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AN EXTENSIVE STUDY ON SUM AND PRODUCT THEOREMS OF RELATIVE (p,q)-TH ORDER AND RELATIVE (p,q)-TH TYPE OF MEROMORPHIC FUNCTIONS WITH RESPECT TO ENTIRE FUNCTIONS

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Abstract. Orders and types of entire and meromorphic functions have been actively investigated by many authors. In this paper, we aim at investigating some basic properties in connection with sum and product of relative (p,q)-th order, relative (p,q)-th type, and relative (p,q)-th weak type of meromorphic functions with respect to entire functions where p and q are any two positive integers.

1. **Introduction, Definitions and Notations**

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [6, 10, 12, 13] and therefore we do not explain those in details. Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to f (see [14]) is defined on $|z| = r$ as $M_f(r) = \max|z| = r\,|f(z)|$. A non-constant entire function f is said to have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds (see [1]). When f is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f (see [6, p. 4]), playing the same role as $M_f(r)$ .We also recall the following definitions due to Juneja, Kapoor and Bajpai [7]. For any two positive integers p and q with $p \geq q$, Juneja et al. [7] defined the (p,q)-th order (resp. (p,q)-th lower order) of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \quad \left(\text{resp. } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}\right),$$

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where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, \cdots$; $\log^{[0]} x = x$ and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ for $k = 1, 2, 3, \cdots$; $\exp^{[0]} x = x$. When f is meromorphic one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \quad \left(\text{resp. } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}\right),$$

where p, q are any two positive integers with $p \geq q$. If $p = l$ and $q = 1$ then we write $\rho_f(l,1) = \rho_f^{[l]}$ and $\lambda_f(l,1) = \lambda_f^{[l]}$ where $\rho_f^{[l]}$ and $\lambda_f^{[l]}$ are respectively known as generalized order and generalized lower order of f. For details about generalized order one may see [11]. Also for $p = 2$ and $q = 1$ we respectively denote $\rho_f(2,1)$ and $\lambda_f(2,1)$ by ρ_f and λ_f. which are classical growth indicators such as order and lower order of meromorphic function f.

In this connection we recall the following definition (see [6]):

Definition 1. An entire function f is said to have index-pair (p,q), $p \geq q \geq 1$ if $b < \rho_f(p,q) < \infty$ and $\rho_f(p-1, q-1)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ if $p > q$. Moreover if $0 < \rho_f(p,q) < \infty$, then

$$\rho_f(p-n,q) = \infty \text{ for } n < p, \ \rho_f(p, q-n) = 0 \text{ for } n < q \text{ and}$$

$$\rho_f(p+n, q+n) = 1 \text{ for } n = 1, 2, ... \,.$$

Similarly for $0 < \lambda_f(p,q) < \infty$, one can easily verify that

$$\lambda_f(p-n,q) = \infty \text{ for } n < p, \ \lambda_f(p, q-n) = 0 \text{ for } n < q \text{ and}$$

$$\lambda_f(p+n, q+n) = 1 \ forn = 1, 2, ... \,.$$

An entire function for which (p,q)-th order and (p,q)-th lower order are same is said to be of regular (p,q) growth. Functions which are not of regular (p,q) growth are said to be of irregular (p,q) growth.

Analogously one can easily verify that Definition 1 of index-pair can also be applicable to a meromorphic function f.

Given a non-constant entire function f defined in the open complex plane $\mathbb{C}$ its Nevanlinna's characteristic function is strictly increasing and continuous. Hence there exists its inverse functions $T_f^{-1}(r) : (|f(0)|, \infty) \to (0, \infty)$ with $\lim_{s\to\infty} T_f^{-1}(s) = \infty$.

Order of a meromorphic function f which is generally used in computational purpose is defined in terms of the growth of f respect to the exponential function as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)} \ .$$

Lahiri and Banerjee [9] introduced the relative order of a meromorphic function with respect to an entire function to avoid comparing growth just with $\exp z$ in the following definition (see [9]).

Definition 2. Let f be any meromorphic function and g be any entire function. The relative order of f with respect to g is defined as

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} \ . \end{aligned}$$

It is known (cf., [9]) that if $g(z) = \exp z$ then Definition 2 coincides with the classical definition of order of a meromorphic function f .

In the case of relative order, it therefore seems reasonable to define suitably the relative (p,q)-th order of meromorphic functions. Banerjee and Jana [2] also introduced such definition in the following manner (see [2]).

Definition 3. Let p and q be any two positive integers with $p > q$. The relative (p,q)-th order of a non-constant meromorphic function f with respect to another non-constant entire function g is defined by

$$\begin{aligned} \rho_g^{(p,q)}(f) &= \inf\left\{ \begin{array}{c} \mu > 0 : T_f(r) < T_g\left(\exp^{[p-1]}\left(\mu \log^{[q]} r\right)\right) \\ \text{for all } r > r_0(\mu) > 0 \end{array} \right\} \\ &= \limsup_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1} T_f(r)}{\log^{[q]} r}. \end{aligned}$$

Recently, Debnath et al. [3] give an alternative definition of relative (p,q)-th order of a meromorphic function with respect to an entire function in the light of index-pair as follows:

Definition 4. Let f be any meromorphic function and g be any entire function with index-pairs (m,q) and (m,p) respectively where p,q,m are all positive integers such that $m \geq p$ and $m \geq q$. Then the relative (p,q) -th order of f with respect to g is defined as

$$\rho_g^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r}.$$

Similarly, one can define the relative (p,q) -th lower order of a meromorphic function f with respect to an entire function g denoted by $\lambda_g^{(p,q)}(f)$ where p and q are any two positive integers in the following way:

$$\lambda_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r}.$$

In fact, Definition 4 improves Definition 3 ignoring the restriction $p > q$.

If a meromorphic function f and an entire function g have the same index-pair $(p,1)$ where p is any positive integer, we may get the definition of relative order of meromorphic function introduced by Lahiri and Banerjee [9] and if $g = \exp^{[m-1]} z$, then $\rho_g(f) = \rho_f^{[m]}$ and $\rho_g^{(p,q)}(f) = \rho_f(m,q)$. Also Definition 4 coincides with the classical one if f is a meromorphic function with index-pair $(2,1)$ and $g = \exp z$.

Further a meromrphic function f for which (p,q)-th relative order and (p,q)-th relative lower order with respect to another entire function g are same is called a function of regular relative (p,q) growth with respect to g. Otherwise, f is said to be irregular relative (p,q) growth with respect to g.

In this connection we also introduce the following definition:

Definition 5. A meromorphic function f is said to have relative index-pair (p,q) with respect to an entire function g where p and q are any two positive integers if $b < \rho_g^{(p,q)}(f) < \infty$ and $\rho_g^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if

$p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho_g^{(p,q)}(f) < \infty$, then

$$\begin{cases} \rho_g^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \rho_g^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho_g^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Similarly for $0 < \lambda_g^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda_g^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \lambda_g^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda_g^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Now in order to refine the above growth scale, now we intend to introduce the definitions of an another growth indicators, such as relative (p, q) -th type and relative (p, q) -th lower type of meromorphic function with respect to another entire function in the light of their index-pair as follows:

Definition 6. Let f be a meromorphic function and g be an entire function with index-pairs (m, q) and (m, p) respectively where p, q, m are all positive integers such that $m \geq \max\{p, q\}$. The relative (p, q) -th type and relative (p, q) -th lower type of f with respect to the entire function g having finite positive relative (p, q) th order $\rho_g^{(p,q)}(f)$ $\left(0 < \rho_g^{(p,q)}(f) < \infty\right)$ are defined as

$$\sigma_g^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1} T_f(r)}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}} \quad \text{and} \quad \overline{\sigma}_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1} T_f(r)}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}}$$

Analogously, to determine the relative growth of two meromorphic functions having same non zero finite relative (p, q) -th lower order with respect to another entire function, one can introduce the definition of relative (p, q)-th weak type of a meromorphic f with respect to an entire g of finite positive relative (p, q)-th lower order $\lambda_g^{(p,q)}(f)$ in the following way:

Definition 7. Let f be a meromorphic function and g be an entire function having finite positive relative (p, q) th lower order $\lambda_g^{(p,q)}(f)$ $\left(0 < \lambda_g^{(p,q)}(f) < \infty\right)$ where p and q are any two positive integers. Then the relative (p, q) -th weak type of f with respect to g is defined as

$$\tau_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}}.$$

Similarly one can define another growth indicator $\overline{\tau}_g^{(p,q)}(f)$ in the following way:

$$\overline{\tau}_g^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}}.$$

If f and g have index-pair $(m, 1)$ and (m, l), respectively, then Definition 6 and Definition 7 reduces to the definition of generalized relative growth indicators such as generalized relative type $\sigma_g^{[l]}(f)$, generalized relative weak type $\tau_g^{[l]}(f)$ etc. If f and g have the same index-pair $(p, 1)$ where p is any positive integer, we get the definitions of relative growth indicators such as relative type $\sigma_g(f)$, relative weak type $\tau_g(f)$ etc. Further if $g = \exp^{[m-1]} z$, then Definition 6 and Definition 7 reduce to the (m, q) th growth indicators of meromorphic f which is analogous the definition as introduced by Juneja et al. [8] for an entire function. Also for $g = \exp^{[m-1]} z$, relative growth indicators reduce to the definition of generalized growth indicators.such as generalized type $\sigma_f^{[m]}$, generalized weak type $\tau_f^{[m]}$ etc. Moreover, if f has index-pair $(2, 1)$ and $g = \exp z$, then Definition 6 and Definition 7 become the classical definitions of f. For details about different type of relative growth indicators, one may see [4, 5].

Here, in this paper, we aim at investigating some basic properties of relative (p, q)-th order, relative (p, q)-th type and relative (p, q)-th weak type of a meromorphic function with respect to an entire function where p and q are any two positive integers under somewhat different conditions. Throughout this paper, we assume that all the growth indicators are all nonzero finite.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [1] Let f be an entire function which satisfies the Property (A) then for any positive integer n and for all sufficiently large r,

$$[M_f(r)]^n \leq M_f\left(r^\delta\right)$$

holds where $\delta > 1$.

Lemma 2. [6, p. 18] Let f be an entire function. Then for all sufficiently large values of r,

$$T_f(r) \leq \log M_f(r) \leq 3T_f(2r) \ .$$

3. Main Results

In this section we present our main results.

Theorem 1. Let f_1, f_2 be meromorphic functions and g_1 be any entire function such that at least f_1 or f_2 is of regular relative (p, q) growth with respect to g_1 where p and q are any two positive integers. Also let g_1 has the Property (A). Then

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\} \ .$$

The equality holds when $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p, q) growth with respect to g_1 where $i = j = 1, 2$ and $i \neq j$.

Proof. If $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) = 0$ then the result is obvious. So we suppose that $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) > 0$. We can clearly assume that $\lambda_{g_1}^{(p,q)}(f_k)$ is finite for $k = 1, 2$.

Further let $\max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\} = \Delta$ and f_2 is of regular relative (p, q) growth with respect to g_1.

Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{g_1}^{(p,q)}(f_1)$, we have for a sequence values of r tending to infinity that

$$T_{f_1}(r) \leq T_{g_1}\left[\exp^{[p]}\left[\left(\lambda_{g_1}^{(p,q)}(f_1) + \varepsilon\right)\log^{[q]} r\right]\right]$$

$$i.e., \ T_{f_1}(r) \leq T_{g_1}\left[\exp^{[p]}\left[(\Delta + \varepsilon)\log^{[q]} r\right]\right] \ . \quad (1)$$

Also for any arbitrary $\varepsilon > 0$ from the definition of $\rho_{g_1}^{(p,q)}(f_2)\left(=\lambda_{g_1}^{(p,q)}(f_2)\right)$, we obtain for all sufficiently large values of r that

$$T_{f_2}(r) \le T_{g_1}\left[\exp^{[p]}\left[\left(\lambda_{g_1}^{(p,q)}(f_2)+\varepsilon\right)\log^{[q]} r\right]\right] \quad (2)$$

$$i.e.,\ T_{f_2}(r) \le T_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]} r\right]\right]\ . \quad (3)$$

Since $T_{f_1\pm f_2}(r) \le T_{f_1}(r)+T_{f_2}(r)+O(1)$ for all large r, so in view of (1) , (3) and Lemma 2, we obtain for a sequence values of r tending to infinity that

$$T_{f_1\pm f_2}(r) \le 2\log M_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]} r\right]\right]+O(1)$$

$$i.e.,\ T_{f_1\pm f_2}(r) \le 3\log M_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]} r\right]\right]\ . \quad (4)$$

Therefore in view of Lemma 1 and Lemma 2, we obtain from (4) for a sequence values of r tending to infinity and $\sigma > 1$ that

$$T_{f_1\pm f_2}(r) \le \frac{1}{3}\log\left[M_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]} r\right]\right]\right]^9$$

$$i.e.,\ T_{f_1\pm f_2}(r) \le \frac{1}{3}\log M_{g_1}\left[\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]} r\right]\right]^{\sigma}\right]$$

$$i.e.,\ T_{f_1\pm f_2}(r) \le T_{g_1}\left[2\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]} r\right]\right]^{\sigma}\right]\ .$$

Now we get from above by letting $\sigma \to 1^+$

$$i.e.,\ \underline{\lim}_{r\to\infty}\frac{\log^{[p]} T_{g_1}^{-1}T_{f_1\pm f_2}(r)}{\log^{[q]} r} < (\Delta+\varepsilon)\ .$$

Since $\varepsilon > 0$ is arbitrary,

$$\lambda_{g_1}^{(p,q)}(f_1\pm f_2) \le \Delta = \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}\ .$$

Similarly, if we consider that f_1 is of regular relative (p,q) growth with respect to g_1 or both f_1 and f_2 are of regular relative (p,q) growth with respect to g_1, then one can easily verify that

$$\lambda_{g_1}^{(p,q)}(f_1\pm f_2) \le \Delta = \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}\ . \quad (5)$$

Further without loss of generality, let $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$ and $f = f_1 \pm f_2$. Then in view of (5) we get that $\lambda_{g_1}^{(p,q)}(f) \le \lambda_{g_1}^{(p,q)}(f_2)$. As, $f_2 = \pm(f - f_1)$ and in this case we obtain that $\lambda_{g_1}^{(p,q)}(f_2) \le \max\left\{\lambda_{g_1}^{(p,q)}(f), \lambda_{g_1}^{(p,q)}(f_1)\right\}$. As we assume that $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$, therefore we have $\lambda_{g_1}^{(p,q)}(f_2) \le \lambda_{g_1}^{(p,q)}(f)$ and hence $\lambda_{g_1}^{(p,q)}(f) = \lambda_{g_1}^{(p,q)}(f_2) = \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}$. Therefore, $\lambda_{g_1}^{(p,q)}(f_1\pm f_2) = \lambda_{g_1}^{(p,q)}(f_i) \mid i = 1,2$ provided $\lambda_{g_1}^{(p,q)}(f_1) \ne \lambda_{g_1}^{(p,q)}(f_2)$. Thus the theorem is established.

Now we state the following theorem due to Debnath et al. [3] and Banerjee et al. [2]:

Theorem 2. Let f_1 and f_2 be any two meromorphic functions with relative index-pair (p,q) with respect to another entire function g_1 where p and q are positive integers. Also let g_1 has the Property (A). Then

$$\rho_{g_1}^{(p,q)}(f_1\pm f_2) \le \max\left\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_1}^{(p,q)}(f_2)\right\}\ .$$

The equality holds when $\rho_{g_1}^{(p,q)}(f_1) \neq \rho_{g_1}^{(p,q)}(f_2)$.

Remark 1. In *Theorem 2* of [2], Banerjee et al. [2] said nothing about the condition of equality but the equality of *Theorem 2* holds when $\rho_{g_1}^{(p,q)}(f_1) \neq \rho_{g_1}^{(p,q)}(f_2)$ which can easily be derived in the line of *Theorem 1.*

Theorem 3. Let f_1 be a meromorphic function and g_1, g_2 be any two entire functions such that $\lambda_{g_1}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_1)$ exists where p and q are positive integers. Also let $g_1 \pm g_2$ has the Property (A). Then

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \min\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\right\} .$$

The equality holds when $\lambda_{g_1}^{(p,q)}(f_1) \neq \lambda_{g_2}^{(p,q)}(f_1)$.

Proof. If $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) = \infty$ then the result is obvious. So we suppose that $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) < \infty$.

We can clearly assume that $\lambda_{g_k}^{(p,q)}(f_1)$ is finite for $k = 1, 2$.

Further let $\Psi = \min\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\right\}$.

Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{g_k}^{(p,q)}(f_1)$, we have for all sufficiently large values of r that

$$T_{g_k}\left[\exp^{[p]}\left[\left(\lambda_{g_k}^{(p,q)}(f_1) - \varepsilon\right)\log^{[q]} r\right]\right] \leq T_{f_1}(r) \quad \text{where } k = 1, 2 \tag{6}$$

$$i.e, \ T_{g_k}\left[\exp^{[p]}\left[(\Psi - \varepsilon)\log^{[q]} r\right]\right] \leq T_{f_1}(r) \quad \text{where } k = 1, 2$$

Since $T_{g_1 \pm g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r,, we obtain from above and Lemma 2 for all sufficiently large values of r that

$$T_{g_1 \pm g_2}\left[\exp^{[p]}\left[(\Psi - \varepsilon)\log^{[q]} r\right]\right] \leq 2T_{f_1}(r) + O(1)$$

$$i.e., \ T_{g_1 \pm g_2}\left[\exp^{[p]}\left[(\Psi - \varepsilon)\log^{[q]} r\right]\right] < 3T_{f_1}(r) .$$

Therefore in view of Lemma 1 and Lemma 2, we obtain from above for all sufficiently large values of r and any $\sigma > 1$ that

Proof.

$$\frac{1}{9}\log M_{g_1 \pm g_2}\left[\frac{\exp^{[p]}\left[(\Psi - \varepsilon)\log^{[q]} r\right]}{2}\right] < T_{f_1}(r)$$

$$i.e., \ \log M_{g_1 \pm g_2}\left[\frac{\exp^{[p]}\left[(\Psi - \varepsilon)\log^{[q]} r\right]}{2}\right]^{\frac{1}{9}} < T_{f_1}(r)$$

$$i.e., \ \log M_{g_1 \pm g_2}\left[\left(\frac{\exp^{[p]}\left[(\Psi - \varepsilon)\log^{[q]} r\right]}{2}\right)^{\frac{1}{\sigma}}\right] < T_{f_1}(r)$$

$$i.e., \ T_{g_1 \pm g_2}\left[\left(\frac{\exp^{[p]}\left[(\Psi - \varepsilon)\log^{[q]} r\right]}{2}\right)^{\frac{1}{\sigma}}\right] < T_{f_1}(r) .$$

Since $\varepsilon > 0$ is arbitrary, we get from above by letting $\sigma \to 1^+$

$$\lambda^{(p,q)}_{g_1 \pm g_2}(f_1) \geq \Psi = \min\left\{\lambda^{(p,q)}_{g_1}(f_1), \lambda^{(p,q)}_{g_2}(f_1)\right\} . \quad (7)$$

Now without loss of generality, we may consider that $\lambda^{(p,q)}_{g_1}(f_1) < \lambda^{(p,q)}_{g_2}(f_1)$ and $g = g_1 \pm g_2$. Then in view of (7) we get that $\lambda^{(p,q)}_{g}(f_1) \geq \lambda^{(p,q)}_{g_1}(f_1)$. Further, $g_1 = (g \pm g_2)$ and in this case we obtain that $\lambda^{(p,q)}_{g_1}(f_1) \geq \min\left\{\lambda^{(p,q)}_{g}(f_1), \lambda^{(p,q)}_{g_2}(f_1)\right\}$. As we assume that $\lambda^{(p,q)}_{g_1}(f_1) < \lambda^{(p,q)}_{g_2}(f_1)$, therefore we have $\lambda^{(p,q)}_{g_1}(f_1) \geq \lambda^{(p,q)}_{g}(f_1)$ and hence $\lambda^{(p,q)}_{g}(f_1) = \lambda^{(p,q)}_{g_1}(f_1) = \min\left\{\lambda^{(p,q)}_{g_1}(f_1), \lambda^{(p,q)}_{g_2}(f_1)\right\}$. Therefore, $\lambda^{(p,q)}_{g_1 \pm g_2}(f_1) = \lambda^{(p,q)}_{g_i}(f_1) \mid i = 1, 2$ provided $\lambda^{(p,q)}_{g_1}(f_1) \neq \lambda^{(p,q)}_{g_2}(f_1)$. Thus the theorem follows. □

Theorem 4. Let f_1 be a meromorphic function and g_1, g_2 be any two entire functions such that the relative index-pair of f_1 with respect to g_1 and g_2 are (p, q) where p and q are positive integers. Also let f_1 is of regular relative (p, q) growth with respect to at least any one of g_1 or g_2. If $g_1 \pm g_2$ has the Property (A), then

$$\rho^{(p,q)}_{g_1 \pm g_2}(f_1) \geq \min\left\{\rho^{(p,q)}_{g_1}(f_1), \rho^{(p,q)}_{g_2}(f_1)\right\} .$$

The equality holds when $\rho^{(p,q)}_{g_i}(f_1) < \rho^{(p,q)}_{g_j}(f_1)$ with at least f_1 is of regular relative (p, q) growth with respect to g_j where $i = j = 1, 2$ and $i \neq j$.

The proof of Theorem 4 would run parallel to that of Theorem 3. We omit the details.

Theorem 5. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $g_1 \pm g_2$ has the Property (A). Then for any two positive integers p and q

$$\begin{aligned}&\rho^{(p,q)}_{g_1 \pm g_2}(f_1 \pm f_2)\\ &\leq \max\left[\min\left\{\rho^{(p,q)}_{g_1}(f_1), \rho^{(p,q)}_{g_2}(f_1)\right\}, \min\left\{\rho^{(p,q)}_{g_1}(f_2), \rho^{(p,q)}_{g_2}(f_2)\right\}\right]\end{aligned}$$

when the following two conditions holds:

(i) $\rho^{(p,q)}_{g_i}(f_1) < \rho^{(p,q)}_{g_j}(f_1)$ with at least f_1 is of regular relative (p, q) growth with respect to g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and

(ii) $\rho^{(p,q)}_{g_i}(f_2) < \rho^{(p,q)}_{g_j}(f_2)$ with at least f_2 is of regular relative (p, q) growth with respect to g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.

The sign of equality holds when $\rho^{(p,q)}_{g_1}(f_i) < \rho^{(p,q)}_{g_1}(f_j)$ and $\rho^{(p,q)}_{g_2}(f_i) < \rho^{(p,q)}_{g_2}(f_j)$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Proof. Let the conditions (i) and (ii) of the theorem hold. Therefore in view of Theorem 2 and Theorem 4 we get

$$\begin{aligned}&\max\left[\min\left\{\rho^{(p,q)}_{g_1}(f_1), \rho^{(p,q)}_{g_2}(f_1)\right\}, \min\left\{\rho^{(p,q)}_{g_1}(f_2), \rho^{(p,q)}_{g_2}(f_2)\right\}\right]\\ &= \max\left[\rho^{(p,q)}_{g_1 \pm g_2}(f_1), \rho^{(p,q)}_{g_1 \pm g_2}(f_2)\right]\\ &\geq \rho^{(p,q)}_{g_1 \pm g_2}(f_1 \pm f_2) . \end{aligned} \quad (8)$$

Since $\rho^{(p,q)}_{g_1}(f_i) < \rho^{(p,q)}_{g_1}(f_j)$ and $\rho^{(p,q)}_{g_2}(f_i) < \rho^{(p,q)}_{g_2}(f_j)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$, we obtain that

$$\text{either } \min\left\{\rho^{(p,q)}_{g_1}(f_1), \rho^{(p,q)}_{g_2}(f_1)\right\} > \min\left\{\rho^{(p,q)}_{g_1}(f_2), \rho^{(p,q)}_{g_2}(f_2)\right\} \text{ or}$$

$$\min\left\{\rho_{g_1}^{(p,q)}(f_2), \rho_{g_2}^{(p,q)}(f_2)\right\} > \min\left\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\right\} \text{ holds.}$$

Now in view of the conditions (i) and (ii) of the theorem, it follows from above argument that

$$\text{either } \rho_{g_1 \pm g_2}^{(p,q)}(f_1) > \rho_{g_1 \pm g_2}^{(p,q)}(f_2) \text{ or } \rho_{g_1 \pm g_2}^{(p,q)}(f_2) > \rho_{g_1 \pm g_2}^{(p,q)}(f_1)$$

which is the condition for holding equality in (8).

Hence the theorem follows.

Theorem 6. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let g_1, g_2 and $g_1 \pm g_2$ satisfy the Poperty (A). Then for any two positive integers p and q,

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2)$$
$$\geq \min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\right\}\right]$$

when the following two conditions holds:
(i) $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) growth with respect to g_1 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and
(ii) $\lambda_{g_2}^{(p,q)}(f_i) > \lambda_{g_2}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) growth with respect to g_2 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.
The sign of equality holds when $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1)$ and $\lambda_{g_i}^{(p,q)}(f_2) < \lambda_{g_j}^{(p,q)}(f_2)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Proof. Suppose that the conditions (i) and (ii) of the theorem holds. Therefore in view of Theorem 1 and Theorem 3, we obtain

$$\min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\right\}\right]$$
$$= \min\left[\lambda_{g_1}^{(p,q)}(f_1 \pm f_2), \lambda_{g_2}^{(p,q)}(f_1 \pm f_2)\right]$$
$$\geq \lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) \ . \tag{9}$$

Since $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1)$ and $\lambda_{g_i}^{(p,q)}(f_2) < \lambda_{g_j}^{(p,q)}(f_2)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$, we get

$$\text{either } \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\} < \max\left\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\right\} \text{ or}$$

$$\max\left\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\right\} < \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\} \text{ holds.}$$

Since condition (i) and (ii) of the theorem holds, it follows from above argument that

$$\text{either } \lambda_{g_1}^{(p,q)}(f_1 \pm f_2) < \lambda_{g_2}^{(p,q)}(f_1 \pm f_2) \text{ or } \lambda_{g_2}^{(p,q)}(f_1 \pm f_2) < \lambda_{g_1}^{(p,q)}(f_1 \pm f_2)$$

which is the condition for holding equality in (9).

Hence the theorem follows.

Theorem 7. Let f_1, f_2 be any two meromorphic functions and g_1 be any entire function such that at least f_1 or f_2 is of regular relative (p,q) growth with respect to g_1 where p and q are any two positive integers. Also let g_1 satisfy the Property (A). Then

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\} \ .$$

The equality holds when $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) growth with respect to g_1 where $i = j = 1, 2$ and $i \neq j$.

Proof. Since $T_{f_1 \cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$ for all large r, therefore applying the same procedure as adopted in Theorem 1 we get

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\} .$$

Now without loss of generality, let $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$ and $f = f_1 \cdot f_2$. Then $\lambda_{g_1}^{(p,q)}(f) \leq \lambda_{g_1}^{(p,q)}(f_2)$. Further, $f_2 = \frac{f}{f_1}$ and and $T_{f_1}(r) = T_{\frac{1}{f_1}}(r) + O(1)$. Therefore $T_{f_2}(r) \leq T_f(r) + T_{f_1}(r) + O(1)$ and in this case we obtain that $\lambda_{g_1}^{(p,q)}(f_2) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f), \lambda_{g_1}^{(p,q)}(f_1)\right\}$. As we assume that $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$, therefore we have $\lambda_{g_1}^{(p,q)}(f_2) \leq \lambda_{g_1}^{(p,q)}(f)$ and hence

$$\lambda_{g_1}^{(p,q)}(f) = \lambda_{g_1}^{(p,q)}(f_2) = \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}.$$

Therefore, $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) = \lambda_{g_1}^{(p,q)}(f_i) \mid i = 1, 2$ provided $\lambda_{g_1}^{(p,q)}(f_1) \neq \lambda_{g_1}^{(p,q)}(f_2)$.

Hence the theorem follows.

Next we prove the result for the quotient $\frac{f_1}{f_2}$, provided $\frac{f_1}{f_2}$ is meromorphic.

Theorem 8. Let f_1, f_2 be any two meromorphic functions and g_1 be any entire function such that at least f_1 or f_2 is of regular relative (p,q) growth with respect to g_1 where p and q are any two positive integers. Also let g_1 satisfy the Property (A). Then

$$\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\},$$

provided $\frac{f_1}{f_2}$ is meromorphic. The sign of equality holds when at least f_2 is of regular relative (p,q) growth with respect to g_1 and $\lambda_{g_1}^{(p,q)}(f_1) \neq \lambda_{g_1}^{(p,q)}(f_2)$.

Proof. Since $T_{f_2}(r) = T_{\frac{1}{f_2}}(r) + O(1)$ and $T_{\frac{f_1}{f_2}}(r) \leq T_{f_1}(r) + T_{\frac{1}{f_2}}(r)$, we get in view of Theorem 1 that

$$\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\} . \tag{10}$$

Now in order to prove the equality conditions, we discuss the following two cases:

Case I. Suppose $\frac{f_1}{f_2}(= h)$ satisfies the following condition

$$\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2),$$

and f_2 is of regular relative (p,q) growth with respect to g_1.

Now if possible, let $\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right) < \lambda_{g_1}^{(p,q)}(f_2)$. Therefore from $f_1 = h \cdot f_2$ we get that $\lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2)$ which is a contradiction. Therefore $\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right) \geq \lambda_{g_1}^{(p,q)}(f_2)$ and in view of (10), we get

$$\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right) = \lambda_{g_1}^{(p,q)}(f_2) .$$

Case II. Suppose $\frac{f_1}{f_2}(= h)$ satisfies the following condition

$$\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2),$$

and f_2 is of regular relative (p,q) growth with respect to g_1.

Now from $f_1 = h{\cdot}f_2$ we get that either $\lambda_{g_1}^{(p,q)}(f_1) \leq \lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right)$ or $\lambda_{g_1}^{(p,q)}(f_1) \leq \lambda_{g_1}^{(p,q)}(f_2)$. But according to our assumption $\lambda_{g_1}^{(p,q)}(f_1) \nleq \lambda_{g_1}^{(p,q)}(f_2)$. Therefore $\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right) \geq \lambda_{g_1}^{(p,q)}(f_1)$ and in view of (10), we get

$$\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right) = \lambda_{g_1}^{(p,q)}(f_1) \ .$$

Hence the theorem follows.

Now we state the following theorem due to Debnath et al. [3] and Banerjee et al. [2]:

Theorem 9. Let f_1 and f_2 be any two meromorphic functions with relative index-pair (p,q) with respect to another entire function g_1 where p and q are positive integers. Also let g_1 satisfy the Property (A). Then

$$\rho_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \max\left\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_1}^{(p,q)}(f_2)\right\} \ .$$

The equality holds when $\rho_{g_1}^{(p,q)}(f_1) \neq \rho_{g_1}^{(p,q)}(f_2)$.
Similar results hold for the quotient $\frac{f_1}{f_2}$, provided $\frac{f_1}{f_2}$ is meromorphic.

Theorem 10. Let f_1 be a meromorphic function and g_1, g_2 be any two entire functions such that $\lambda_{g_1}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_1)$ exists where p and q are positive integers. Also let $g_1 \cdot g_2$ satisfy the Property (A). Then

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) \geq \min\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\right\} \ .$$

The equality holds when $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1)$ where $i = j = 1, 2$ and $i \neq j$ and g_i satisfy the Property (A).
Similar results hold for the quotient $\frac{g_1}{g_2}$, provided $\frac{g_1}{g_2}$ is entire and satisfy the Property (A). The sign of equality holds when $\lambda_{g_1}^{(p,q)}(f_1) \neq \lambda_{g_2}^{(p,q)}(f_1)$ and g_1 satisfy the Property (A).

Proof. Since $T_{g_1 \cdot g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r)$ for all large r, therefore applying the same procedure as adopted in Theorem 3 we get

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) \geq \min\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\right\} \ .$$

Now without loss of generality, we may consider that $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$ and $g = g_1 \cdot g_2$. Then $\lambda_{g}^{(p,q)}(f_1) \geq \lambda_{g_1}^{(p,q)}(f_1)$. Further, $g_1 = \frac{g}{g_2}$ and and $T_{g_2}(r) = T_{\frac{1}{g_2}}(r) + O(1)$. Therefore $T_{g_1}(r) \leq T_g(r) + T_{g_2}(r) + O(1)$ and in this case we obtain $\lambda_{g_1}^{(p,q)}(f_1) \geq \min\left\{\lambda_{g}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\right\}$. As we assume that $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$, so we have $\lambda_{g_1}^{(p,q)}(f_1) \geq \lambda_{g}^{(p,q)}(f_1)$ and hence $\lambda_{g}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_1) = \min\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\right\}$. Therefore, $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) = \lambda_{g_i}^{(p,q)}(f_1) \mid i = 1, 2$ provided $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$ and g_1 satisfy the Property (A).

Hence the first part of the theorem follows.

Now we prove our results for the quotient $\frac{g_1}{g_2}$, provided $\frac{g_1}{g_2}$ is entire and $\lambda_{g_1}^{(p,q)}(f_1) \neq \lambda_{g_2}^{(p,q)}(f_1)$.

Since $T_{g_2}(r) = T_{\frac{1}{g_2}}(r) + O(1)$ and $T_{\frac{g_1}{g_2}}(r) \leq T_{g_1}(r) + T_{\frac{1}{g_2}}(r)$, we get in view of Theorem 3 that

$$\lambda^{(p,q)}_{\frac{g_1}{g_2}}(f_1) \geq \min\left\{\lambda^{(p,q)}_{g_1}(f_1), \lambda^{(p,q)}_{g_2}(f_1)\right\} \ . \tag{11}$$

Now in order to prove the equality conditions, we discuss the following two cases:

Case I. Suppose $\frac{g_1}{g_2}\,(= h)$ satisfies the following condition

$$\lambda^{(p,q)}_{g_1}(f_1) > \lambda^{(p,q)}_{g_2}(f_1) \ .$$

Now if possible, let $\lambda^{(p,q)}_{\frac{g_1}{g_2}}(f_1) > \lambda^{(p,q)}_{g_2}(f_1)$. Therefore from $g_1 = h \cdot g_2$ we get $\lambda^{(p,q)}_{g_1}(f_1) = \lambda^{(p,q)}_{g_2}(f_1)$, which is a contradiction. Therefore $\lambda^{(p,q)}_{\frac{g_1}{g_2}}(f_1) \leq \lambda^{(p,q)}_{g_2}(f_1)$ and in view of (11), we get

$$\lambda^{(p,q)}_{\frac{g_1}{g_2}}(f_1) = \lambda^{(p,q)}_{g_2}(f_1) \ .$$

Case II. Suppose that $\frac{g_1}{g_2}\,(= h)$ satisfies the following condition

$$\lambda^{(p,q)}_{g_1}(f_1) < \lambda^{(p,q)}_{g_2}(f_1) \ .$$

Therefore from $g_1 = h \cdot g_2$, we get that either $\lambda^{(p,q)}_{g_1}(f_1) \geq \lambda^{(p,q)}_{\frac{g_1}{g_2}}(f_1)$ or $\lambda^{(p,q)}_{g_1}(f_1) \geq \lambda^{(p,q)}_{g_2}(f_1)$. But according to our assumption $\lambda^{(p,q)}_{g_1}(f_1) \ngeq \lambda^{(p,q)}_{g_2}(f_1)$. Therefore $\lambda^{(p,q)}_{\frac{g_1}{g_2}}(f_1) \leq \lambda^{(p,q)}_{g_1}(f_1)$ and in view of (11), we get

$$\lambda^{(p,q)}_{\frac{g_1}{g_2}}(f_1) = \lambda^{(p,q)}_{g_1}(f_1) \ .$$

Hence the theorem follows.

Theorem 11. Let f_1 be any meromorphic function and g_1, g_2 be any two entire functions such that the relative index-pair of f_1 with respect to g_1 and g_2 is (p, q) where p and q are positive integers. Further let f_1 is of regular relative (p, q) growth with respect to at least any one of g_1 or g_2. Also let $g_1 \cdot g_2$ satisfy the Property (A). Then

$$\rho^{(p,q)}_{g_1 \cdot g_2}(f_1) \geq \min\left\{\rho^{(p,q)}_{g_1}(f_1), \rho^{(p,q)}_{g_2}(f_1)\right\} \ .$$

The equality holds when $\rho^{(p,q)}_{g_i}(f_1) < \rho^{(p,q)}_{g_j}(f_1)$ with at least f_1 is of regular relative (p, q) growth with respect to g_j where $i = j = 1, 2$ and $i \neq j$ and g_i satisfy the Property (A).

Theorem 12. Let f_1 be any meromorphic function and g_1, g_2 be any two entire functions such that the relative index-pair of f_1 with respect to g_1 and g_2 is (p, q) where p and q are positive integers. Further let f_1 is of regular relative (p, q) growth with respect to at least any one of g_1 or g_2. Then

$$\rho^{(p,q)}_{\frac{g_1}{g_2}}(f_1) \geq \min\left\{\rho^{(p,q)}_{g_1}(f_1), \rho^{(p,q)}_{g_2}(f_1)\right\},$$

provided $\frac{g_1}{g_2}$ is entire and satisfy the Property (A). The equality holds when at least f_1 is of regular relative (p, q) growth with respect to g_2, $\rho^{(p,q)}_{g_1}(f_1) \neq \rho^{(p,q)}_{g_2}(f_1)$ and g_1 satisfy the Property (A).

A similar argument in the proof of Theorem 10 will establish the results in Theorem 11 and Theorem 12. We omit the details.

Now we state the following four theorems without their proofs as those can easily be carried out in the line of Theorem 5 and Theorem 6 respectively.

Theorem 13. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $g_1 \cdot g_2$ be satisfy the Property (A). Then for any two positive integers p and q,

$$\begin{aligned}&\rho_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2)\\&\leq \max\left[\min\left\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\right\}, \min\left\{\rho_{g_1}^{(p,q)}(f_2), \rho_{g_2}^{(p,q)}(f_2)\right\}\right],\end{aligned}$$

when the following two conditions holds:
(i) $\rho_{g_i}^{(p,q)}(f_1) < \rho_{g_j}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_j and g_i satisfy the Property (A) for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and
(ii) $\rho_{g_i}^{(p,q)}(f_2) < \rho_{g_j}^{(p,q)}(f_2)$ with at least f_2 is of regular relative (p,q) growth with respect to g_j and g_i satisfy the Property (A) for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.
The equality holds when $\rho_{g_1}^{(p,q)}(f_i) < \rho_{g_1}^{(p,q)}(f_j)$ and $\rho_{g_2}^{(p,q)}(f_i) < \rho_{g_2}^{(p,q)}(f_j)$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Theorem 14. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $g_1 \cdot g_2$, g_1 and g_2 be satisfy the Property (A). Then for any two positive integers p and q,

$$\begin{aligned}&\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2)\\&\geq \min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\right\}\right]\end{aligned}$$

when the following two conditions holds:
(i) $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) growth with respect to g_1 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and
(ii) $\lambda_{g_2}^{(p,q)}(f_i) > \lambda_{g_2}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) growth with respect to g_2 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.
The equality holds when $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1)$ and $\lambda_{g_i}^{(p,q)}(f_2) < \lambda_{g_j}^{(p,q)}(f_2)$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Theorem 15. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions such that $\frac{f_1}{f_2}$ is meromorphic and $\frac{g_1}{g_2}$ is entire. Also let $\frac{g_1}{g_2}$ satisfy the Property (A). Then for any two positive integers p and q,

$$\begin{aligned}&\rho_{\frac{g_1}{g_2}}^{(p,q)}\left(\frac{f_1}{f_2}\right)\\&\leq \max\left[\min\left\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\right\}, \min\left\{\rho_{g_1}^{(p,q)}(f_2), \rho_{g_2}^{(p,q)}(f_2)\right\}\right]\end{aligned}$$

when the following two conditions holds:
(i) At least f_1 is of regular relative (p,q) growth with respect to g_2 and $\rho_{g_1}^{(p,q)}(f_1) \neq \rho_{g_2}^{(p,q)}(f_1)$; and
(ii) At least f_2 is of regular relative (p,q) growth with respect to g_2 and $\rho_{g_1}^{(p,q)}(f_2) \neq \rho_{g_2}^{(p,q)}(f_2)$.
The equality holds when $\rho_{g_1}^{(p,q)}(f_i) < \rho_{g_1}^{(p,q)}(f_j)$ and $\rho_{g_2}^{(p,q)}(f_i) < \rho_{g_2}^{(p,q)}(f_j)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Theorem 16. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions such that $\frac{f_1}{f_2}$ is meromorphic and $\frac{g_1}{g_2}$ is entire. Also let $\frac{g_1}{g_2}$, g_1 and g_2 be satisfy the Property (A). Then for any two positive integers p and q,

$$\lambda^{(p,q)}_{\frac{g_1}{g_2}}\left(\frac{f_1}{f_2}\right)$$
$$\geq \min\left[\max\left\{\lambda^{(p,q)}_{g_1}(f_1), \lambda^{(p,q)}_{g_1}(f_2)\right\}, \max\left\{\lambda^{(p,q)}_{g_2}(f_1), \lambda^{(p,q)}_{g_2}(f_2)\right\}\right]$$

when the following two conditions hold:
(i) At least f_2 is of regular relative (p,q) growth with respect to g_1 and $\lambda^{(p,q)}_{g_1}(f_1) \neq \lambda^{(p,q)}_{g_1}(f_2)$; and
(ii) At least f_2 is of regular relative (p,q) growth with respect to g_2 and $\lambda^{(p,q)}_{g_2}(f_1) \neq \lambda^{(p,q)}_{g_2}(f_2)$.
The equality holds when $\lambda^{(p,q)}_{g_i}(f_1) < \lambda^{(p,q)}_{g_j}(f_1)$ and $\lambda^{(p,q)}_{g_i}(f_2) < \lambda^{(p,q)}_{g_j}(f_2)$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Next we intend to find out the sum and product theorems of relative (p,q)-th type (respectively relative (p,q)-th lower type) and relative (p,q)-th weak type of meromorphic function with respect to an entire function taking into consideration of the above theorems.

Theorem 17. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $\rho^{(p,q)}_{g_1}(f_1)$, $\rho^{(p,q)}_{g_1}(f_2)$, $\rho^{(p,q)}_{g_2}(f_1)$ and $\rho^{(p,q)}_{g_2}(f_2)$ be all non zero and finite where p and q are positive integers.
(A) If $\rho^{(p,q)}_{g_1}(f_i) > \rho^{(p,q)}_{g_1}(f_j)$ for $i = j = 1,2$; $i \neq j$, and g_1 has the Property (A), then

$$\sigma^{(p,q)}_{g_1}(f_1 \pm f_2) = \sigma^{(p,q)}_{g_1}(f_i) \mid i = 1,2 \text{ and } \overline{\sigma}^{(p,q)}_{g_1}(f_1 \pm f_2) = \overline{\sigma}^{(p,q)}_{g_1}(f_i) \mid i = 1,2 \ .$$

(B) If $\rho^{(p,q)}_{g_i}(f_1) < \rho^{(p,q)}_{g_j}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_j for $i = j = 1,2$; $i \neq j$ and $g_1 \pm g_2$ has the Property (A), then

$$\sigma^{(p,q)}_{g_1 \pm g_2}(f_1) = \sigma^{(p,q)}_{g_i}(f_1) \mid i = 1,2 \text{ and } \overline{\sigma}^{(p,q)}_{g_1 \pm g_2}(f_1) = \overline{\sigma}^{(p,q)}_{g_i}(f_1) \mid i = 1,2 \ .$$

(C) Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:
(i) $\rho^{(p,q)}_{g_i}(f_1) < \rho^{(p,q)}_{g_j}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_j for $i = 1$, 2, $j = 1,2$ and $i \neq j$;
(ii) $\rho^{(p,q)}_{g_i}(f_2) < \rho^{(p,q)}_{g_j}(f_2)$ with at least f_2 is of regular relative (p,q) growth with respect to g_j for $i = 1$, 2, $j = 1,2$ and $i \neq j$;
(iii) $\rho^{(p,q)}_{g_1}(f_i) > \rho^{(p,q)}_{g_1}(f_j)$ and $\rho^{(p,q)}_{g_2}(f_i) > \rho^{(p,q)}_{g_2}(f_j)$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$;
(iv) $\rho^{(p,q)}_{g_m}(f_l) = \max\left[\min\left\{\rho^{(p,q)}_{g_1}(f_1), \rho^{(p,q)}_{g_2}(f_1)\right\}, \min\left\{\rho^{(p,q)}_{g_1}(f_2), \rho^{(p,q)}_{g_2}(f_2)\right\}\right] \mid l = m = 1,2$, and $g_1 \pm g_2$ has the Property (A);
then we have

$$\sigma^{(p,q)}_{g_1 \pm g_2}(f_1 \pm f_2) = \sigma^{(p,q)}_{g_m}(f_l) \mid l = m = 1,2$$

and

$$\overline{\sigma}^{(p,q)}_{g_1 \pm g_2}(f_1 \pm f_2) = \overline{\sigma}^{(p,q)}_{g_m}(f_l) \mid l = m = 1,2 \ .$$

Proof. From the definition of relative (p,q)-th type and relative (p,q)-th lower type of meromorphic function with respect to an entire function, we have for all sufficiently large values of r that

$$T_{f_k}(r) \leq T_{g_l}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_l}^{(p,q)}(f_k)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_{g_l}^{(p,q)}(f_k)}\right\}\right], \quad (12)$$

$$T_{f_k}(r) \geq T_{g_l}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_l}^{(p,q)}(f_k)-\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_{g_l}^{(p,q)}(f_k)}\right\}\right], \quad (13)$$

$$i.e., \ T_{g_l}(r) \leq T_{f_k}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r}{\left(\overline{\sigma}_{g_l}^{(p,q)}(f_k)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_l}^{(p,q)}(f_k)}}\right)\right], \quad (14)$$

and for a sequence of values of r tending to infinity, we obtain that

$$T_{f_k}(r) \geq T_{g_l}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_l}^{(p,q)}(f_k)-\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_{g_l}^{(p,q)}(f_k)}\right\}\right], \quad (15)$$

$$i.e., \ T_{g_l}(r) \leq T_{f_k}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r}{\left(\sigma_{g_l}^{(p,q)}(f_k)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_l}^{(p,q)}(f_k)}}\right)\right], \quad (16)$$

and

$$T_{f_k}(r) \leq T_{g_l}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_l}^{(p,q)}(f_k)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_{g_l}^{(p,q)}(f_k)}\right\}\right], \quad (17)$$

where $\varepsilon > 0$ is any arbitrary positive number $k = 1, 2$ and $l = 1, 2$.

Case I. Suppose that $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ hold. Also let $\varepsilon (> 0)$ be arbitrary.

Since $T_{f_1 \pm f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1)$ for all large r, so in view of (12), we get for all sufficiently large values of r that

$$T_{f_1 \pm f_2}(r) \leq T_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right\}\right](1+A) \ . \quad (18)$$

where $A = \frac{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_2)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_{g_1}^{(p,q)}(f_2)}\right\}\right]+O(1)}{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right\}\right]}$, and in view of $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, and for all sufficiently large values of r, we can make the term A sufficiently small .

Hence for any $\alpha = 1 + \varepsilon_1$, it follows from (18) for all sufficiently large values of r that

$$T_{f_1 \pm f_2}(r) \leq T_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right\}\right] \cdot (1+\varepsilon_1)$$

$$i.e., \ T_{f_1 \pm f_2}(r) \leq T_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right\}\right] \cdot \alpha \ .$$

Hence making $\alpha \to 1+$, we get in view of Theorem 2, $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ and above for all sufficiently large values of r that

$$\limsup_{r \to \infty} \frac{\log^{[p-1]} T_{g_1}^{-1} T_{f_1 \pm f_2}(r)}{\left[\log^{[q-1]} r\right]^{\rho_{g_1}^{(p,q)}(f_1 \pm f_2)}} \leq \sigma_{g_1}^{(p,q)}(f_1)$$

$$i.e., \ \sigma_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \sigma_{g_1}^{(p,q)}(f_1) \ . \quad (19)$$

Now we may consider that $f = f_1 \pm f_2$. Since $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ hold. Then $\sigma_{g_1}^{(p,q)}(f) = \sigma_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \sigma_{g_1}^{(p,q)}(f_1)$. Further, let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 2 and $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, we obtain that $\rho_{g_1}^{(p,q)}(f) > \rho_{g_1}^{(p,q)}(f_2)$ holds. Hence in view of (19) $\sigma_{g_1}^{(p,q)}(f_1) \leq \sigma_{g_1}^{(p,q)}(f) = \sigma_{g_1}^{(p,q)}(f_1 \pm f_2)$. Therefore $\sigma_{g_1}^{(p,q)}(f) = \sigma_{g_1}^{(p,q)}(f_1) \Rightarrow \sigma_{g_1}^{(p,q)}(f_1 \pm f_2) = \sigma_{g_1}^{(p,q)}(f_1)$.

Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2)$, then one can easily verify that $\sigma_{g_1}^{(p,q)}(f_1 \pm f_2) = \sigma_{g_1}^{(p,q)}(f_2)$.

Case II. Let us consider that $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ hold. Also let $\varepsilon (> 0)$ be arbitrary.

Since $T_{f_1 \pm f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1)$ for all large r, from (12) and (17), we get for a sequence of values of r tending to infinity that

$$T_{f_1 \pm f_2}(r_n) \leq T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q)}(f_1) + \varepsilon\right)\left[\log^{[q-1]} r_n\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right\}\right](1 + B) \ . \tag{20}$$

where $B = \frac{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_2)+\varepsilon\right)\left[\log^{[q-1]} r_n\right]^{\rho_{g_1}^{(p,q)}(f_2)}\right\}\right]+O(1)}{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r_n\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right\}\right]}$, and in view of $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, we can make the term B sufficiently small by taking n sufficiently large and therefore using the similar technique for as executed in the proof of Case I we get from (20) that $\overline{\sigma}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\sigma}_{g_1}^{(p,q)}(f_1)$ when $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ hold.

Likewise, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2)$, then one can easily verify that $\overline{\sigma}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\sigma}_{g_1}^{(p,q)}(f_2)$.

Thus combining Case I and Case II, we obtain the first part of the theorem.

Case III. Let us consider that $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_2.

As $T_{g_1 \pm g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r, in view of (14) and (16), we obtain for a sequence of values of r tending to infinity that

$$T_{g_1 \pm g_2}(r_n) \leq T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r_n}{\left(\sigma_{g_1}^{(p,q)}(f_1) - \varepsilon\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right](1 + C) \ , \tag{21}$$

where $C = \frac{T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r_n}{\left(\overline{\sigma}_{g_2}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}\right)\right]+O(1)}{T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r_n}{\left(\sigma_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right]}$, and since $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, we can make the term C sufficiently small by taking n sufficiently large. Hence for any $\alpha = 1 + \varepsilon_1$, we get from (21) and Theorem 4, for a sequence of values of r tending to infinity that

$$T_{g_1 \pm g_2}(r_n) < T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r_n}{\left(\sigma_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right](1+\varepsilon_1)$$

$$i.e.,\ T_{g_1 \pm g_2}(r_n) < T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r_n}{\left(\sigma_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right]\alpha,$$

Hence, making $\alpha \to 1+$, we obtain from above for a sequence of values of r tending to infinity that

$$\left(\sigma_{g_1}^{(p,q)}(f_1)-\varepsilon\right)\left[\log^{[q-1]}(r_n)\right]^{\rho_{g_1 \pm g_2}^{(p,q)}(f_1)} < \log^{[p-1]} T_{g_1 \pm g_2}^{-1} T_{f_1}(r_n)$$

Since $\varepsilon > 0$ is arbitrary, we find

$$\sigma_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \sigma_{g_1}^{(p,q)}(f_1)\ . \tag{22}$$

Now we may consider that $g = g_1 \pm g_2$. Also $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ and at least f_1 is of regular relative (p,q) growth with respect to g_2. Then $\sigma_{g}^{(p,q)}(f_1) = \sigma_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \sigma_{g_1}^{(p,q)}(f_1)$. Further let $g_1 = (g \pm g_2)$. Therefore in view of Theorem 4 and $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, we obtain that $\rho_{g}^{(p.q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ as at least f_1 is of regular relative (p,q) growth with respect to g_2. Hence in view of (22), $\sigma_{g_1}^{(p,q)}(f_1) \geq \sigma_{g}^{(p,q)}(f_1) = \sigma_{g_1 \pm g_2}^{(p,q)}(f_1)$. Therefore $\sigma_{g}^{(p,q)}(f_1) = \sigma_{g_1}^{(p,q)}(f_1) \Rightarrow \sigma_{g_1 \pm g_2}^{(p,q)}(f_1) = \sigma_{g_1}^{(p,q)}(f_1)$.

Similarly if we consider $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_1, then $\sigma_{g_1 \pm g_2}^{(p,q)}(f_1) = \sigma_{g_2}^{(p,q)}(f_1)$.

Case IV. In this case suppose that $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_2.

As $T_{g_1 \pm g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r, therefore from (14), we get for all sufficiently large values of r that

$$T_{g_1 \pm g_2}(r) \leq T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r}{\left(\overline{\sigma}_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right](1+D)\ , \tag{23}$$

where $D = \dfrac{T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r}{\left(\overline{\sigma}_{g_2}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}\right)\right]+O(1)}{T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r}{\left(\overline{\sigma}_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right]}$ and in view of $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, we can make the term D sufficiently small by taking r sufficiently large and therefore using the similar technique for as executed in the proof of Case III we get from (23) that $\overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_1) = \overline{\sigma}_{g_1}^{(p,q)}(f_1)$ where $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ and at least f_1 is of regular relative (p,q) growth with respect to g_2.

Likewise if we consider $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_1, then $\overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_1) = \overline{\sigma}_{g_2}^{(p,q)}(f_1)$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The third part of the theorem is a natural consequence of Theorem 5 and the first part and second part of the theorem. Hence its proof is omitted.

Theorem 18. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $\lambda_{g_1}^{(p,q)}(f_1)$, $\lambda_{g_1}^{(p,q)}(f_2)$, $\lambda_{g_2}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_2)$ be all non zero and finite where p and q are positive integers.
(A) If $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) growth with respect to g_1 for $i = j = 1, 2$; $i \neq j$, and g_1 has the Property (A), then

$$\tau_{g_1}^{(p,q)}(f_1 \pm f_2) = \tau_{g_1}^{(p,q)}(f_i) \mid i = 1, 2 \text{ and } \overline{\tau}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\tau}_{g_1}^{(p,q)}(f_i) \mid i = 1, 2 .$$

(B) If $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1)$ for $i = j = 1, 2$; $i \neq j$ and $g_1 \pm g_2$ has the Property (A), then

$$\tau_{g_1 \pm g_2}^{(p,q)}(f_1) = \tau_{g_i}^{(p,q)}(f_1) \mid i = 1, 2 \text{ and } \overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_i}^{(p,q)}(f_1) \mid i = 1, 2 .$$

(C) Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:
(i) $\rho_{g_1}^{(p,q)}(f_i) > \rho_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) growth with respect to g_1 for $i = j = 1, 2$ and $i \neq j$;
(ii) $\rho_{g_2}^{(p,q)}(f_i) > \rho_{g_2}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) growth with respect to g_2 for $i = j = 1, 2$ and $i \neq j$;
(iii) $\rho_{g_i}^{(p,q)}(f_1) < \rho_{g_j}^{(p,q)}(f_1)$ and $\rho_{g_i}^{(p,q)}(f_2) < \rho_{g_j}^{(p,q)}(f_2)$ hold simultaneously for $i = j = 1, 2$ and $i \neq j$;
(iv) $\lambda_{g_m}^{(p,q)}(f_l) = \min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\right\}\right] \mid l = m = 1, 2$ and $g_1 \pm g_2$ has the Property (A)
then we have

$$\tau_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \tau_{g_m}^{(p,q)}(f_l) \mid l = m = 1, 2$$

and

$$\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \overline{\tau}_{g_m}^{(p,q)}(f_l) \mid l = m = 1, 2 .$$

Proof. For any arbitrary positive number $\varepsilon(> 0)$, we have for all sufficiently large values of r that

$$T_{f_k}(r) \leq T_{g_l}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_l}^{(p,q)}(f_k) + \varepsilon\right)\left[\log^{[q-1]} r\right]^{\lambda_{g_l}^{(p,q)}(f_k)}\right\}\right], \tag{24}$$

$$T_{f_k}(r) \geq T_{g_l}\left[\exp^{[p-1]}\left\{\left(\tau_{g_l}^{(p,q)}(f_k) - \varepsilon\right)\left[\log^{[q-1]} r\right]^{\lambda_{g_l}^{(p,q)}(f_k)}\right\}\right], \tag{25}$$

$$i.e., \ T_{g_l}(r) \leq T_{f_k}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r}{\left(\tau_{g_l}^{(p,q)}(f_k) - \varepsilon\right)}\right)^{\frac{1}{\lambda_{g_l}^{(p,q)}(f_k)}}\right)\right], \tag{26}$$

and for a sequence of values of r tending to infinity we obtain

$$T_{f_k}(r) \geq T_{g_l}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_l}^{(p,q)}(f_k) - \varepsilon\right)\left[\log^{[q-1]} r\right]^{\lambda_{g_l}^{(p,q)}(f_k)}\right\}\right], \tag{27}$$

$$i.e., \ T_{g_l}(r) \leq T_{f_k}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r}{\left(\overline{\tau}_{g_l}^{(p,q)}(f_k) - \varepsilon\right)}\right)^{\frac{1}{\lambda_{g_l}^{(p,q)}(f_k)}}\right)\right], \tag{28}$$

and

$$T_{f_k}(r) \leq T_{g_l}\left[\exp^{[p-1]}\left\{\left(\tau_{g_l}^{(p,q)}(f_k)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\lambda_{g_l}^{(p,q)}(f_k)}\right\}\right], \tag{29}$$

where $k = 1, 2$ and $l = 1, 2$.

Case I. Let $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ with at least f_2 is of regular relative (p, q) growth with respect to g_1. Also let $\varepsilon\,(> 0)$ be arbitrary.

Since $T_{f_1 \pm f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1)$ for all large r, we get from (24) and (29), for a sequence $\{r_n\}$ of values of r tending to infinity that

$$T_{f_1\pm f_2}(r_n) \leq T_{g_1}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r_n\right]^{\lambda_{g_1}^{(p,q)}(f_1)}\right\}\right](1+E)\ . \tag{30}$$

where $E = \dfrac{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q)}(f_2)+\varepsilon\right)\left[\log^{[q-1]} r_n\right]^{\lambda_{g_1}^{(p,q)}(f_2)}\right\}\right]+O(1)}{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r_n\right]^{\lambda_{g_1}^{(p,q)}(f_1)}\right\}\right]}$ and in view of $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$, we can make the term E sufficiently small by taking n sufficiently large. Now with the help of Theorem 1 and using the similar technique of Case I of Theorem 17, we get from (30) that

$$\tau_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \tau_{g_1}^{(p,q)}(f_1)\ . \tag{31}$$

Further, we may consider that $f = f_1 \pm f_2$. Also suppose that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ and at least f_2 is of regular relative (p, q) growth with respect to g_1. Then $\tau_{g_1}^{(p,q)}(f) = \tau_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \tau_{g_1}^{(p,q)}(f_1)$. Now let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 1, $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ and at least f_2 is of regular relative (p, q) growth with respect to g_1, we obtain that $\lambda_{g_1}^{(p,q)}(f) > \lambda_{g_1}^{(p,q)}(f_2)$ holds. Hence in view of (31), $\tau_{g_1}^{(p,q)}(f_1) \leq \tau_{g_1}^{(p,q)}(f) = \tau_{g_1}^{(p,q)}(f_1 \pm f_2)$. Therefore $\tau_{g_1}^{(p,q)}(f) = \tau_{g_1}^{(p,q)}(f_1) \Rightarrow \tau_{g_1}^{(p,q)}(f_1 \pm f_2) = \tau_{g_1}^{(p,q)}(f_1)$.

Similarly, if we consider $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$ with at least f_1 is of regular relative (p, q) growth with respect to g_1 then one can easily verify that $\tau_{g_1}^{(p,q)}(f_1 \pm f_2) = \tau_{g_1}^{(p,q)}(f_2)$.

Case II. Let us consider that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ with at least f_2 is of regular relative (p, q) growth with respect to g_1. Also let $\varepsilon\,(> 0)$ be arbitrary.

As $T_{f_1 \pm f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1)$ for all large r, we obtain from (24) for all sufficiently large values of r that

$$T_{f_1\pm f_2}(r) \leq T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\lambda_{g_1}^{(p,q)}(f_1)}\right\}\right](1+F)\ . \tag{32}$$

where $F = \dfrac{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q)}(f_2)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\lambda_{g_i}^{(p,q)}(f_2)}\right\}\right]+O(1)}{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r\right]^{\lambda_{g_1}^{(p,q)}(f_1)}\right\}\right]}$, and in view of $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$, we can make the term F sufficiently small by taking r sufficiently large and therefore for similar reasoning of Case I we get from (32) that $\overline{\tau}_{g_1}^{(p_i,q)}(f_1 \pm f_2) = \overline{\tau}_{g_1}^{(p,q)}(f_1)$ when $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ and at least f_2 is of regular relative (p, q) growth with respect to g_1.

Likewise, if we consider $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$ with at least f_1 is of regular relative (p,q) growth with respect to g_1 then one can easily verify that $\overline{\tau}_{g_1}^{(p,q)}(f_1 \pm f_2)$ $= \overline{\tau}_{g_1}^{(p,q)}(f_2)$

Thus combining Case I and Case II, we obtain the first part of the theorem.

Case III. Let us consider that $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$. Since $T_{g_1 \pm g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r, we get from (26) for all sufficiently large values of r that

$$T_{g_1 \pm g_2}(r) \leq T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r}{\left(\tau_{g_1}^{(p,q)}(f_1) - \varepsilon\right)}\right)^{\frac{1}{\tau_{g_1}^{(p,q)}(f_1)}}\right)\right](1+G) \ , \tag{33}$$

where $G = \dfrac{T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r}{\left(\tau_{g_2}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}\right)\right]+O(1)}{T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r}{\left(\tau_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\tau_{g_1}^{(p,q)}(f_1)}}\right)\right]}$, and since $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$, we can make the term G sufficiently small by taking r sufficiently large. Therefore in view of Theorem 3 and using the similar technique of Case III of Theorem 17, we get from (33) that

$$\tau_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \tau_{g_1}^{(p,q)}(f_1) \ . \tag{34}$$

Further, we may consider that $g = g_1 \pm g_2$. As $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$, so $\tau_g^{(p,q)}(f_1) = \tau_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \tau_{g_1}^{(p,q)}(f_1)$. Further let $g_1 = (g \pm g_2)$. Therefore in view of Theorem 3 and $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$ we obtain that $\lambda_g^{(p.q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$ holds. Hence in view of (34) $\tau_{g_1}^{(p,q)}(f_1) \geq \tau_g^{(p,q)}(f_1) = \tau_{g_1 \pm g_2}^{(p,q)}(f_1)$. Therefore $\tau_g^{(p,q)}(f_1) = \tau_{g_1}^{(p,q)}(f_1) \Rightarrow \tau_{g_1 \pm g_2}^{(p,q)}(f_1) = \tau_{g_1}^{(p,q)}(f_1)$.

Likewise, if we consider that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_2}^{(p,q)}(f_1)$, then one can easily verify that $\tau_{g_1 \pm g_2}^{(p,q)}(f_1) = \tau_{g_2}^{(p,q)}(f_1)$.

Case IV. In this case further we consider $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$.

As $T_{g_1 \pm g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r, we obtain from (26) and (28), for a sequence $\{r_n\}$ of values of r tending to infinity that

$$T_{g_1 \pm g_2}(r_n) \leq T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r_n}{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1) - \varepsilon\right)}\right)^{\frac{1}{\lambda_{g_1}^{(p,q)}(f_1)}}\right)\right](1+H) \ , \tag{35}$$

where $H = \dfrac{T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r_n}{\left(\tau_{g_2}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\lambda_{g_2}^{(p,q)}(f_1)}}\right)\right]+O(1)}{T_{f_1}\left[\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} r_n}{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\lambda_{g_1}^{(p,q)}(f_1)}}\right)\right]}$. Now in view of $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$, we can make the term H sufficiently small by taking n sufficiently large and therefore using the similar technique for as executed in the proof of Case IV of Theorem 17, we get from (35) that $\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_1}^{(p,q)}(f_1)$ when $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$.

Similarly, if we consider that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_2}^{(p,q)}(f_1)$, then one can easily verify that $\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_2}^{(p,q)}(f_1)$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 6 and the above cases.

In the next two theorems we reconsider the equalities in Theorem 1 to Theorem 4 under somewhat different conditions.

Theorem 19. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let p and q be two positive integers.
(A) The following condition is assumed to be satisfied:
(i) Either $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_1}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2)$ holds and g_1 has the Property (A), then

$$\rho_{g_1}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) \ .$$

(B) The following conditions are assumed to be satisfied:
(i) Either $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_2}^{(p,q)}(f_1)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1)$ holds and $g_1 \pm g_2$ has the Property (A);
(ii) f_1 is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2, then

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1) \ .$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions satisfying the conditions of the theorem.

Case I. Suppose that $\rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2)$ $(0 < \rho_{g_1}^{(p,q)}(f_1), \rho_{g_1}^{(p,q)}(f_2) < \infty)$. Now in view of Theorem 2 it is easy to see that $\rho_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2)$. If possible let

$$\rho_{g_1}^{(p,q)}(f_1 \pm f_2) < \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) \ . \tag{36}$$

Let $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_1}^{(p,q)}(f_2)$. Then in view of the first part of Theorem 17 and (36) we obtain that $\sigma_{g_1}^{(p,q)}(f_1) = \sigma_{g_1}^{(p,q)}(f_1 \pm f_2 \mp f_2) = \sigma_{g_1}^{(p,q)}(f_2)$ which is a contradiction. Hence $\rho_{g_1}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2)$. Similarly with the help of the first part of Theorem 17, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2)$. This proves the first part of the theorem.

Case II. Let us consider that $\rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1)$ $(0 < \rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1) < \infty)$, f_1 is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2 and $(g_1 \pm g_2)$ and $g_1 \pm g_2$ satisfy the Property (A). Therefore in view of Theorem 4, it follows that $\rho_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1)$ and if possible let

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1) \ . \tag{37}$$

Let us consider that $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_2}^{(p,q)}(f_1)$. Then. in view of the proof of the second part of Theorem 17 and (37) we obtain that $\sigma_{g_1}^{(p,q)}(f_1) = \sigma_{g_1 \pm g_2 \mp g_2}^{(p,q)}(f_1) = \sigma_{g_2}^{(p,q)}(f_1)$ which is a contradiction. Hence $\rho_{g_1 \pm g_2}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1)$. Also in view of the proof of second part of Theorem 17 one can derive the same conclusion for the condition $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1)$ and therefore the second part of the theorem is established.

Theorem 20. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let p and q be any two positive integers.
(A) The following conditions are assumed to be satisfied:
(i) $(f_1 \pm f_2)$ is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2, and g_1, g_2 , $g_1 \pm g_2$ have the Property (A);
(ii) Either $\sigma_{g_1}^{(p,q)}(f_1 \pm f_2) \neq \sigma_{g_2}^{(p,q)}(f_1 \pm f_2)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1 \pm f_2) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1 \pm f_2)$;
(iii) Either $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_1}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2)$;
(iv) Either $\sigma_{g_2}^{(p,q)}(f_1) \neq \sigma_{g_2}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_2}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_2)$; then

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) = \rho_{g_2}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_2) \; .$$

(B) The following conditions are assumed to be satisfied:
(i) f_1 and f_2 are of regular relative (p,q) growth with respect to at least any one of g_1 or g_2, and $g_1 \pm g_2$ has the Property (A);
(ii) Either $\sigma_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \sigma_{g_1 \pm g_2}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_2)$;
(iii) Either $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_2}^{(p,q)}(f_1)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1)$;
(iv) Either $\sigma_{g_1}^{(p,q)}(f_2) \neq \sigma_{g_2}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_2) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_2)$; then

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) = \rho_{g_2}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_2) \; .$$

We omit the proof of Theorem 20 as it is a natural consequence of Theorem 19.

Theorem 21. Let f_1, f_2 be ant two meromorphic functions and g_1,g_2 be any two entire functions.
(A) The following conditions are assumed to be satisfied:
(i) At least any one of f_1 or f_2 is of regular relative (p,q) growth with respect to g_1 where p and q are positive integers;
(ii) Either $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_1}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2)$ holds and g_1 has the Property (A), then

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) \; .$$

(B) The following conditions are assumed to be satisfied:
(i) f_1, g_1 and g_2 be any three entire functions such that $\lambda_{g_1}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_1)$ exists where p and q are positive integers;
(ii) Either $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_1)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ holds and $g_1 \pm g_2$ has the Property (A), then

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1) \; .$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions satisfying the conditions of the theorem.

Case I. Let $\lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2)$ $(0 < \lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2) < \infty)$ and at least f_1 or f_2 and $(f_1 \pm f_2)$ are of regular relative (p,q) growth with respect to g_1. Now, in view of Theorem 1, it is easy to see that $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2)$. If possible let

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) < \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) \; . \tag{38}$$

Let $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_1}^{(p,q)}(f_2)$. Then in view of the proof of the first part of Theorem 18 and (38) we obtain that $\tau_{g_1}^{(p,q)}(f_1) = \tau_{g_1}^{(p,q)}(f_1 \pm f_2 \mp f_2) = \tau_{g_1}^{(p,q)}(f_2)$ which is a contradiction. Hence $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2)$. Similarly in view of the proof of the first part of Theorem 18 , one can establish the same conclusion under the hypothesis $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2)$. This proves the first part of the theorem.

Case II. Let us consider that $\lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1)$ $(0 < \lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1) < \infty$. Therefore in view of Theorem 3, it follows that $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1)$ and if possible let

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1) \ . \tag{39}$$

Suppose $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_1)$. Then in view of the second part of Theorem 18 and (39), we obtain that $\tau_{g_1}^{(p,q)}(f_1) = \tau_{g_1 \pm g_2 \mp g_2}^{(p,q)}(f_1) = \tau_{g_2}^{(p,q)}(f_1)$ which is a contradiction. Hence $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1)$. Analogously with the help of the second part of Theorem 18, the same conclusion can also be derived under the condition $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ and therefore the second part of the theorem is established.

Theorem 22. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.
(A) The following conditions are assumed to be satisfied:
(i) At least any one of f_1 or f_2 is of regular relative (p,q) growth with respect to g_1 and g_2 where p and q are positive integers, and g_1, g_2, $g_1 \pm g_2$ have satisfy the Property (A);
(ii) Either $\tau_{g_1}^{(p,q)}(f_1 \pm f_2) \neq \tau_{g_2}^{(p,q)}(f_1 \pm f_2)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1 \pm f_2) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1 \pm f_2)$;
(iii) Either $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_1}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2)$;
(iv) Either $\tau_{g_2}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_2}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_2)$; then

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) = \lambda_{g_2}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_2) \ .$$

(B) The following conditions are assumed to be satisfied:
(i) At least any one of f_1 or f_2 are of regular relative (p,q) growth with respect to $g_1 \pm g_2$ where p and q are any two positive integers, and $g_1 \pm g_2$ has satisfy the Property (A);
(ii) Either $\tau_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \tau_{g_1 \pm g_2}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_2)$ holds;
(iii) Either $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_1)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ holds;
(iv) Either $\tau_{g_1}^{(p,q)}(f_2) \neq \tau_{g_2}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_2) \neq \overline{\tau}_{g_2}^{(p,q)}(f_2)$ holds, then

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) = \lambda_{g_2}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_2) \ .$$

We omit the proof of Theorem 22 as it is a natural consequence of Theorem 21.

Theorem 23. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $\rho_{g_1}^{(p,q)}(f_1)$, $\rho_{g_1}^{(p,q)}(f_2)$, $\rho_{g_2}^{(p,q)}(f_1)$ and $\rho_{g_2}^{(p,q)}(f_2)$ are all non zero and finite where p and q are positive integers.
(A) Assume the functions f_1, f_2 and g_1 satisfy the following conditions:
(i) $\rho_{g_1}^{(p,q)}(f_i) > \rho_{g_1}^{(p,q)}(f_j)$ for $i = j = 1,2$ and $i \neq j$;
(ii) g_1 satisfies the Property (A), then

$$\sigma_{g_1}^{(p,q)}(f_1 \cdot f_2) = \sigma_{g_1}^{(p,q)}(f_i) \mid i = 1,2 \text{ and } \overline{\sigma}_{g_1}^{(p,q)}(f_1 \cdot f_2) = \overline{\sigma}_{g_1}^{(p,q)}(f_i) \mid i = 1,2 \ .$$

Similarly,

$$\sigma_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right)=\sigma_{g_1}^{(p,q)}(f_i)\mid i=1,2 \text{ and } \overline{\sigma}_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right)=\overline{\sigma}_{g_1}^{(p,q)}(f_i)\mid i=1,2$$

holds provided (i) $\frac{f_1}{f_2}$ is meromorphic, (ii) $\rho_{g_1}^{(p,q)}(f_i)>\rho_{g_1}^{(p,q)}(f_j)\mid i=1,2;\ j=1,2;$ $i\neq j$ and (iii) g_1 satisfy the Property (A).

(B) Assume the functions g_1, g_2 and f_1 satisfy the following conditions:
(i) $\rho_{g_i}^{(p,q)}(f_1)<\rho_{g_j}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_j for $i=j=1,2$ and $i\neq j$, and g_i satisfy the Property (A);
(ii) $g_1\cdot g_2$ satisfy the Property (A), then

$$\sigma_{g_1\cdot g_2}^{(p,q)}(f_1)=\sigma_{g_i}^{(p,q)}(f_1)\mid i=1,2 \text{ and } \overline{\sigma}_{g_1\cdot g_2}^{(p,q)}(f_1)=\overline{\sigma}_{g_i}^{(p,q)}(f_1)\mid i=1,2\ .$$

Similarly,

$$\sigma_{\frac{g_1}{g_2}}^{(p,q)}(f_1)=\sigma_{g_i}^{(p,q)}(f_1)\mid i=1,2 \text{ and } \overline{\sigma}_{\frac{g_1}{g_2}}^{(p,q)}(f_1)=\overline{\sigma}_{gi}^{(p,q)}(f_1)\mid i=1,2$$

hold provided (i) $\frac{g_1}{g_2}$ is entire and satisfy the Property (A), (ii) At least f_1 is of regular relative (p,q) growth with respect to g_2, (iii) $\rho_{g_i}^{(p,q)}(f_1)<\rho_{g_j}^{(p,q)}(f_1)\mid i=1,2;\ j=1,2;\ i\neq j$ and (iv) g_1 satisfy the Property (A).

(C) Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:
(i) $g_1\cdot g_2$ satisfy the Property (A);
(ii) $\rho_{g_i}^{(p,q)}(f_1)<\rho_{g_j}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_j for $i=1,\ 2,\ j=1,2$ and $i\neq j$;
(iii) $\rho_{g_i}^{(p,q)}(f_2)<\rho_{g_j}^{(p,q)}(f_2)$ with at least f_2 is of regular relative (p,q) growth with respect to g_j for $i=1,\ 2,\ j=1,2$ and $i\neq j$;
(iv) $\rho_{g_1}^{(p,q)}(f_i)>\rho_{g_1}^{(p,q)}(f_j)$ and $\rho_{g_2}^{(p,q)}(f_i)>\rho_{g_2}^{(p,q)}(f_j)$ hold simultaneously for $i=1,2;\ j=1,2$ and $i\neq j$;
(v) $\rho_{g_m}^{(p,q)}(f_l)=\max\left[\min\left\{\rho_{g_1}^{(p,q)}(f_1),\rho_{g_2}^{(p,q)}(f_1)\right\},\min\left\{\rho_{g_1}^{(p,q)}(f_2),\rho_{g_2}^{(p,q)}(f_2)\right\}\right]\mid$ $l=m=1,2$; then

$$\sigma_{g_1\cdot g_2}^{(p,q)}(f_1\cdot f_2)=\sigma_{g_m}^{(p,q)}(f_l)\mid l=m=1,2 \text{ and}$$
$$\overline{\sigma}_{g_1\cdot g_2}^{(p,q)}(f_1\cdot f_2)=\overline{\sigma}_{g_m}^{(p,q)}(f_l)\mid l=m=1,2\ .$$

Similarly,

$$\sigma_{\frac{g_1}{g_2}}^{(p,q)}\left(\frac{f_1}{f_2}\right)=\sigma_{g_m}^{(p,q)}(f_l)\mid l=m=1,2 \text{ and}$$
$$\overline{\sigma}_{\frac{g_1}{g_2}}^{(p,q)}\left(\frac{f_1}{f_2}\right)=\overline{\sigma}_{g_m}^{(p,q)}(f_l)\mid l=m=1,2.$$

holds provided $\frac{f_1}{f_2}$ is meromorphic function and $\frac{g_1}{g_2}$ is entire function which satisfy the following conditions:
(i) $\frac{g_1}{g_2}$ satisfy the Property (A);
(ii) At least f_1 is of regular relative (p,q) growth with respect to g_2 and $\rho_{g_1}^{(p,q)}(f_1)\neq\rho_{g_2}^{(p,q)}(f_1)$;
(iii) At least f_2 is of regular relative (p,q) growth with respect to g_2 and $\rho_{g_1}^{(p,q)}(f_2)\neq\rho_{g_2}^{(p,q)}(f_2)$;
(iv) $\rho_{g_1}^{(p,q)}(f_i)<\rho_{g_1}^{(p,q)}(f_j)$ and $\rho_{g_2}^{(p,q)}(f_i)<\rho_{g_2}^{(p,q)}(f_j)$ holds simultaneously for

$i = 1, 2$; $j = 1, 2$ and $i \neq j$;
(v) $\rho_{g_m}^{(p,q)}(f_l) = \max\left[\min\left\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\right\}, \min\left\{\rho_{g_1}^{(p,q)}(f_2), \rho_{g_2}^{(p,q)}(f_2)\right\}\right] \mid l = m = 1, 2$.

Proof. Let us consider that $\rho_{g_1}^{(p,q)}(f_1)$, $\rho_{g_1}^{(p,q)}(f_2)$, $\rho_{g_2}^{(p,q)}(f_1)$ and $\rho_{g_2}^{(p,q)}(f_2)$ are all non zero and finite.

Case I. Suppose that $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$. Also let g_1 satisfy the Property (A). Since $T_{f_1 \cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$ for all large r, therefore applying the same procedure as adopted in Case I of Theorem 17 we get that

$$\sigma_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \sigma_{g_1}^{(p,q)}(f_1) \ . \tag{40}$$

Further without loss of any generality, let $f = f_1 \cdot f_2$ and $\rho_{g_1}^{(p,q)}(f_2) < \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f)$. Then in view of (40), we obtain that $\sigma_{g_1}^{(p,q)}(f) = \sigma_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \sigma_{g_1}^{(p,q)}(f_1)$. Also $f_1 = \frac{f}{f_2}$ and $T_{f_2}(r) = T_{\frac{1}{f_2}}(r) + O(1)$. Therefore $T_{f_1}(r) \leq T_f(r) + T_{f_2}(r) + O(1)$ and in this case also we obtain from (40) that $\sigma_{g_1}^{(p,q)}(f_1) \leq \sigma_{g_1}^{(p,q)}(f) = \sigma_{g_1}^{(p,q)}(f_1 \cdot f_2)$. Hence $\sigma_{g_1}^{(p,q)}(f) = \sigma_{g_1}^{(p,q)}(f_1) \Rightarrow \sigma_{g_1}^{(p,q)}(f_1 \cdot f_2) = \sigma_{g_1}^{(p,q)}(f_1)$.

Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2)$, then one can verify that $\sigma_{g_1}^{(p,q)}(f_1 \cdot f_2) = \sigma_{g_1}^{(p,q)}(f_2)$.

Next we may suppose that $f = \frac{f_1}{f_2}$ with f_1, f_2 and f are all meromorphic functions.

Sub Case $\mathbf{I_A}$. Let $\rho_{g_1}^{(p,q)}(f_2) < \rho_{g_1}^{(p,q)}(f_1)$. Therefore in view of Theorem 9, $\rho_{g_1}^{(p,q)}(f_2) < \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f)$. We have $f_1 = f \cdot f_2$. So, $\sigma_{g_1}^{(p,q)}(f_1) = \sigma_{g_1}^{(p,q)}(f) = \sigma_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right)$.

Sub Case $\mathbf{I_B}$. Let $\rho_{g_1}^{(p,q)}(f_2) > \rho_{g_1}^{(p,q)}(f_1)$. Therefore in view of Theorem 9, $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2) = \rho_{g_1}^{(p,q)}(f)$. Since $T_f(r) = T_{\frac{1}{f}}(r) + O(1) = T_{\frac{f_2}{f_1}}(r) + O(1)$, So $\sigma_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right) = \sigma_{g_1}^{(p,q)}(f_2)$.

Case II. Let $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$. Also let g_1 satisfy the Property (A). As $T_{f_1 \cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$ for all large r, therefore applying the same procedure as explored in Case II of Theorem 17, one can easily verify that $\overline{\sigma}_{g_1}^{(p,q)}(f_1 \cdot f_2) = \overline{\sigma}_{g_1}^{(p,q)}(f_1)$ and $\overline{\sigma}_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right) = \overline{\sigma}_{g_1}^{(p,q)}(f_i) \mid i = 1, 2$ under the conditions specified in the theorem.

Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2)$, then one can verify that $\overline{\sigma}_{g_1}^{(p,q)}(f_1 \cdot f_2) = \overline{\sigma}_{g_1}^{(p,q)}(f_2)$ and $\overline{\sigma}_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right) = \overline{\sigma}_{g_1}^{(p,q)}(f_2)$.

Therefore the first part of theorem follows from Case I and Case II.

Case III. Let $g_1 \cdot g_2$ satisfy the Property (A) and $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p, q) growth with respect to g_2. Since $T_{g_1 \cdot g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r)$ for all large r, therefore applying the same procedure as adopted in Case III of Theorem 17 we get that

$$\sigma_{g_1 \cdot g_2}^{(p,q)}(f_1) \geq \sigma_{g_1}^{(p,q)}(f_1) \ . \tag{41}$$

Further without loss of generality, let $g = g_1 \cdot g_2$ and $\rho_g^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$. Then in view of (41), we obtain that $\sigma_g^{(p,q)}(f_1) = \sigma_{g_1 \cdot g_2}^{(p,q)}(f_1) \geq \sigma_{g_1}^{(p,q)}(f_1)$. Also $g_1 = \frac{g}{g_2}$ and $T_{g_2}(r) = T_{\frac{1}{g_2}}(r) + O(1)$. Therefore $T_{g_1}(r) \leq T_g(r) + T_{g_2}(r) + O(1)$ and in this case we obtain from (41) that $\sigma_{g_1}^{(p,q)}(f_1) \geq \sigma_g^{(p,q)}(f_1) = \sigma_{g_1 \cdot g_2}^{(p,q)}(f_1)$. Hence $\sigma_g^{(p,q)}(f_1) = \sigma_{g_1}^{(p,q)}(f_1) \Rightarrow \sigma_{g_1 \cdot g_2}^{(p,q)}(f_1) = \sigma_{g_1}^{(p,q)}(f_1)$.

Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_1, then one can verify that $\sigma_{g_1 \cdot g_2}^{(p,q)}(f_1) = \sigma_{g_2}^{(p,q)}(f_1)$.

Next we may suppose that $g = \frac{g_1}{g_2}$, g_1, g_2, g are all entire functions satisfying the conditions specified in the theorem.

Sub Case $\mathrm{III_A}$. Let $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$. Therefore in view of Theorem 12, $\rho_g^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$. We have $g_1 = g \cdot g_2$. So $\sigma_{g_1}^{(p,q)}(f_1) = \sigma_g^{(p,q)}(f_1)$ $= \sigma_{\frac{g_1}{g_2}}^{(p,q)}(f_1)$.

Sub Case $\mathrm{III_B}$. Let $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_2}^{(p,q)}(f_1)$. Therefore in view of Theorem 12, $\rho_g^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_1)$. Since $T_g(r) = T_{\frac{1}{g}}(r) + O(1) = T_{\frac{g_2}{g_1}}(r) + O(1)$, So $\sigma_{\frac{g_1}{g_2}}^{(p,q)}(f_1) = \sigma_{g_2}^{(p,q)}(f_1)$.

Case IV. Suppose $g_1 \cdot g_2$ satisfy the Property (A). Also let $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_2. As $T_{g_1 \cdot g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r)$ for all large r, the same procedure as explored in Case IV of Theorem 17, one can easily verify that $\overline{\sigma}_{g_1 \cdot g_2}^{(p,q)}(f_1) = \overline{\sigma}_{g_1}^{(p,q)}(f_1)$ and $\overline{\sigma}_{\frac{g_1}{g_2}}^{(p,q)}(f_1) = \overline{\sigma}_{g_i}^{(p,q)}(f_1) \mid i = 1, 2$ under the conditions specified in the theorem.

Likewise, if we consider $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_1, then one can verify that $\overline{\sigma}_{g_1 \cdot g_2}^{(p,q)}(f_1) = \overline{\sigma}_{g_2}^{(p,q)}(f_1)$ and $\overline{\sigma}_{\frac{g_1}{g_2}}^{(p,q)}(f_1) = \overline{\sigma}_{g_2}^{(p,q)}(f_1)$. Therefore the second part of theorem follows from Case III and Case IV.

Proof of the third part of the theorem is omitted as it can be carried out in view of Theorem 13 and Theorem 15 and the above cases.

Theorem 24. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $\lambda_{g_1}^{(p,q)}(f_{1i})$, $\lambda_{g_1}^{(p,q)}(f_2)$, $\lambda_{g_2}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_2)$ be all non zero and finite where p and q are all positive integers.

(A) Assume the functions f_1, f_2 and g_1 satisfy the following conditions:

(i) $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) growth with respect to g_1 for $i = j = 1, 2$ and $i \neq j$;

(ii) g_1 satisfy the Property (A), then

$$\tau_{g_1}^{(p,q)}(f_1 \cdot f_2) = \tau_{g_1}^{(p,q)}(f_i) \mid i = 1, 2 \text{ and } \overline{\tau}_{g_1}^{(p,q)}(f_1 \cdot f_2) = \overline{\tau}_{g_1}^{(p,q)}(f_i) \mid i = 1, 2 .$$

Similarly,

$$\tau_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right) = \tau_{g_1}^{(p,q)}(f_i) \mid i = 1, 2 \text{ and } \overline{\tau}_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right) = \overline{\tau}_{g_1}^{(p,q)}(f_i) \mid i = 1, 2$$

holds provided $\frac{f_1}{f_2}$ is meromorphic, at least f_2 is of regular relative (p,q) growth with respect to g_1 where g_1 satisfy the Property (A) and $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j) \mid i$

$= 1, 2$; $j = 1, 2$; $i \neq j$.
(B) Assume the functions g_1, g_2 and f_1 satisfy the following conditions:
(i) $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1)$ for $i = j = 1, 2$, $i \neq j$; and g_i satisfy the Property (A)
(ii) $g_1 \cdot g_2$ satisfy the Property (A), then

$$\tau_{g_1 \cdot g_2}^{(p,q)}(f_1) = \tau_{g_i}^{(p,q)}(f_1) \mid i = 1, 2 \text{ and } \overline{\tau}_{g_1 \cdot g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_i}^{(p,q)}(f_1) \mid i = 1, 2 \ .$$

Similarly,

$$\tau_{\frac{g_1}{g_2}}^{(p,q)}(f_1) = \tau_{g_i}^{(p,q)}(f_1) \mid i = 1, 2 \text{ and } \overline{\tau}_{\frac{g_1}{g_2}}^{(p,q)}(f_1) = \overline{\tau}_{gi}^{(p,q)}(f_1) \mid i = 1, 2$$

holds provided $\frac{g_1}{g_2}$ is entire and satisfy the Property (A), g_1 satisfy the Property (A) and $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1) \mid i = 1, 2$; $j = 1, 2$; $i \neq j$.
(C) Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:
(i) $g_1 \cdot g_2$, g_1 and g_2 are satisfy the Property (A);
(ii) $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p, q) growth with respect to g_1 for $i = 1$, 2, $j = 1, 2$ and $i \neq j$;
(iii) $\lambda_{g_2}^{(p,q)}(f_i) > \lambda_{g_2}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p, q) growth with respect to g_2 for $i = 1$, 2, $j = 1, 2$ and $i \neq j$;
(iv) $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1)$ and $\lambda_{g_i}^{(p,q)}(f_2) < \lambda_{g_j}^{(p,q)}(f_2)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$;
(v) $\lambda_{g_m}^{(p,q)}(f_l) = \min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\right\}\right] \mid$ $l = m = 1, 2$; then

$$\tau_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2) = \tau_{g_m}^{(p,q)}(f_l) \mid l = m = 1, 2 \text{ and}$$
$$\overline{\tau}_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2) = \overline{\tau}_{g_m}^{(p,q)}(f_l) \mid l = m = 1, 2 \ .$$

Similarly,

$$\tau_{\frac{g_1}{g_2}}^{(p,q)}\left(\frac{f_1}{f_2}\right) = \tau_{g_m}^{(p,q)}(f_l) \mid l = m = 1, 2 \text{ and}$$
$$\overline{\tau}_{\frac{g_1}{g_2}}^{(p,q)}\left(\frac{f_1}{f_2}\right) = \overline{\tau}_{g_m}^{(p,q)}(f_l) \mid l = m = 1, 2 \ .$$

holds provided $\frac{f_1}{f_2}$ is meromorphic and $\frac{g_1}{g_2}$ is entire functions which satisfy the following conditions:
(i) $\frac{g_1}{g_2}$, g_1 and g_2 satisfy the Property (A);
(ii) At least f_2 is of regular relative (p, q) growth with respect to g_1 and $\lambda_{g_1}^{(p,q)}(f_1) \neq \lambda_{g_1}^{(p,q)}(f_2)$;
(iii) At least f_2 is of regular relative (p, q) growth with respect to g_2 and $\lambda_{g_2}^{(p,q)}(f_1) \neq \lambda_{g_2}^{(p,q)}(f_2)$;
(iv) $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1)$ and $\lambda_{g_i}^{(p,q)}(f_2) < \lambda_{g_j}^{(p,q)}(f_2)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$;
(v) $\lambda_{g_m}^{(p,q)}(f_l) = \min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\right\}\right] \mid$ $l = m = 1, 2$.

Proof. Let us consider that $\lambda_{g_1}^{(p,q)}(f_1)$, $\lambda_{g_1}^{(p,q)}(f_2)$, $\lambda_{g_2}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_2)$ are all non zero and finite.
Case I. Suppose $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ with at least f_2 is of regular relative (p, q) growth with respect to g_1 and g_1 satisfy the Property (A). Since $T_{f_1 \cdot f_2}(r) \leq$

$T_{f_1}(r)+T_{f_2}(r)$ for all large r, therefore applying the same procedure as adopted in Case I of Theorem 18 we get that

$$\tau_{g_1}^{(p,q)}(f_1\cdot f_2)\leq\tau_{g_1}^{(p,q)}(f_1)\ . \tag{42}$$

Further without loss of generality, let $f=f_1\cdot f_2$ and $\lambda_{g_1}^{(p,q)}(f_2)<\lambda_{g_1}^{(p,q)}(f_1) =\lambda_{g_1}^{(p,q)}(f)$. Then in view of (42), we obtain that $\tau_{g_1}^{(p,q)}(f)=\tau_{g_1}^{(p,q)}(f_1\cdot f_2)\leq \tau_{g_1}^{(p,q)}(f_1)$. Also $f_1=\frac{f}{f_2}$ and $T_{f_2}(r)=T_{\frac{1}{f_2}}(r)+O(1)$. Therefore $T_{f_1}(r)\leq T_f(r)+T_{f_2}(r)+O(1)$ and in this case we obtain from the above arguments that $\tau_{g_1}^{(p,q)}(f_1) \leq\tau_{g_1}^{(p,q)}(f)=\tau_{g_1}^{(p,q)}(f_1\cdot f_2)$. Hence $\tau_{g_1}^{(p,q)}(f)=\tau_{g_1}^{(p,q)}(f_1)\Rightarrow\tau_{g_1}^{(p,q)}(f_1\cdot f_2)= \tau_{g_1}^{(p,q)}(f_1)$.

Similarly, if we consider $\lambda_{g_1}^{(p,q)}(f_1)<\lambda_{g_1}^{(p,q)}(f_2)$ with at least f_1 is of regular relative (p,q) growth with respect to g_1, then one can easily verify that $\tau_{g_1}^{(p,q)}(f_1\cdot f_2) =\tau_{g_1}^{(p,q)}(f_2)$.

Next we may suppose that $f=\frac{f_1}{f_2}$ with f_1, f_2 and f are all meromorphic functions satisfying the conditions specified in the theorem.

Sub Case $\mathbf{I_A}$. Let $\lambda_{g_1}^{(p,q)}(f_2)<\lambda_{g_1}^{(p,q)}(f_1)$. Therefore in view of Theorem 8, $\lambda_{g_1}^{(p,q)}(f_2)<\lambda_{g_1}^{(p,q)}(f_1)=\lambda_{g_1}^{(p,q)}(f)$. We have $f_1=f\cdot f_2$. So $\tau_{g_1}^{(p,q)}(f_1)=\tau_{g_1}^{(p,q)}(f) =\tau_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right)$.

Sub Case $\mathbf{I_B}$. Let $\lambda_{g_1}^{(p,q)}(f_2)>\lambda_{g_1}^{(p,q)}(f_1)$. Therefore in view of Theorem 8, $\lambda_{g_1}^{(p,q)}(f_1)<\lambda_{g_1}^{(p,q)}(f_2)=\lambda_{g_1}^{(p,q)}(f)$. Since $T_f(r)=T_{\frac{1}{f}}(r)+O(1)=T_{\frac{f_2}{f_1}}(r)+O(1)$, So $\tau_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}\right)=\tau_{g_1}^{(p,q)}(f_2)$.

Case II. Let $\lambda_{g_1}^{(p,q)}(f_1)>\lambda_{g_1}^{(p,q)}(f_2)$ with at least f_2 is of regular relative (p,q) growth with respect to g_1 where g_1 satisfy the Property (A). As $T_{f_1\cdot f_2}(r)\leq T_{f_1}(r)+T_{f_2}(r)$ for all large r, so applying the same procedure as adopted in Case II of Theorem 18 we can easily verify that $\overline{\tau}_{g_1}^{(p,q)}(f_1\cdot f_2)=\overline{\tau}_{g_1}^{(p,q)}(f_1)$ and $\overline{\tau}_{\frac{g_1}{g_2}}^{(p,q)}(f_1)= \overline{\tau}_{gi}^{(p,q)}(f_1)\mid i=1,2$ under the conditions specified in the theorem.

Similarly, if we consider $\lambda_{g_1}^{(p,q)}(f_1)<\lambda_{g_1}^{(p,q)}(f_2)$ with at least f_1 is of regular relative (p,q) growth with respect to g_1, then one can easily verify that $\overline{\tau}_{g_1}^{(p,q)}(f_1\cdot f_2) =\overline{\tau}_{g_1}^{(p,q)}(f_2)$.

Therefore the first part of theorem follows Case I and Case II.

Case III. Let $\lambda_{g_1}^{(p,q)}(f_1)<\lambda_{g_2}^{(p,q)}(f_1)$ and $g_1\cdot g_2$ satisfy the Property (A).Since $T_{g_1\cdot g_2}(r)\leq T_{g_1}(r)+T_{g_2}(r)$ for all large r, therefore applying the same procedure as adopted in Case III of Theorem 18 we get that

$$\tau_{g_1\cdot g_2}^{(p,q)}(f_1)\leq\tau_{g_1}^{(p,q)}(f_1)\ . \tag{43}$$

Further without loss of generality, let $g=g_1\cdot g_2$ and $\lambda_{g}^{(p,q)}(f_1)=\lambda_{g_1}^{(p,q)}(f_1)< \lambda_{g_2}^{(p,q)}(f_1)$. Then in view of (43), we obtain that $\tau_{g}^{(p,q)}(f_1)=\tau_{g_1\cdot g_2}^{(p,q)}(f_1)\geq\tau_{g_1}^{(p,q)}(f_1)$. Also $g_1=\frac{g}{g_2}$ and $T_{g_2}(r)=T_{\frac{1}{g_2}}(r)+O(1)$. Therefore $T_{g_1}(r)\leq T_g(r)+T_{g_2}(r)+O(1)$ and in this case we obtain from above arguments that $\tau_{g_1}^{(p,q)}(f_1)\geq\ \tau_{g}^{(p,q)}(f_1)= \tau_{g_1\cdot g_2}^{(p,q)}(f_1)$. Hence $\tau_{g}^{(p,q)}(f_1)=\tau_{g_1}^{(p,q)}(f_1)\Rightarrow\tau_{g_1\cdot g_2}^{(p,q)}(f_1)=\tau_{g_1}^{(p,q)}(f_1)$.

If $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_2}^{(p,q)}(f_1)$, then one can easily verify that $\tau_{g_1 \cdot g_2}^{(p,q)}(f_1) = \tau_{g_2}^{(p,q)}(f_1)$.

Next we may suppose that $g = \frac{g_1}{g_2}$ with g_1, g_2, g are all entire functions satisfying the conditions specified in the theorem.

Sub Case III$_\mathbf{A}$. Let $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$. Therefore in view of Theorem 10, $\lambda_{g}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$. We have $g_1 = g \cdot g_2$. So $\tau_{g_1}^{(p,q)}(f_1) = \tau_{g}^{(p,q)}(f_1)$ $= \tau_{\frac{g_1}{g_2}}^{(p,q)}(f_1)$.

Sub Case III$_\mathbf{B}$. Let $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_2}^{(p,q)}(f_1)$. Therefore in view of Theorem 10, $\lambda_{g}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_1)$. Since $T_g(r) = T_{\frac{1}{g}}(r) + O(1) = T_{\frac{g_2}{g_1}}(r) + O(1)$, So $\tau_{\frac{g_1}{g_2}}^{(p,q)}(f_1) = \tau_{g_2}^{(p,q)}(f_1)$.

Case IV. Suppose $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$ and $g_1 \cdot g_2$ satisfy the Property (A). Since $T_{g_1 \cdot g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r)$ for all large r, then adopting the same procedure as of Case IV of Theorem 18, we obtain that $\overline{\tau}_{g_1 \cdot g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_1}^{(p,q)}(f_1)$ and $\overline{\tau}_{\frac{g_1}{g_2}}^{(p,q)}(f_1) = \overline{\tau}_{gi}^{(p,q)}(f_1) \mid i = 1, 2$.

Similarly if we consider that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_2}^{(p,q)}(f_1)$, then one can easily verify that $\overline{\tau}_{g_1 \cdot g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_2}^{(p,q)}(f_1)$.

Therefore the second part of the theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 14 , Theorem 16 and the above cases.

Theorem 25. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let p and q be any two positive integers.

(A) The following condition is assumed to be satisfied:

(i) Either $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_1}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2)$ holds;

(ii) g_1 satisfies the Property (A), then

$$\rho_{g_1}^{(p,q)}(f_1 \cdot f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) \ .$$

(B) The following conditions are assumed to be satisfied:

(i) Either $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_2}^{(p,q)}(f_1)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1)$ holds;

(ii) f_1 is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2. Also $g_1 \cdot g_2$ satisfy the Property (A). Then we have

$$\rho_{g_1 \cdot g_2}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1) \ .$$

Proof. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions satisfying the conditions of the theorem.

Case I. Suppose that $\rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2)$ $(0 < \rho_{g_1}^{(p,q)}(f_1), \rho_{g_1}^{(p,q)}(f_2) < \infty)$ and g_1 satisfy the Property (A). Now in view of Theorem 9, it is easy to see that $\rho_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2)$. If possible let

$$\rho_{g_1}^{(p,q)}(f_1 \cdot f_2) < \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) \ . \tag{44}$$

Let $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_1}^{(p,q)}(f_2)$. Now in view of the first part of Theorem 23 and (44) we obtain that $\sigma_{g_1}^{(p,q)}(f_1) = \sigma_{g_1}^{(p,q)}\left(\frac{f_1 \cdot f_2}{f_2}\right) = \sigma_{g_1}^{(p,q)}(f_2)$ which is a contradiction.

Hence $\rho_{g_1}^{(p,q)}(f_1 \cdot f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2)$. Similarly with the help of the first part of Theorem 23, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2)$. This prove the first part of the theorem.

Case II. Let us consider that $\rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1)$ $(0 < \rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1) < \infty)$, f_1 is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2. Also $g_1 \cdot g_2$ satisfy the Property (A). Therefore in view of Theorem 11, it follows that $\rho_{g_1 \cdot g_2}^{(p,q)}(f_1) \geq \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1)$ and if possible let

$$\rho_{g_1 \cdot g_2}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1) \ . \tag{45}$$

Further suppose that $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_2}^{(p,q)}(f_1)$. Therefore in view of the proof of the second part of Theorem 23 and (45), we obtain that $\sigma_{g_1}^{(p,q)}(f_1) = \sigma_{\frac{g_1 \cdot g_2}{g_2}}^{(p,q)}(f_1)$ $= \sigma_{g_2}^{(p,q)}(f_1)$ which is a contradiction. Hence $\rho_{g_1 \cdot g_2}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1)$. Likewise in view of the proof of second part of Theorem 23, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1)$. This proves the second part of the theorem.

Theorem 26. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let p and q be any two positive integers.

(A) The following conditions are assumed to be satisfied:

(i) $(f_1 \cdot f_2)$ is of regular relative (p,q) growth with respect to at least any one g_1 or g_2;

(ii) $(g_1 \cdot g_2)$, g_1 and g_2 all satisfy the Property (A);

(iii) Either $\sigma_{g_1}^{(p,q)}(f_1 \cdot f_2) \neq \sigma_{g_2}^{(p,q)}(f_1 \cdot f_2)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1 \cdot f_2) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1 \cdot f_2)$;

(iv) Either $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_1}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2)$;

(v) Either $\sigma_{g_2}^{(p,q)}(f_1) \neq \sigma_{g_2}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_2}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_2)$; then

$$\rho_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) = \rho_{g_2}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_2) \ .$$

(B) The following conditions are assumed to be satisfied:

(i) $(g_1 \cdot g_2)$ satisfy the Property (A);

(ii) f_1 and f_2 are of regular relative (p,q) growth with respect to at least any one g_1 or g_2;

(iii) Either $\sigma_{g_1 \cdot g_2}^{(p,q)}(f_1) \neq \sigma_{g_1 \cdot g_2}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_1 \cdot g_2}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_1 \cdot g_2}^{(p,q)}(f_2)$;

(iv) Either $\sigma_{g_1}^{(p,q)}(f_1) \neq \sigma_{g_2}^{(p,q)}(f_1)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1)$;

(v) Either $\sigma_{g_1}^{(p,q)}(f_2) \neq \sigma_{g_2}^{(p,q)}(f_2)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_2) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_2)$; then

$$\rho_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) = \rho_{g_2}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_2) \ .$$

We omit the proof of Theorem 26 as it is a natural consequence of Theorem 25.

Theorem 27. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.

(A) The following conditions are assumed to be satisfied:

(i) At least any one of f_1 or f_2 are of regular relative (p,q) growth with respect to g_1 where p and q are any two positive integers;

(ii) Either $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_1}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2)$ holds.

(iii) g_1 satisfy the Property (A), then

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) \ .$$

(B) The following conditions are assumed to be satisfied:
(i) f_1 be any meromorphic function and g_1, g_2 be any two entire functions such that $\lambda_{g_1}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_1)$ exist where p and q are any two positive integers and $g_1 \cdot g_2$ satisfy the Property (A);
(ii) Either $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_1)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ holds, then

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1) .$$

Proof. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions satisfy the conditions of the theorem.

Case I. Let $\lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2)$ $(0 < \lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2) < \infty)$, g_1 satisfy the Property (A) and at least f_1 or f_2 is of regular relative (p,q) growth with respect to g_1. Now in view of Theorem 7 it is easy to see that $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \lambda_{g_1}^{(p,q)}(f_1)$ $= \lambda_{g_1}^{(p,q)}(f_2)$. If possible let

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) < \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) . \tag{46}$$

Also let $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_1}^{(p,q)}(f_2)$. Then in view of the proof of first part of Theorem 24 and (46), we obtain that $\tau_{g_1}^{(p,q)}(f_1) = \tau_{g_1}^{(p,q)}\left(\frac{f_1 \cdot f_2}{f_2}\right) = \tau_{g_1}^{(p,q)}(f_2)$ which is a contradiction. Hence $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2)$. Analogously, in view of the proof of first part of Theorem 24, one can derive the same conclusion under the hypothesis $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2)$. Hence the first part of the theorem is established.
Case II. Let us consider that $\lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1)$ $(0 < \lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1) <$ ∞ and $g_1 \cdot g_2$ satisfy the Property (A). Therefore in view of Theorem 10, it follows that $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) \geq \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1)$ and if possible let

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1) . \tag{47}$$

Further let $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_1)$. Then in view of second part of Theorem 24 and (47), we obtain that $\tau_{g_1}^{(p,q)}(f_1) = \tau_{\frac{g_1 \cdot g_2}{g_2}}^{(p,q)}(f_1) = \tau_{g_2}^{(p,q)}(f_1)$ which is a contradiction. Hence $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1)$. Similarly by the second part of Theorem 24, we get the same conclusion when $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ and therefore the second part of the theorem follows.

Theorem 28. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.
(A) The following conditions are assumed to be satisfied:
(i) $g_1 \cdot g_2$, g_1 and g_2 satisfy the Property (A);
(ii) At least any one of f_1 or f_2 are of regular relative (p,q) growth with respect to g_1 and g_2 where p and q are positive integers;
(iii) Either $\tau_{g_1}^{(p,q)}(f_1 \cdot f_2) \neq \tau_{g_2}^{(p,q)}(f_1 \cdot f_2)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1 \cdot f_2) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1 \cdot f_2)$;
(iv) Either $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_1}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2)$;
(v) Either $\tau_{g_2}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_2}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_2)$; then

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) = \lambda_{g_2}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_2) .$$

(B) The following conditions are assumed to be satisfied:
(i) $g_1 \cdot g_2$ satisfy the Property (A);
(ii) At least any one of f_1 or f_2 are of regular relative (p,q) growth with respect to

$g_1 \cdot g_2$ where p and q are positive integers;
(iii) Either $\tau^{(p,q)}_{g_1 \cdot g_2}(f_1) \neq \tau^{(p,q)}_{g_1 \cdot g_2}(f_2)$ or $\overline{\tau}^{(p,q)}_{g_1 \cdot g_2}(f_1) \neq \overline{\tau}^{(p,q)}_{g_1 \cdot g_2}(f_2)$ holds;
(iv) Either $\tau^{(p,q)}_{g_1}(f_1) \neq \tau^{(p,q)}_{g_2}(f_1)$ or $\overline{\tau}^{(p,q)}_{g_1}(f_1) \neq \overline{\tau}^{(p,q)}_{g_2}(f_1)$ holds;
(v) Either $\tau^{(p,q)}_{g_1}(f_2) \neq \tau^{(p,q)}_{g_2}(f_2)$ or $\overline{\tau}^{(p,q)}_{g_1}(f_2) \neq \overline{\tau}^{(p,q)}_{g_2}(f_2)$ holds, then

$$\lambda^{(p,q)}_{g_1 \cdot g_2}(f_1 \cdot f_2) = \lambda^{(p,q)}_{g_1}(f_1) = \lambda^{(p,q)}_{g_1}(f_2) = \lambda^{(p,q)}_{g_2}(f_1) = \lambda^{(p,q)}_{g_2}(f_2) \ .$$

We omit the proof of Theorem 28 as it is a natural consequence of Theorem 27.

Remark 2. If we take $\frac{f_1}{f_2}$ instead of $f_1 \cdot f_2$ and $\frac{g_1}{g_2}$ instead of $g_1 \cdot g_2$ where $\frac{f_1}{f_2}$ is meromorphic and $\frac{g_1}{g_2}$ is entire function, and the other conditions of *Theorem 25*, *Theorem 26*, *Theorem 27* and *Theorem 28* remain the same, then conclusion of *Theorem 25*, *Theorem 26*, *Theorem 27* and *Theorem 28* remains valid.

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References

[1] L. Bernal, Orden relative de crecimiento de funciones enteras , *Collect. Math.* 39 (1988), 209-229.
[2] D. Banerjee and S. Jana, Meromorphic functions of relative order (p, q), *Soochow J. Math.*, 33(3) (2007), 343-357.
[3] L. Debnath, S. K. Datta, T. Biswas and A. Kar, Growth of meromorphic functions depending on (p,q)-th relative order, *Facta Univ. Ser. Math. Inform.*, 31(3) (2016), 691-705.
[4] S. K. Datta and T. Biswas, On the generalized relative type and generalized relative weak type related growth analysis of entire functions, *J. Fract. Calc. Appl.*, 8(2) (2017), 101-113.
[5] S. K. Datta and T. Biswas, On the growth measures of entire and meromorphic functions focusing their generalized relative type and generalized relative weak type, *Acta Univ. Sapientiae, Mathematica*, 9(1) (2017), 53-73.
[6] W.K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford (1964).
[7] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, On the (p,q)-order and lower (p,q)-order of an entire function, *J. Reine Angew. Math.*, 282 (1976), 53-67.
[8] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, On the (p,q)-type and lower (p,q)-type of an entire function, *J. Reine Angew. Math.*, 290 (1977), 180-190.
[9] B. K. Lahiri and D. Banerjee, Relative order of entire and meromorphic functions, *Proc. Nat. Acad. Sci. India Ser. A.*, 69(A) (3) (1999), 339-354.
[10] I. Laine, Nevanlinna Theory and Complex Differential Equations. De Gruyter, Berlin, 1993.
[11] D. Sato, On the rate of growth of entire functions of fast growth, *Bull. Amer. Math. Soc.*, 69 (1963), 411-414.
[12] C. C. Yang and H. X. Yi, Uniqueness theory of meromorphic functions, Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht, 2003.
[13] L. Yang, Value distribution theory, Springer-Verlag, Berlin, 1993.
[14] G. Valiron, Lectures on the general theory of integral functions, Chelsea Publishing Company, 1949.

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