $\beta_{2,0}(\zeta,\eta)$ into (5.9)–(5.12), we obtain the first approximation of the VIM which are the same as the two terms of the ADM for (5)–(8). A comparison between the numerical, the ADM and the VIM solutions are demonstrated in Figures 1-3 for $\gamma = 0.1, k = 0.01, a_1 = 0.1, a_2 = 0.2, b_1 = 001, b_2 = 0.002$. Figures 1-3 show the comparison of the three terms of the ADM solutions and the second approximation of the VIM with the numerical solutions using the command NDsolve of MATHEMATICA 9 respectively. It can be seen from Figures 2-3 that the absolute errors obtained by ADM and VIM are close to each other. The two-terms ADM and the first approximation by VIM are identical. So the errors for them are of the same order. In order to get small error, more terms need to be considered for ADM and high approximation for VIM solutions. Also we show the absolute error of the three terms of the ADM solutions and the second approximation of the VIM in Tables 1-2. We note when we compare our results via ADM and VIM methods then found their results show that errors are little less in VIM method. But both methods are very efficient and accurate that can be used to provide approximate analytical solutions of partial differential equations. Figures 4-5 show the behaviour of 3-terms ADM solutions and the second approximation of VIM for (5)–(8) with the same caption of Figures 1-2.

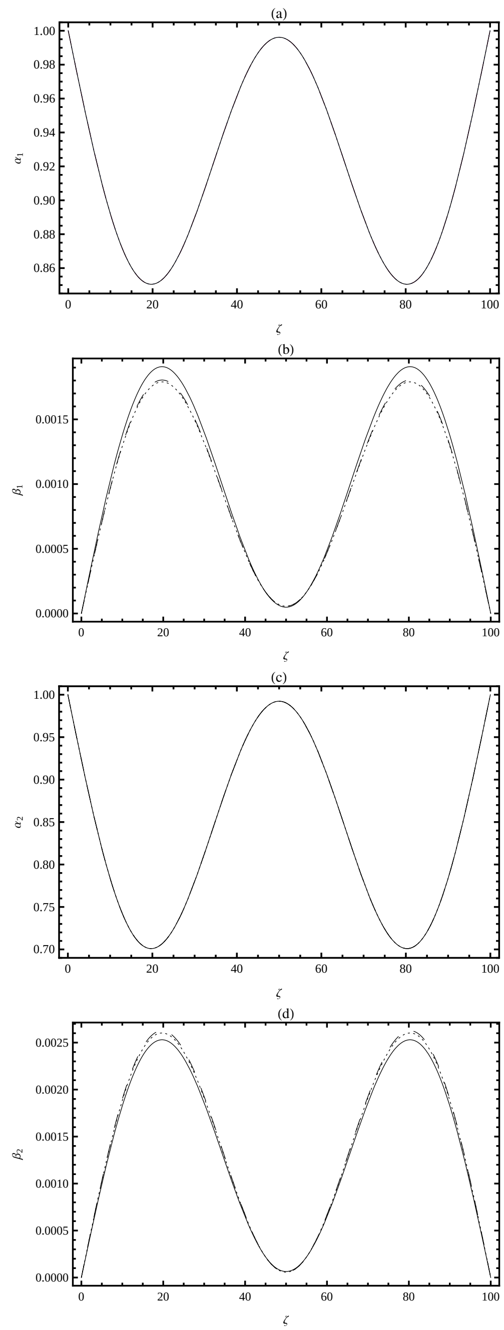

FIGURE 1. The comparison between the 3-terms ADM solutions , the second approximation by VIM and numerical method using Mathematica for (5)–(8) with $k = 0.01, \gamma = 0.2, a_1 = 0.1, a_2 = 0.2, b_1 = 0.001, b_2 = 0.002$. Dashing-tiny for VIM; dashing-large for ADM ; solid line for numerical using Mathematica

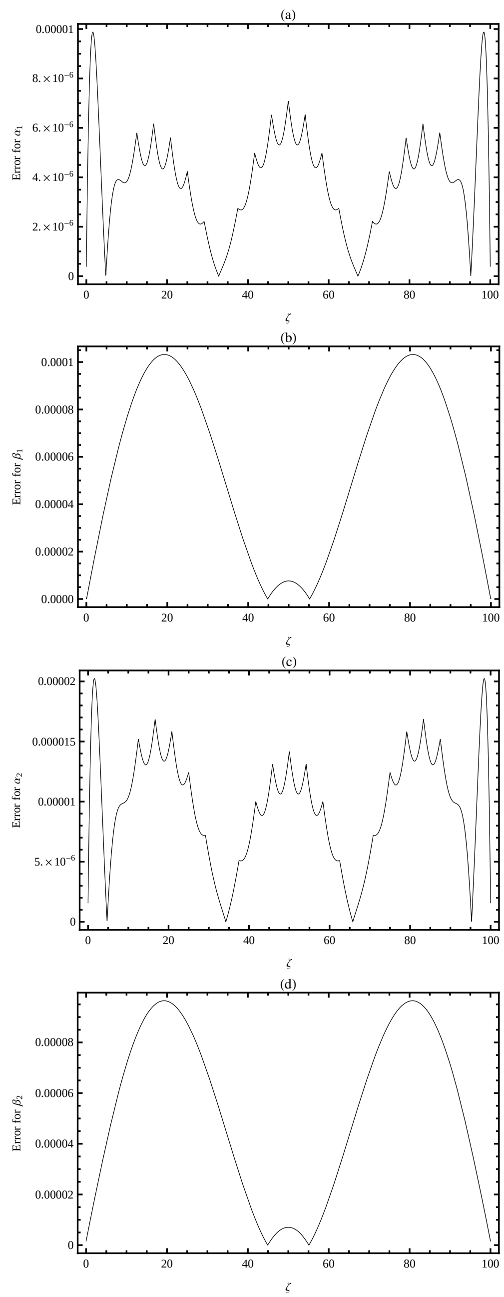

FIGURE 2. The absolute error between the 3-terms ADM solutions and numerical method using Mathematica for (5)–(8) with $k = 0.01, \gamma = 0.2, a_1 = 0.1, a_2 = 0.2, b_1 = 0.001, b_2 = 0.002$.

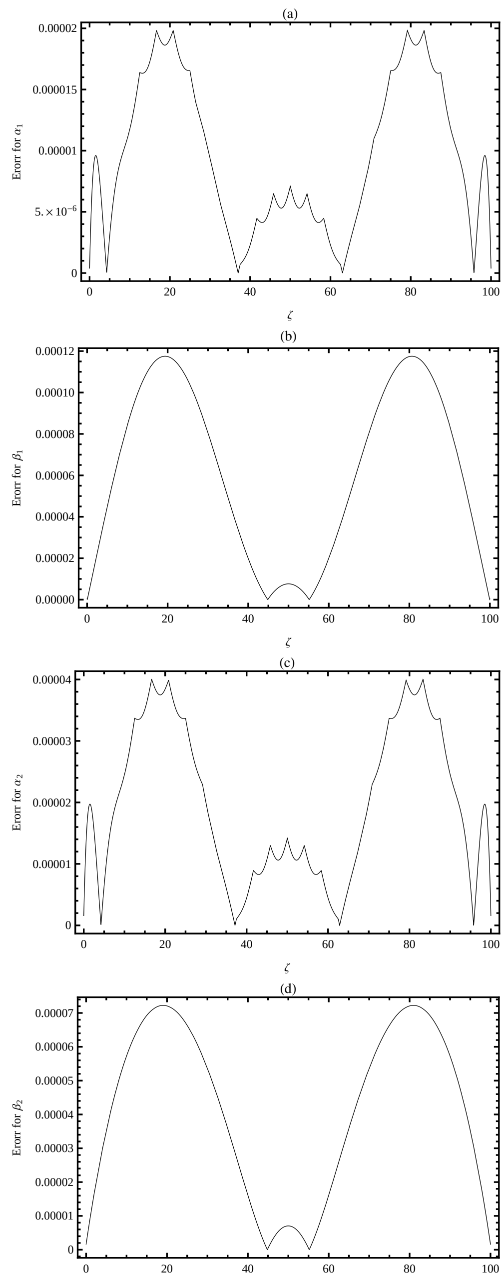

FIGURE 3. The absolute error between the second approximation by VIM and the numerical method using Mathematica for (5)–(8) with $k = 0.01, \gamma = 0.2, a_1 = 0.1, a_2 = 0.2, b_1 = 0.001, b_2 = 0.002$.

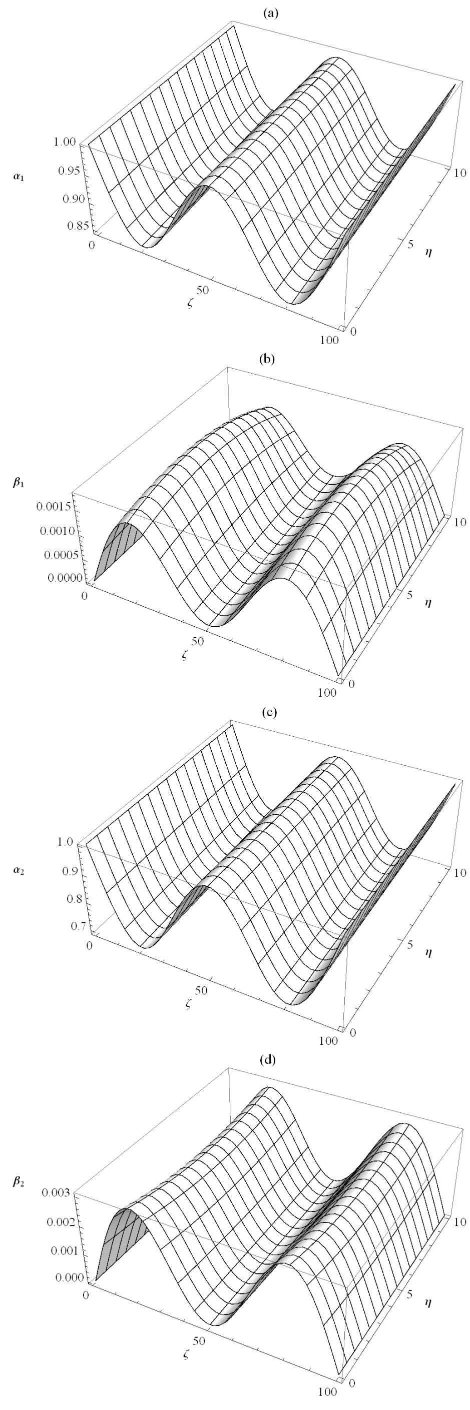

FIGURE 4. The 3-terms ADM solutions for (5)–(8) with $k = 0.01, \gamma = 0.2, a_1 = 0.1, a_2 = 0.2, b_1 = 0.001, b_2 = 0.002$.

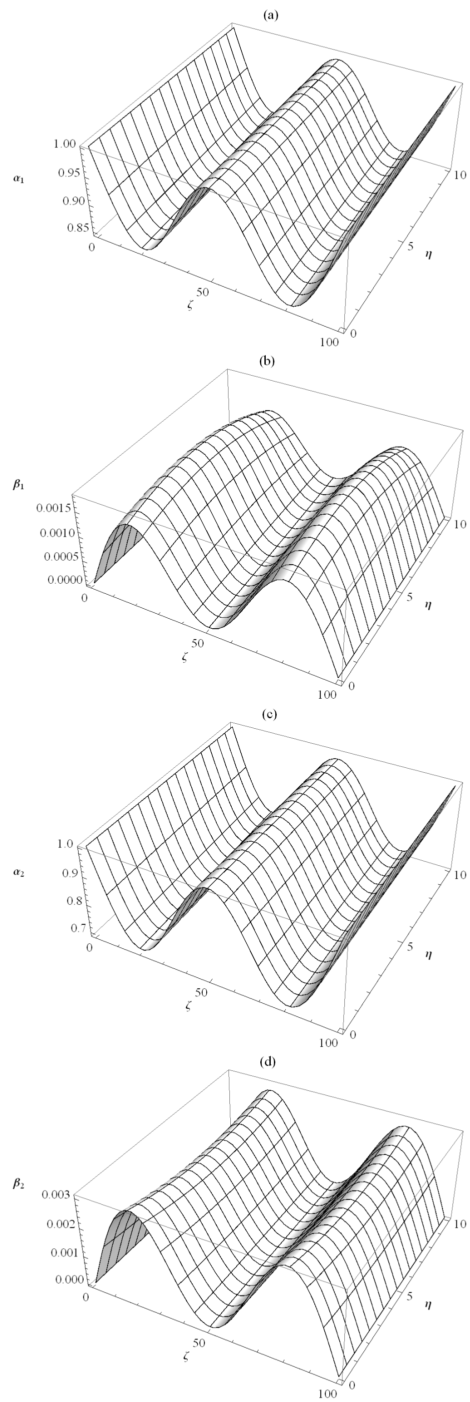

FIGURE 5. The second approximation by VIM for (5)–(8) with $k = 0.01, \gamma = 0.2, a_1 = 0.1, a_2 = 0.2, b_1 = 0.001, b_2 = 0.002$.

ζ	Error for α_1	Error for β_1	Error for α_2	Error for β_2
0	1.57914×10^{-6}	1.57914×10^{-6}	6.31655×10^{-6}	6.31655×10^{-6}
10	8.36053×10^{-6}	5.12974×10^{-4}	1.6844×10^{-6}	4.82716×10^{-4}
20	1.09907×10^{-5}	6.93902×10^{-4}	9.85207×10^{-6}	6.51928×10^{-4}
30	4.94389×10^{-6}	4.90958×10^{-4}	7.29188×10^{-6}	4.62038×10^{-4}
40	2.08871×10^{-6}	1.32108×10^{-4}	4.90642×10^{-6}	1.25168×10^{-4}
50	2.54392×10^{-6}	444787×10^{-5}	5.08785×10^{-6}	4.34379×10^{-5}
60	2.08871×10^{-6}	1.32108×10^{-4}	4.90642×10^{-6}	1.25168×10^{-4}
70	4.94389×10^{-6}	490958×10^{-4}	7.29188×10^{-6}	4.62038×10^{-4}
80	1.09907×10^{-5}	6.93902×10^{-4}	9.85207×10^{-6}	6.51928×10^{-4}
90	8.36053×10^{-6}	5.12974×10^{-4}	1.6844×10^{-6}	4.82716×10^{-4}
100	1.57914×10^{-6}	1.57914×10^{-6}	6.31655×10^{-6}	6.31655×10^{-6}

TABLE 1. The absolute error of 3-terms of ADM solutions (3.26)–(3.29) for $k = 0.01, \gamma = 0.2, a_1 = 0.1, a_2 = 0.2, b_1 = 0.001, b_2 = 0.002, L = 100, \eta = 10$.

ζ	Error for α_1	Error for β_1	Error for α_2	Error for β_2
0	1.57914×10^{-6}	1.57914×10^{-6}	6.31655×10^{-6}	6.31655×10^{-6}
10	1.21472×10^{-5}	5.33482×10^{-4}	1.95407×10^{-5}	4.6486×10^{-4}
20	$2.62704 times 10^{-5}$	7.31163×10^{-4}	410388×10^{-5}	6.20742×10^{-4}
30	159312×10^{-5}	5.11833×10^{-4}	263215×10^{-5}	4.43008×10^{-4}
40	4.65652×10^{-6}	1.34675×10^{-4}	7.94141×10^{-6}	1.22133×10^{-4}
50	2.54392×10^{-6}	4.44787×10^{-5}	5.08785×10^{-6}	4.34379×10^{-5}
60	4.65652×10^{-6}	1.34675×10^{-4}	7.94141×10^{-6}	1.22133×10^{-4}
70	1.59312×10^{-5}	5.11833×10^{-4}	2.63215×10^{-5}	4.43008×10^{-4}
80	262704×10^{-5}	7.31163×10^{-4}	4.10388×10^{-5}	6.20742×10^{-4}
90	121472×10^{-5}	5.33482×10^{-4}	1.95407×10^{-5}	4.6486×10^{-4}
100	1.57914×10^{-6}	1.57914×10^{-6}	6.31655×10^{-6}	6.31655×10^{-6}

TABLE 2. The absolute error of second VIM solutions (4.43)–(4.46) for $k = 0.01, \gamma = 0.2, a_1 = 0.1, a_2 = 0.2, b_1 = 0.001, b_2 = 0.002, L = 100, \eta = 10$.

6. Conclusion

In this paper, two powerful techniques, namely ADM and VIM, have been efficiently applied to obtain the approximate solutions for cubic isothermal autocatalytic chemical system (CIACS). Unlike many other methods, ADM and VIM are very simple, as it does not need any discretization. Our results also show that VIM is superior to ADM in solving CIACS. In fact, two terms of ADM solutions and the first approximation by VIM are identical. We also sketch some figures which prove that ADM and VIM solutions are very close to each other. This fact is also clear in those figures which compare new solutions with numerical results obtained by Mathematica. Besides the results demonstrate that ADM and VIM are accurate for solving CIACS, by increasing the number of iterations one can reach any desired accuracy. Finally, this work confirms that the VIM and ADM are powerful and efficient methods and also we note that after a few iterations, a symbolic program is necessary for successive calculations. We have made use of MATHEMATICA 9 to overcome the complicated calculations.

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