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SOLUTIONS FOR SINGULAR KIRCHHOFF PROBLEM INVOLVING CRITICAL NONLINEARITY

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Abstract. This paper deals with a class of singular Kirchhoff problem involving a critical nonlinearity. The existence and multiplicity of solutions for this problem are obtained by the variational methods.

1. Introduction

In this paper, we are concerned with the existence and multiplicity of solutions to the following Kirchhoff problem with the critical Sobolev exponent

$$(\mathcal{P}_\lambda)\begin{cases} -M(\|u\|_\mu^2)\left(\Delta u+\mu\dfrac{u}{|x|^2}\right)=u^5+\lambda g(x) & \text{in } \mathbb{R}^3\\ u\in H_\mu\left(\mathbb{R}^3\right), \end{cases}$$

where $M(t)=at+b$, a and b are two positive constants, λ is a positive parameter, $\mu<1/4$, $\|u\|_\mu^2=\int_{\mathbb{R}^3}(|\nabla u|^2-\mu\dfrac{u^2}{|x|^2})dx$ is the norm in $H_\mu\left(\mathbb{R}^3\right)$ and g belongs to $H^{-1}\left(\mathbb{R}^3\right)$, ($H^{-1}\left(\mathbb{R}^3\right)$ is the dual of $H_\mu\left(\mathbb{R}^3\right)$).

Such problems are frequently called nonlocal because the function M contains an integral over the domain $\mathbb{R}^3$ which implies that the equation in $(\mathcal{P}_\lambda)$ is no longer a pointwise identity.

The original one-dimensional Kirchhoff equation was first introduced by Kirchhoff [11] in 1883, he take into account the changes in length of the strings produced by transverse vibrations.

The problem $(\mathcal{P}_\lambda)$ is also related to the stationary analogue of the following evolutionary higher order problem which can been considered as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings:

$$\begin{cases} u_{tt}-(a\int_\Omega|\nabla u|^2\,dx+b)\triangle u=h(x,u) & \text{in } \Omega\times(0,\ T),\\ u=0 & \text{on } \partial\Omega\times(0,\ T),\\ u(x,\ 0)=u_0(x),\ u_t(x,\ 0)=u_1(x), \end{cases}$$

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where $\Omega \subset \mathbb{R}^N$ is an open bounded domain $(N \geq 1)$, T is a positive constant, $u_0,$ u_1 are given functions. In such problems, u denotes the displacement, $h(x, u)$ the external force, b is the initial tension and a is related to the intrinsic properties of the strings (such as Young's modulus). For more details, we refer the readers to the works [3], [4] and the references therein.

It is well known that the Kirchhoff type problem has mechanical and biological motivations; for example when an elastic string with fixed ends is subjected to transverse vibrations. They also serve as model in biological systems where u describes a process depending on the average of itself as population density. The presence of the nonlocal term makes the theorical study of these problems so difficult, then they have attracted the attention of many researchers in particular after the work of Lions [12], where a functional analysis approach was proposed to attack them.

In recent years, the existence and multiplicity of solutions for stationary problems of Kirchhoff type were also investigated in some papers, via variational methods like the Ekeland variational principle and the Mountain Pass Theorem. Some interesting results in bounded domains can be found in [1, 5, 7, 9, 14, 15].

In the regular case and in the unbounded domain $\mathbb{R}^N$, some earlier classical investigations of the following Kirchhoff equations

$$(\mathcal{P}_{V,g}) \left\{ \; -M\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + V\left(x\right) u = g(x,u) \quad \text{in } \mathbb{R}^N \right.$$

have been done, where $N \geq 3$, $M(t) = at + b$, $a > 0$, b is a positive constants, $V \in C\left(\mathbb{R}^N, \mathbb{R}\right)$ and $g \in C\left(\mathbb{R}^N \times \mathbb{R},\ \mathbb{R}\right)$ is subcritical and satisfies sufficient conditions to ensure the boundedness of any Palais Smale or Cerami sequence. Such problems become more complicated since the Sobolev embedding $\left(H^1(\mathbb{R}^N),\ \|.\|\right) \hookrightarrow \left(L^p\left(\mathbb{R}^N\right),\ |.|_p\right)$ is not compact for all $p \in [2,\ 2_*\left(N\right)]$, where $\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{1/2}$ is the standard norm in $H^1\left(\mathbb{R}^N\right)$, $|u|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{1/p}$ is the norm in $L^p\left(\mathbb{R}^N\right)$ and $2_*\left(N\right)$ is the critical Sobolev exponent.

To overcome the lack of compactness of the Sobolev embedding, many authors imposed some conditions on the potential function $V\left(x\right)$ for example in [18], Wu used the following assumption:

$$(*) \quad \inf_{\mathbb{R}^N} V\left(x\right) \geq c > 0 \text{ and for all } d > 0;\ meas\left\{x \in \mathbb{R}^N : V\left(x\right) \leq d\right\} < \infty,$$

to show the existence of nontrivial solutions to $(\mathcal{P}_{V,h})$. On the other hand, Chena and Li in [8] studied $(\mathcal{P}_{V,g})$ where $g(x,u) = h\left(x,u\right) + k\left(x\right)$, h satisfies the Ambrosetti–Rabinowitz type condition, $k \in L^2\left(\mathbb{R}^3\right)$ and V verifies $(*)$. They proved the existence of multiple solutions by using Ekeland's variational principle [10] and the Mountain Pass Theorem [2]. Recently, Li et al. [13] studied $(\mathcal{P}_{V,g})$ where $V \equiv 0$, they proved the existence of a constant $a_0 > 0$ such that $(\mathcal{P}_{0,g})$ admits a positive solution for all $a \in (0, a_0)$.

However, from the results mentioned above, there are very few existence results for singular nonlocal type problems (when $\mu > 0$) in particular for those who contain singularity in the diverge operator. This is a more difficult and interesting situation comparing with the regular case (when $\mu = 0$). Moreover, the main difficulties in such problem appear in the fact that for nonlocal problems with critical exponent, to overcome the lack of compactness, we need to determine a good level of the Palais-Smale and have to verify that the critical value is contained in the range of this level.

The main result of this paper is given in the following theorems.

Theorem 1 . Let $a > 0$, $b > 0$, $\mu < 1/4$ and $g \not\equiv 0$. Then there exists $\Lambda_1 > 0$ such that problem $(\mathcal{P}_\lambda)$ has at least one nontrivial solution for any $\lambda \in (0, \Lambda_1)$.

Theorem 2 . Let $a > 0$, $b > 0$, $\mu < 1/4$ and $g \not\equiv 0$. Then there exists $\Lambda_2 > 0$ with $\Lambda_2 \leq \Lambda_1$ such that problem $(\mathcal{P}_\lambda)$ has at least two nontrivial solutions for any $\lambda \in (0, \Lambda_2)$.

Here we give a brief sketch of the way how we get two distinct critical points of the energy functional. First, we minimize the functional in a neighborhood of zero and use the Ekeland variational principle to find the first critical point which achieves a local minimum. Moreover, the level of this local minimum is negative. Next around the zero point, using the Mountain Pass Theorem we also obtain a critical point whose level is positive.

This paper is organized as follows. In Section 2, we give some notations and technical results which allow us to give a variational approach of our main results that we prove in Section 3.

2. Auxiliary results

To start this section, we need to introduce the following notation.
$||.||_-$ denotes the norm in $H^{-1}(\mathbb{R}^3)$, B_ρ is the ball centred at 0 and of radius ρ, and $\circ_n(1)$ denotes $\circ_n(1) \to 0$ as $n \to \infty$.

Define the constant

$$S_\mu := \inf \left\{ \int_{\mathbb{R}^3} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx; \ u \in H_\mu(\mathbb{R}^3), \ \int_{\mathbb{R}^3} u^6 dx = 1 \right\}$$

It is well known that the embedding $\left(H_\mu(\mathbb{R}^3), \ \|.\|_\mu\right) \hookrightarrow \left(L^6(\mathbb{R}^3), \ |.|_p\right)$ is continuous but not compact and S_μ is achieved by a family of functions

$$U_\varepsilon(x) = \frac{\left[12\varepsilon\left(\frac{1}{4} - \mu\right)\right]^{\frac{1}{4}}}{\left[\varepsilon |x|^{1-2\sqrt{\frac{1}{4}-\mu}} + |x|^{1+2\sqrt{\frac{1}{4}-\mu}}\right]^{\frac{1}{2}}}, \ \varepsilon > 0,$$

see [17]. Moreover, there holds

$$\Delta U_\varepsilon + \mu \frac{U_\varepsilon}{|x|^2} = U_\varepsilon^5 \quad \text{in } \mathbb{R}^3 / \{0\},$$

and

$$\int_{\mathbb{R}^3} \left(|\nabla U_\varepsilon|^2 - \mu \frac{U_\varepsilon^2}{|x|^2} \right) dx = \int_{\mathbb{R}^3} U_\varepsilon^6 dx = S_\mu^{\frac{3}{2}}.$$

Since our approach is variational, we define the energy functional associated to the problem $(\mathcal{P}_\lambda)$ by

$$I_\lambda(u) = \frac{a}{4}||u||_\mu^4 + \frac{b}{2}||u||_\mu^2 - \frac{1}{6}|u|_6^6 - \lambda \int_{\mathbb{R}^3} gu \, dx, \text{ for all } u \in H_\mu(\mathbb{R}^3).$$

It is clear that I_λ is well defined in $H_\mu(\mathbb{R}^3)$ and belongs to $C^1(H_\mu(\mathbb{R}^3), \mathbb{R})$.
$u \in H_\mu(\mathbb{R}^3)$ is said to be a weak solution of problem $(\mathcal{P}_\lambda)$ if it satisfies

$$\left(a||u||_{\mu}^{2}+b\right)\int_{\mathbb{R}^{3}}\left(\nabla u\nabla\varphi-\mu\frac{u\varphi}{|x|^{2}}\right)dx-\int_{\mathbb{R}^{3}}\left(u^{5}\varphi-\lambda g\varphi\right)dx=0,\ \forall\ \varphi\in H_{\mu}\left(\mathbb{R}^{3}\right).$$

We recall the following standard definitions.

Definition 1. Let $c\in\mathbb{R}$, a sequence $(u_n)\subset H_{\mu}\left(\mathbb{R}^{3}\right)$ is called a Palais Smale sequence ($(PS)_c$ sequence for short), if

$$I_{\lambda}\left(u_{n}\right)\rightarrow c\quad\text{and}\quad I_{\lambda}^{\prime}\left(u_{n}\right)\rightarrow 0\ \text{ in }\ H^{-1}\left(\mathbb{R}^{3}\right)\ \text{ as } n\rightarrow\infty. \tag{1}$$

Definition 2. Let $c\in\mathbb{R}$. We say that I_{λ} satisfies the Palais Smale condition at level c, if any $(PS)_c$ sequence contains a convergent subsequence in $H_{\mu}\left(\mathbb{R}^{3}\right)$.

In order to prove our main result, we give the following lemmas.

Lemma 1. Let $(u_n)\subset H_{\mu}\left(\mathbb{R}^{3}\right)$ be a $(\mathrm{PS})_c$ sequence of I_{λ} for some $c\in\mathbb{R}$. Then $u_n\rightharpoonup u$ in $H_{\mu}\left(\mathbb{R}^{3}\right)$ for some u with $I_{\lambda}^{\prime}(u)=0$.

Proof . By (1) we get

$$\begin{aligned}c+\circ_{n}(1) &= I_{\lambda}\left(u_{n}\right)-\frac{1}{6}\left\langle I_{\lambda}^{\prime}\left(u_{n}\right),u_{n}\right\rangle\\ &\geq \frac{a}{12}||u_{n}||_{\mu}^{4}+\frac{b}{3}||u_{n}||_{\mu}^{2}-\lambda\frac{5}{6}\left\|g\right\|_{-}\left\|u_{n}\right\|_{\mu}.\end{aligned}$$

Then (u_n) is bounded in $H_{\mu}\left(\mathbb{R}^{3}\right)$. Up to a subsequence if necessary, we obtain

$$u_{n}\rightharpoonup u\text{ in }H_{\mu}\left(\mathbb{R}^{3}\right),\ u_{n}\rightharpoonup u\text{ in }L_{6}\left(\mathbb{R}^{N}\right)\text{ and }u_{n}\rightarrow u\text{ a.e. in }\mathbb{R}^{N},$$

then $\left\langle I_{\lambda}^{\prime}\left(u_{n}\right),\varphi\right\rangle=0$ for all $\varphi\in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$, which means that $I_{\lambda}^{\prime}(u)=0$. □

Lemme 2. There exist positive constants Λ_1, ρ_1 and δ_1 such that for all $\lambda\in(0,\Lambda_1)$ we have

$$I_{\lambda}(u)|_{\partial B_{\rho_{1}}}\geq\delta_{1}\text{ and }\ I_{\lambda}(u)|_{B_{\rho_{1}}}\geq-\frac{\lambda^{3/2}}{2}\left\|g\right\|_{-}^{2}.$$

Proof . Let $u\in H_{\mu}\left(\mathbb{R}^{3}\right)\setminus\{0\}$ and $\rho=\|u\|_{\mu}$. Then by the Sobolev and Hölder inequalities, we have

$$I_{\lambda}(u)\geq\frac{a}{4}\rho^{4}+\frac{b}{2}\rho^{2}-\frac{1}{6S_{\mu}^{3}}\rho^{6}-\left(\lambda^{3/4}\left\|g\right\|_{-}\right)\left(\lambda^{1/4}\rho\right),$$

On the other hand, by applying the inequality $\alpha\beta<\frac{\alpha^{2}}{2}+\frac{\beta^{2}}{2}$ for any α, $\beta>0$ we get

$$\begin{aligned}I_{\lambda}(u) &\geq \frac{a}{4}\rho^{4}+\frac{b-\lambda^{1/2}}{2}\rho^{2}-\frac{1}{6S_{\mu}^{3}}\rho^{6}-\frac{\lambda^{3/2}}{2}\left\|g\right\|_{-}^{2},\\ &\geq \frac{a}{4}\rho^{4}-\frac{1}{6S_{\mu}^{3}}\rho^{6}-\frac{\lambda^{3/2}}{2}\left\|g\right\|_{-}^{2},\ \text{for all }\lambda\leq b^{2}.\end{aligned}$$

Let

$$\Psi(\rho)=\frac{a}{4}\rho^{4}-\frac{1}{6S_{\mu}^{3}}\rho^{6},$$

direct calculation shows that

$$\Psi(\rho)\geq 0\text{ for all }\rho\leq\rho_{1}\text{ with }\rho_{1}=\left(\frac{3}{2}aS_{\mu}^{3}\right)^{1/2},$$

from this and for all $\lambda \leq b^2$, we immediately derive that

$$I_\lambda (u)|_{B_{\rho_1}} \geq -\frac{\lambda^{3/2}}{2} \|g\|_-^2 ,$$

and also

$$I_\lambda (u)|_{\partial B_{\rho_1}} \geq \frac{\Psi(\rho_1)}{2} \text{ for all } \lambda \leq \left(\frac{\Psi(\rho_1)}{\|g\|_-^2}\right)^{2/3} .$$

Taking

$$\Lambda_1 = \min\left\{b^2, \ \left(\frac{\Psi(\rho_1)}{\|g\|_-^2}\right)^{2/3}\right\} \text{ and } \delta_1 = \frac{\Psi(\rho_1)}{2},$$

the conclusion holds. □

Lemma 3. Let $(u_n) \subset H_\mu(\mathbb{R}^3)$ be a $(\mathrm{PS})_c$ sequence of I_λ for some $c \in \mathbb{R}$ such that $u_n \rightharpoonup u$ in $H_\mu(\mathbb{R}^3)$. Then

$$\text{either } u_n \to u \text{ or } c \geq I_\lambda(u) + C_*,$$

where $C_* = \frac{ab}{4}S_\mu^3 + \frac{a^3}{24}S_\mu^6 + \left(\frac{a^2}{24}S_\mu^3 + \frac{b}{6}\right)\left(a^2 S_\mu^6 + 4bS_\mu^3\right)^{1/2}$.

Proof . By the proof of Lemma 1 we have (u_n) is a bounded sequence in $H_\mu(\mathbb{R}^3)$, furthermore, if we write $v_n = u_n - u$; we derive that $v_n \rightharpoonup 0$ in H_μ, then by Brezis-Lieb Lemma [6] we have

$$\|u_n\|_\mu^2 = \|v_n\|_\mu^2 + \|u\|_\mu^2 + o_n(1) \text{ and } |u_n|_6^6 = |v_n|_6^6 + |u|_6^6 + o_n(1). \tag{2}$$

Using together (1) and (2) , we get

$$c + o_n(1) = I_\lambda(u) + \frac{a}{4}\|v_n\|_\mu^4 + \frac{b}{2}\|v_n\|_\mu^2 + \frac{a}{2}\|v_n\|_\mu^2\|u\|_\mu^2 - \frac{1}{6}|v_n|_6^6 ,$$

and

$$o_n(1) = a\|v_n\|_\mu^4 + b\|v_n\|_\mu^2 + 2a\|v_n\|_\mu^2\|u\|_\mu^2 - |v_n|_6^6 . \tag{3}$$

Therefore,

$$c + o_n(1) = I_\lambda(u) + \frac{a}{12}\|v_n\|_\mu^4 + \frac{b}{3}\|v_n\|_\mu^2 + \frac{a}{6}\|v_n\|_\mu^2\|u\|_\mu^2 \tag{4}$$

Assume that $\|v_n\| \to l > 0$, then by (3) and the Sobolev inequality we obtain

$$S_\mu^{-3} l^6 \geq al^4 + bl^2,$$

this implies that

$$l^2 \geq \frac{a}{2}S_\mu^3 + \frac{1}{2}S_\mu\left(a^2 S_\mu^4 + 4S_\mu b)\right)^{1/2} .$$

From the above inequality and (4) we conclude

$$\begin{aligned} c &\geq I_\lambda(u) + \frac{a}{12}l^4 + \frac{b}{3}l^2 \\ &\geq I_\lambda(u) + \frac{ab}{4}S_\mu^3 + \frac{a^3}{24}S_\mu^6 + \left(\frac{a^2}{24}S_\mu^3 + \frac{b}{6}\right)\left(a^2 S_\mu^6 + 4bS_\mu^3\right)^{1/2} . \end{aligned}$$

This finishes the proof of lemma 3. □

3. Proof of the main results

3.1. **Proof of Theorem 1.** First, by Lemma 2 we define

$$c_1 = \inf \left\{ I_\lambda (u) ; \; u \in \bar{B}_{\rho_1} \right\} .$$

Since $g \not\equiv 0$, we can choose $\Phi \in C_0^\infty \left(\mathbb{R}^3 \backslash \{0\} \right)$ such that $\int_{\mathbb{R}^3} g\Phi \; dx > 0$. Hence, there exists $t_0 > 0$ small enough such that $\|t_0 \Phi\|_\mu < \rho_1$ and

$$I_\lambda (t_0 \Phi) = \frac{a}{4} t_0^4 \|\Phi\|_\mu^4 + \frac{b}{2} t_0^2 \|\Phi\|_\mu^2 - \frac{t_0^6}{6} |\Phi|_6^6 - \lambda t_0 \int_{\mathbb{R}^3} g\Phi \; dx < 0,$$

which implies that $c_1 < 0 = I_\lambda(0)$. Using the Ekeland variational principle, for the complete metric space $\bar{B}_{\rho_1}$ with respect to the norm of $H_\mu \left(\mathbb{R}^3 \right)$, we obtain that there exists a $(\mathrm{PS})_{c_1}$ sequence $(u_n) \subset \bar{B}_{\rho_1}$ such that $u_n \rightharpoonup u_1$ in $H_\mu \left(\mathbb{R}^3 \right)$ for some u_1 with $\|u_1\|_\mu \leq \rho_1$. Assume that $u_n \nrightarrow u_1$ in $H_\mu \left(\mathbb{R}^3 \right)$, then it follows from Lemma 3 that

$$c_1 \geq I_\lambda (u_1) + C_* > c_1,$$

which is a contradiction. Thus u_1 is a nontrivial solution of $(\mathcal{P}_\lambda)$ with negative energy.

3.2. **Proof of Theorem 2.** The existence of the second solution follows immediately from the following lemma.

Lemma 4. Let $\Lambda_2 > 0$ such that

$$-\frac{\lambda^{3/2}}{2} \|g\|_-^2 + C_* > 0, \; \forall \lambda \in (0, \Lambda_2).$$

Then there exist $u_\varepsilon \in H_\mu \left(\mathbb{R}^3 \right)$ and $0 < \Lambda_* \leq \Lambda_2$ such that

$$\sup_{t \geq 0} I_\lambda (t u_\varepsilon) < c_1 + C_*, \quad \text{for all } \lambda \in (0, \Lambda_*).$$

Proof . Since $g \not\equiv 0$, we can choose $\varepsilon > 0$ and $u_\varepsilon (x) = \pm U_\varepsilon (x)$ such that

$$\int_{\mathbb{R}^3} g U_\varepsilon dx > 0.$$

We consider the functions

$$\Phi_1(t) = \frac{at^4}{4} \|u_\varepsilon\|_\mu^4 + \frac{bt^2}{2} \|u_\varepsilon\|_\mu^2 - \frac{t^6}{6} |u_\varepsilon|_6^6 .$$

and

$$\Phi_2(t) = \Phi_1(t) - \lambda t \int_{\mathbb{R}^3} g u_\varepsilon \; dx.$$

So, for all $\lambda \in (0, \Lambda_2)$ we have

$$\Phi_2(0) = 0 < -\frac{5}{24} \lambda^2 \|g\|_- + C_*.$$

Hence, by the continuity of $\Phi_2(t)$, there exists $t_1 > 0$ small enough such that

$$\Phi_2(t) < -\frac{5}{24} \lambda^2 \|g\|_- + C_* \;\; \text{for all } t \in (0, t_1) .$$

On the other hand, the function $\Phi_1(t)$ attains its maximum at

$$t_\varepsilon = \left[\frac{a\left\|u_\varepsilon\right\|_\mu^4 + \left(a^2\left\|u_\varepsilon\right\|_\mu^8 + 4b\left\|u_\varepsilon\right\|_\mu^2\left|u_\varepsilon\right|_6^6\right)^{1/2}}{2\left|u_\varepsilon\right|_6^6}\right]^{1/2}.$$

From the definition of S_μ we have

$$\begin{aligned}
\frac{at_\varepsilon^4}{4}\left\|u_\varepsilon\right\|_\mu^4 &= \frac{a}{4}\left\|u_\varepsilon\right\|_\mu^4\left[\frac{a\left\|u_\varepsilon\right\|_\mu^4 + \left(a^2\left\|u_\varepsilon\right\|_\mu^8 + 4b\left\|u_\varepsilon\right\|_\mu^2\left|u_\varepsilon\right|_6^6\right)^{1/2}}{2\left|u_\varepsilon\right|_6^6}\right]^2 \\
&= \frac{a}{16}\left[\frac{a\left\|u_\varepsilon\right\|_\mu^6}{\left|u_\varepsilon\right|_6^6} + \left[\frac{a^2\left\|u_\varepsilon\right\|_\mu^{12} + 4b\left\|u_\varepsilon\right\|_\mu^6\left|u_\varepsilon\right|_6^6}{\left|u_\varepsilon\right|_6^{12}}\right]^{1/2}\right]^2 \\
&= \frac{a}{16}\left[aS_\mu^3 + \left[a^2S_\mu^6 + 4bS_\mu^3\right]^{1/2}\right]^2 \\
&= \frac{a^3}{8}S_\mu^6 + \frac{ab}{4}S_\mu^3 + \frac{a^2}{8}\left(a^2S_\mu^{12} + 4bS_\mu^9\right)^{1/2}.
\end{aligned}$$

Similarly, we obtain

$$\frac{bt_\varepsilon^2}{2}\left\|u_\varepsilon\right\|_\mu^2 = \frac{ab}{4}S_\mu^3 + \frac{b}{4}\left(a^2S_\mu^6 + 4bS_\mu^3\right)^{1/2},$$

and

$$\frac{t_\varepsilon^6}{6}\left|u_\varepsilon\right|_6^6 = \frac{a^3}{12}S_\mu^6 + \frac{ab}{4}S_\mu^3 + \frac{1}{12}\left(a^2S_\mu^3 + b\right)\left(a^2S_\mu^6 + 4bS_\mu^3\right)^{1/2}.$$

By the above estimates, we deduce that $\sup\limits_{t\geq 0}\Phi_1(t) \leq C_*$.

On the other hand, using Lemma 2 we see that

$$c_1 \geq -\frac{\lambda^{3/2}}{2}\left\|g\right\|_-^2 \quad \text{for all} \quad \lambda \in (0, \Lambda_1),$$

furthermore

$$c_1 > -t_1\lambda\int_{\mathbb{R}^3} gu_\varepsilon dx \quad \text{if } \lambda < 4\left(t_1\int_{\mathbb{R}^3} gu_\varepsilon\ dx\right)^2 / \left\|g\right\|_-^4.$$

Taking $\Lambda_* = \min\left\{\Lambda_2,\ 4\left(t_1\int_{\mathbb{R}^3} gu_\varepsilon\ dx\right)^2 / \left\|g\right\|_-^4\right\}$, then we deduce that

$\sup\limits_{t\geq 0} I_\lambda(tu_\varepsilon) < c_1 + C_*$ for all $\lambda \in (0, \Lambda_*)$. □

Note that $I_\lambda(0) = 0$ and $I_\lambda(Tu_\varepsilon) < 0$ for T large enough, also from Lemma 2, we know that

$$I_\lambda(u)|_{\partial B_{\rho_1}} \geq \delta_1 > 0 \text{ for all } \lambda \in (0, \Lambda_1).$$

Then, by the Mountain Pass Theorem, there exists a $(\mathrm{PS})_{c_2}$ sequence, where

$$c_2 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

with

$$\Gamma = \left\{\gamma \in C([0,\ 1],\ H_\mu(\mathbb{R}^3)),\ \gamma(0) = 0 \text{ and } \gamma(1) = Tu_\varepsilon\right\}.$$

Using Lemma 1 we have (u_n) has a subsequence, still denoted by (u_n), such that $u_n \rightharpoonup u_2$ in $H_\mu\left(\mathbb{R}^3\right)$, for some u_2. Furthermore, we know by Lemma 4 that

$$\sup_{t\geq 0} I_\lambda(tu_\varepsilon) < c_1 + C_*, \text{ for all } \lambda \in (0, \Lambda_*),$$

then from Lemma 3 we deduce that $u_n \to u_2$ in $H_\mu\left(\mathbb{R}^3\right)$. Thus we obtain a critical point u_2 of I_λ satisfying $I_\lambda(u_2) > 0$.

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