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# STRICT COINCIDENCE AND STRICT COMMON FIXED POINT VIA STRONGLY TANGENTIAL PROPERTY WITH AN APPLICATION

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Abstract. In this paper, we prove two strict coincidence and strict common fixed point theorems for weakly compatible hybrid pairs of strongly tangential mappings satisfying F-contraction, in a metric space. An example and an application to functional equations arising in dynamic programming is given to illustrate our results. In the sequel several known results are extended, generalized and improved.

## 1. Introduction

The contraction principle due to Banach has been generalized in different directions and one of such generalizations is due to Nadler [13], where he used the Hausdorff metric to prove existence of a fixed point of multivalued mapping in metric space. Later many authors established some results in non linear analysis concerning the multivalued / hybrid fixed point theory and its applications using two types of distances. One is the Hausdorff distance and another is the $\delta-$ distance. Although $\delta-$ distance is not a metric like the Hausdorff distance, but shares most of the properties of a metric. In this paper we utilize a Ćirić type F-contraction and Hardy-Rogers type F-contraction inequality introduced by Minak et al. [12] (independently by Wardowski and Dung [23] as F-weak contraction) and Cosentino and Vetro [7] respectively, using $\delta-$ distance to establish the existence of a strict coincidence and strict common fixed point of a weakly compatible hybrid pair of mappings which are strongly tangential. However it is worth mentioning here that idea of F-contraction was initiated by Wardowski [22] which has again been generalized by several authors in different directions. In the last section, an application to functional equation arising in dynamic programming is given to demonstrate applicability of results obtained. We also present some remarks to show that our results provide extensions as well as substantial generalizations and improvements of several well known results existing in literature.

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## 2. Preliminaries

Let $(X,d)$ be a metric space and $B(X)$ be the family of all non-empty bounded subsets of $X$. For all $A,B \in B(X)$, the two functions: $D,\delta : B(X)\times B(X)\to \mathbb{R}^+$ are defined as: $D(A,B)=\inf\{d(a,b); a\in A, b\in B\}$ and $\delta(A,B)=\sup\{d(a,b); a\in A, b\in B\}$. If $A=\{a\}$, we write $\delta(A,B)=\delta(a,B)$ and $D(A,B)=D(a,B)$. Further if $B=\{b\}$, $\delta(A,B)=D(A,B)=d(a,b)$.
Clearly

(1) $\delta(A,B)=\delta(B,A)>0$,
(2) $\delta(A,B)\leq \delta(A,C)+\delta(C,B)$,
(3) $\delta(A,A)=\text{diam}A$,
(4) $\delta(A,B)=0$ implies $A=B=\{a\}$, for all $A,B,C\in B(X)$.

Notice that $D(A,B)\leq d(a,b)\leq \delta(A,B)$, for all $a\in A$ and $b\in B$, where $A,B\in B(X)$.
If $f: X\to X$ is single-valued and $T: X\to B(X)$ is multivalued mapping of a metric space $(X,d)$, then $(f,T)$ is called a hybrid pair of mapping.

For a hybrid pair $(f,T)$, a point $x\in X$ is a coincidence point if $fx\in Tx$; strict coincidence point if $Tx=\{fx\}$; common fixed point if $x=fx\in Tx$; strict (or a stationary or absolute) common fixed point if $Tx=\{fx\}=\{x\}$.

In all that follows $f,g: X\to X$ are single valued and $S,T: X\to B(X)$ are multivalued mappings unless specifically mentioned.

**Definition 1** [9] A hybrid pair of mappings $(f,T)$ of a metric space $(X,d)$ is weakly compatible if it commute at their coincidence point, i.e., if $Tu=\{gu\}$ for some $u\in X$, then $gTu=Tgu$.

**Example 1** Let $X=[0,2]$ be equipped with the euclidian metric. Let self mappings $f$ and $T$ are defined as follows:

$$fx=\begin{cases} 2-x, & 0\leq x\leq 1\\ 0, & 1<x\leq 2,\end{cases} \qquad Tx=\begin{cases}\{1\}, & 0\leq x\leq 1\\ [1,x], & 1<x\leq 2.\end{cases}$$

Clearly 1 is the unique coincidence point as $T1=\{f1\}=\{1\}$. Also $fT1=Tf1=\{1\}$, i.e. $f$ and $T$ are weakly compatible.

Here it is worth mentioning that however several authors claimed to introduce some weaker notions of commuting mappings but still weak compatibility is the minimal and most widely used notion among all weaker variants of commutativity. For brief development of weaker forms of commuting mappings one may refer to Singh and Tomar [15].

**Definition 2** [16] A pair of single-valued self mappings $(f,g)$ of a metric space $(X,d)$ is tangential with respect to a pair of multivalued self mapping $(S,T)$ if

$$\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Ty_n=A\in B(X),$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that

$$\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gy_n=z\in A$$

for some $z\in X$.
In a well known review M. Balaj pointed out that the results of Sintunavarat and

Kumam [16] are not valid under given conditions without closedness of suitable image subspaces. Motivated by this observation, Chauhan et al. [5] introduced the notion of strongly tangential property which is slightly more restrictive than tangential property.

**Definition 3** [5] A pair of single-valued self mappings $(f, g)$ of a metric space $(X, d)$ is strongly tangential with respect to a pair of multivalued self mappings $(S, T)$ if

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = A \in B(X),$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = z \in A$$

and $z \in fX \cap gX$.

**Example 2** Let $X = [0, \infty]$ be equipped with the euclidian metric. Let self mappings $f$, $g$, $S$ and $T$ be defined as:

$$fx = x + 2, \quad gx = 2x,$$

$$Sx = [x + 1, x + 3] \quad Tx = [x, 3x].$$

So $fX \cap gX = [2, \infty)$. Consider two sequences $\{x_n\}$ and $\{y_n\}$ defined as:
$x_n = \frac{1}{n}$ and $y_n = 1 + \frac{1}{n}$ for all $n \geq 1$.
Clearly, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = 2 \in [1, 3] = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n$
and $2 \in [2, \infty) = fX \cap gX$. Hence $(f, g)$ is strongly tangential with respect to $(S, T)$.

It is interesting to see that tangential hybrid pair of mappings is strongly tangential however reverse implication may not be true.
For $S = T$ and $f = g$, definition of strongly tangential reduces to:

**Definition 4** [5] A single-valued self mapping $f$ of a metric space $(X, d)$ is strongly tangential with respect to multivalued self mapping $T$ if

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ty_n = A \in B(X),$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} fy_n = z \in A$$

and $z \in fX$.

**Example 3** Let $X = [0, 2]$ be equipped with the euclidian metric. Let $f$ and $S$ be defined as:

$$fx = \begin{cases} 0, & 0 \leq x < 1 \\ \frac{x+1}{2}, & 1 \leq x \leq 2, \end{cases} \quad Tx = \begin{cases} \{\frac{1}{2}\}, & 0 \leq x < 1 \\ [1, x] & 1 \leq x \leq 2. \end{cases}$$

$$f(X) = \{0\} \cup [1, \frac{3}{2}].$$

Consider two sequences $\{x_n\}$ and $\{y_n\}$ defined by:
$x_n = 1 + \frac{1}{n}$ and $y_n = 1$ for $n \geq 1$. So

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ty_n = \{1\},$$

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} fy_n = 1 \in \{1\},$$

and $1 \in fX$. Hence $f$ is strongly tangential with respect to $T$.

Following Chauhan et al. [5], Tomar et al. [20] introduced the notion of strongly tangential for single-valued mappings.

**Definition 5** A pair of single-valued self mappings $(f, g)$ on a metric space $(X, d)$ is strongly tangential with respect to a pair of single-valued self mappings $(S, T)$ if

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z \in X,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that
$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = z$ and $z \in fX \cap gX$.

It is worth mentioning that if $\{x_n\} = \{y_n\}$, then the notion of strongly tangential for a hybrid pair of mappings reduces to property (E.A.) introduced by Kamran [10].

Let $\mathcal{F}$ be the family of all continuous functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying:

- $F$ is strictly increasing.
- For each sequence $\{\alpha_n\}$ in $X$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$, $n \in \mathbb{N}$.
- There exists $k \in (0, 1)$ satisfying $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

**Example 4** (1) $F(t) = \ln t$,
(2) $F(t) = t + \ln t$,
(3) $F(t) = -\frac{1}{\sqrt{t}}$.

**Definition 6** [22] A single -valued self mapping $T$ of a metric space $(X, d)$ is an $F$-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that $\forall\ x, y \in X$,

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

**Definition 7** [7] Two pairs of single-valued self mappings $(f, S)$ and $(g, T)$ of a metric space $(X, d)$ are said to satisfy Hardy-Rogers type $F$-contraction condition if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that $\forall\ x, y \in X$, $d(Sx, Ty) > 0 \implies$

$$\tau + F(d(Sx, Ty)) \leq F(\alpha d(fx, gy) + \beta d(fx, Sx) + \gamma d(gy, Sy) + \lambda d(fx, Sy) + L d(gy, Tx)),$$

where $\alpha + \beta + \gamma + \lambda + L < 1$ and $\alpha,\ \beta,\ \gamma,\ \lambda,\ L \geq 0$.

**Definition 8**[12] A single -valued self mapping $T$ of a metric space $(X, d)$ is a Ćirić type $F$-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that $\forall x, y \in X$,

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(M(x, y)),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$.

Notice that every F-contraction is a Ćirić type F-contraction or Hardy-Rogers type F-contraction but the reverse implication does not hold.

## 3. Main results

Now we state and prove our first main result using Ćirić type F-contraction to establish strict coincidence and strict common fixed point of two hybrid pairs of self mappings.

**Theorem 1** Let $f, g : X \to X$ be single-valued self mappings and $S, T : X \to B(X)$ be multivalued self mappings of a metric space $(X, d)$. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that $\forall x, y \in X$, we have:

$$\delta(Sx, Ty) > 0 \implies \tau + F(\delta(Sx, Ty)) \leq F(M(x, y)), \tag{1}$$

where $M(x, y) = \max\{d(fx, gy), D(fx, Sx), D(gy, Ty), \frac{1}{2}[D(fx, Ty) + D(gy, Sx)]\}$ and pair $(f, g)$ is strongly tangential with respect to $(S, T)$. Then pairs $(f, S)$ and $(g, T)$ have a strict coincidence point. Moreover, $f, g, S$ and $T$ have a unique strict common fixed point if hybrid pairs $(f, S)$ and $(g, T)$ are weakly compatible.

**Proof** Suppose that $(f, g)$ is strongly tangential with respect to $(S, T)$.
Hence there exist sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$\lim_n fx_n = \lim_n gy_n = z \in A = \lim_n Sx_n = \lim_n Ty_n$$

where $A \in B(X)$ and $z \in fX \cap gX$, i.e., there exist $u, v \in X$ such that $fu = gv = z$.
Now we claim that $z \in Su$, if not using $x = u$ and $y = y_n$ in condition (1),
$\delta(Su, Ty_n) > 0$ (Otherwise if $\delta(Su, Ty_n) = 0$, then as $n \to \infty$, $\delta(Su, A) = 0$ which implies $\delta(Su, fu) \leq \delta(Su, A) = 0$ which implies $d(Su, fu) = 0$ i.e., $fu \in Su$ or $d(Su, fu) < 0$, a contradiction) and
$\tau + F(\delta(Su, Ty_n))$

$$\leq F\{\max(d(fu, gy_n), D(fu, Su), D(gy_n, Ty_n), \frac{1}{2}[D(fu, Ty_n) + D(gy_n, Su)])\}.$$

Taking $n \to \infty$

$$\tau + F(\delta(Su, A)) \leq F\{\max(d(z, z), D(z, Su), D(z, A), \frac{1}{2}[(D(z, A) + D(z, Su)])\},$$

or $\tau + F(\delta(Su, A)) \leq F(D(z, Su))$.
Hence $F(\delta(Su, A)) < \tau + F(\delta(Su, A)) \leq F(D(z, Su)) \leq F(\delta(z, Su))$,
a contradiction, since $z \in A$, F is strictly increasing function and $\tau > 0$.
Hence $\delta(Su, A) = 0$, i.e., $Su = A = \{z\} = \{fu\}$. i.e. $f$ and $S$ have a strict coincidence point.
Now we claim that $z \in Tv$, if not using $x = x_n$ and $y = v$ in condition (1), $\delta(Sx_n, Tv) > 0$ (Otherwise if $\delta(Sx_n, Tv) = 0$, then as $n \to \infty$, $\delta(A, Tv) = 0$ which implies $d(gv, Tv) \leq \delta(A, Tv) = 0$, a contradiction) and
$\tau + F(\delta(Sx_n, Tv))$

$$\leq F\{\max(d(fx_n, gv), D(fx_n, Sx_n), D(gv, Tv), \frac{1}{2}[D(fx_n, Tv) + D(gv, Sx_n)])\}.$$

Taking $n \to \infty$

$$\tau + F(\delta(A, Tv)) \leq F\{\max(d(z, z), D(z, A), D(z, Tv), \frac{1}{2}[D(z, Tv) + D(z, A)])\},$$

i.e., $\tau + F(\delta(A, Tv)) \leq F(D(z, Tv))$.
Hence $F(\delta(A, Tv)) < \tau + F(\delta(A, Tv)) \leq F(D(z, Tv)) \leq F(\delta(z, Tv))$,
a contradiction, since $z \in A$, $F$ is strictly increasing function and $\tau > 0$.
Hence $\delta(A, Tv) = 0$, i.e., $Tv = A = \{z\} = \{gv\}$.
Hence $g$ and $T$ have a strict coincidence point.
Since $(f, S)$ is weakly compatible, therefore $fSu = Sfu$, i.e., $Sz = \{fz\}$.
Similarly, $(g, T)$ is weakly compatible then $Tz = \{gz\}$.
Now we claim that $gz = z$, if not using $x = z$ and $y = y_n$ in condition (1),

$\delta(Sz, Ty_n) > 0$, we get $F(\delta(Sz, Ty_n)) > 0$ (Otherwise if $\delta(Sz, Ty_n) = 0$, then as $n \to \infty$, $\delta(Sz, A) = 0$ which implies $d(Sz, z) \leq \delta(Sz, A) = 0$, a contradiction) and
$\tau + F(\delta(Sz, Ty_n))$

$$\leq F\{\max(d(fz, gy_n), D(fz, Sz), D(gy_n, Ty_n), \frac{1}{2}[D(fz, Ty_n) + D(gy_n, Sz)])\}.$$

Taking $n \to \infty$, we get

$$\tau + F(\delta(Sz, A)) \leq F\{\max(d(fz, z), D(fz, Sz), D(z, A), \frac{1}{2}[D(fz, A) + D(z, Sz)])\},$$

or $\tau + F(\delta(Sz, A)) \leq F\{\max(d(fz, z), 0, 0, \frac{1}{2}[d(fz, z) + d(z, fz)])\}$,
or $\tau + F(\delta(Sz, A)) \leq F\{d(fz, z)\}$,
i.e., $F(d(fz, z)) = F(\delta(Sz, z)) < \tau + F(\delta(Sz, A)) \leq F(d(fz, z))$, a contradiction, since $z \in A$, $F$ is strictly increasing function and $\tau > 0$.
Hence $\delta(Sz, z) = 0$ so that $Sz = \{fz\} = \{z\}$.
Now we claim that $gz = z$, if not using $x = x_n$ and $y = z$ in condition (1), $\delta(Sx_n, Tz) > 0$ (Otherwise if $\delta(Sx_n, Tz) = 0$, then as $n \to \infty$, $\delta(A, Tz) = 0$ which implies $d(z, Tz) \leq \delta(A, Tz) = 0$, a contradiction) and
$\tau + F(\delta(Sx_n, Tz))$

$$\leq F\{\max(d(fx_n, gz), D(fx_n, Sx_n), D(gz, Tz), \frac{1}{2}[D(fx_n, Tz) + D(gz, Sx_n)])\}.$$

Taking $n \to \infty$,we get

$$\tau + F(\delta(A, Tz)) \leq F\{\max(d(z, gz), D(z, A), 0, \frac{1}{2}[D(z, Tz) + D(gz, A)])\},$$

i.e., $\tau + F(\delta(A, Tz)) \leq F(d(z, gz))$
or $F(d(z, Tz)) = F(\delta(A, Tz)) < \tau + F(\delta(z, Tz)) \leq F(d(z, gz))$, a contradiction, since $z \in A$, $F$ is strictly increasing function and $\tau > 0$.
Hence $\delta(z, Tz) = 0$ so that $Tz = \{gz\} = \{z\}$.
Therefore $z$ is a strict common fixed point of $f, g, T$ and $S$.
Now we claim the uniqueness of $z$, if not let $z$ and $w$ be two strict common fixed point.
Now putting $x = z$ and $y = w$ in condition (1),
$\delta(Sz, Tw) > 0$ (Otherwise if $\delta(Sz, Tw) = 0$ which implies $Sz = Tw = \{z\} = \{w\}$, i.e., $z = w$, a contradiction) and
$\tau + F(\delta(Sz, Tw))$

$$\leq F\{\max(d(fz, gw), D(fz, Sz), D(gz, Tz), \frac{1}{2}[(D(fz, Tw) + D(gw, Sz)])\}$$

$$\leq F\{\max(d(fz, gw), 0, 0, \frac{1}{2}[(d(z, w) + d(w, z)])\}.$$

or $F(d(z, w)) < \tau + F(\delta(Sz, Tw)) \leq F(d(z, gz))$, a contradiction, since $Sz = \{z\}$, $Tw = \{w\}$, $F$ is strictly increasing function and $\tau > 0$.
Therefore $z$ is a unique strict common fixed point of $f, g, T$ and $S$.

If $f = g$ and $S = T$ we obtain the following corollary:

**Corollary 1** Let $f : X \to X$ be a single-valued self mapping and $T : X \to B(X)$ be a multivalued self mapping of a metric space $(X, d)$. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that $\forall x, y \in X$ we have:

$$\delta(Tx, Ty) > 0 \implies \tau + F(\delta(Tx, Ty)) \leq F(M(x, y)), \tag{2}$$

where $M(x, y) = \max\{d(fx, fy), D(fx, Tx), D(fy, Ty), \frac{1}{2}[D(fx, Ty) + D(fy, Tx)]\}$ and $f$ is strongly tangential with respect to $T$. Then $f$ and $T$ have a strict coincidence point. Moreover, $f$ and $T$ have a unique strict common fixed point if hybrid pair $(f, T)$ is weakly compatible.

**Example 5** Let $X = [0, 5]$, $d(x, y) = |x - y|$ and $f$ and $T$ defined by

$$fx = \begin{cases} 2 - x, & 0 \leq x \leq 1 \\ \frac{x+5}{2}, & 1 < x \leq 5, \end{cases} \qquad Tx = \begin{cases} \{1\}, & 0 \leq x \leq 1 \\ [\frac{1}{2}, 1], & 1 < x \leq 5. \end{cases}$$

Consider the two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n = 1 - \frac{1}{n}$ and $y_n = 1$ for all $n \geq 1$. Clearly $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} fy_n = 1 \in \{1\} = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ty_n$, i.e., the pair $f$ is strongly tangential with respect to $T$. The point $z = 1$ is a strict coincidence point and $fT1 = Tf1 = \{1\}$, i.e., $(f, T)$ is weakly compatible. Further $f$ and $T$ satisfy Ćirić type $F$-contraction (2) for $\tau = \frac{1}{2}$, $F(x) = \frac{-1}{\sqrt{x}}$. Hence all the conditions of Corollary 1 are satisfied and $z = 1$ is a unique strict common fixed point of $f$ and $T$. One may notice that $f$ and $T$ are discontinuous mappings and $fX \not\subseteq TX$.

If $S$ and $T$ are single-valued mappings, we get the following corollary:

**Corollary 2** Let $(X, d)$ be a metric space and let $f, g, S$ and $T$ be self mappings on $X$. Assume that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that $\forall x, y \in X$ we have:

$$d(Sx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(M(x, y)), \tag{3}$$

where $M(x, y) = \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Sx)]\}$ and $(f, g)$ is strongly tangential with respect to $(S, T)$. Then $f$, $g$, $S$ and $T$ have a coincidence point. Moreover, $f, g, S$ and $T$ have a unique common fixed point if pairs of self mappings $(f, S)$ and $(g, T)$ are weakly compatible.

Now we state and prove our second main result using Hardy-Rogers type $F$-contractions to establish strict coincidence and common fixed point of two hybrid pairs of self mappings.

**Theorem 2** Let $f, g : X \to X$ be single-valued self mappings and $S, T : X \to B(X)$ be multivalued self mappings of a metric space $(X, d)$. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that $\forall x, y \in X$ we have:

$$d(Sx, Ty) > 0 \implies \tau + F(d(Sx, Ty)) \leq F(\alpha d(fx, gy) + \beta d(fx, Sx) + \gamma d(gy, Sy) + \lambda d(fx, Sy) + L d(gy, Tx)), \tag{4}$$

where $\alpha + \beta + \gamma + \lambda + L < 1$, $\alpha$, $\beta$, $\gamma$, $\lambda$, $L \geq 0$ and pair $(f, g)$ is strongly tangential with respect to $(S, T)$. Then pairs $(f, S)$ and $(g, T)$ have a strict coincidence point. Moreover, $f$, $g$, $S$ and $T$ have a unique strict common fixed point if hybrid pairs $(f, S)$ and $(g, T)$ are weakly compatible.

**Proof** Suppose that $(f, g)$ is strongly tangential with respect to $(S, T)$. Hence, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$\lim_n fx_n = \lim_n gy_n = z \in A = \lim_n Sx_n = \lim_n Ty_n$$

where $A \in B(X)$ and $z \in fX \cap gX$, i.e., there exist $u, v \in X$ such that $fu = gv = z$. Now, we claim that $Su = \{z\}$, if not using $x = u$ and $y = y_n$ in condition (4), $\delta(Su, Ty_n) > 0$ (Otherwise if $\delta(Su, Ty_n) = 0$, then as $n \to \infty$, $\delta(Su, A) = 0$ which implies $d(Su, fu) \leq \delta(Su, A) = 0$, a contradiction) and
$\tau + F(\delta(Su, Ty_n))$

$$\leq F\{\alpha d(fu, gy_n) + \beta D(fu, Su) + \gamma D(gy_n, Ty_n) + \lambda D(fu, Ty_n) + LD(gy_n, Su)\}.$$

Taking $n \to \infty$, we get

$$\tau + F(\delta(Su, A)) \leq F\{\alpha d(z, z) + \beta D(fu, Su) + \gamma D(z, A) + \lambda D(fu, z) + LD(z, Su)\}$$

i.e., $F(\delta(Su, A)) < \tau + F(\delta(Su, A)) \leq F\{(\beta + L)D(z, Su)\} \leq F(\delta(Su, A))$,
a contradiction, since $z \in A$, $F$ is strictly increasing function and $\tau > 0$.
Hence, $\delta(Su, A) = 0$, i.e. $Su = A = \{z\} = \{fu\}$, i.e., $f$ and $S$ have a strict coincidence point.
Now we claim that $Tv = \{z\}$, if not using $x = x_n$ and $y = v$ in condition (4), $\delta(Sx_n, Tv) > 0$ (Otherwise if $\delta(Sx_n, Tv) = 0$, then as $n \to \infty$, $\delta(A, Tv) = 0$ which implies $d(gv, Tv) \leq \delta(A, Tv) = 0$, a contradiction) and
$\tau + F(\delta(Sx_n, Tv))$

$$\leq F\{\alpha d(fx_n, gv) + \beta D(fx_n, Sx_n) + \gamma D(gy_n, Tv) + \lambda D(fx_n, Tv) + LD(gv, Sx_n)\}.$$

Taking $n \to \infty$, we get

$$\tau + F(\delta(A, Tv)) \leq F\{\alpha d(z, z) + \beta D(z, A) + \gamma D(z, Tv) + \lambda D(z, Tv) + LD(z, A)\},$$

i.e., $F(\delta(A, Tv)) < \tau + F(\delta(A, Tv)) \leq F\{(\gamma + \lambda)D(z, Tv)\} \leq F(\delta(A, Tv)\}$,
a contradiction, since $z \in A$, $F$ is strictly increasing function and $\tau > 0$.
Hence $\delta(A, Tv) = 0$, i.e. $Tv = A = \{z\} = \{gv\}$, i.e., $g$ and $T$ have a strict coincidence point.
Since $(f, S)$ is weakly compatible, therefore $fSu = Sfu$, i.e, $Sz = \{fz\}$.
Similarly, $(g, T)$ is weakly compatible,therefore $Tz = \{gz\}$.
Now we claim that $fz = z$, if not using $x = z$ and $y = y_n$ in condition (4), $\delta(Sz, Ty_n) > 0$ (Otherwise if $\delta(Sz, Ty_n) = 0$, then as $n \to \infty$, $\delta(Sz, A) = 0$ which implies $d(Sz, z) \leq \delta(Sz, A) = 0$, a contradiction) and
$\tau + F(\delta(Sz, Ty_n))$

$$\leq F\{\alpha d(fz, gy_n) + \beta D(fz, Sz) + \gamma D(gy_n, Ty_n) + \lambda D(fz, Ty_n) + LD(gy_n, Sz)\}.$$

Taking $n \to \infty$,

$$\tau + F(\delta(Sz, A)) \leq F\{\alpha d(fz, z) + 0 + \gamma D(z, A) + \lambda D(fz, A) + LD(z, Sz)\},$$

or $F(\delta(Sz, A)) < \tau + F(\delta(Sz, A)) \leq F\{(\alpha + \lambda + L)d(fz, z)\} \leq F\{\delta(Sz, A)\}$, a contradiction, since $z \in A$, $F$ is strictly increasing function and $\tau > 0$.
Hence, $\delta(fz, z) = 0$, i.e., $Sz = \{fz\} = \{z\}$.
Now we claim that $gz = z$, if not using $x = x_n$ and $y = z$ in condition (4),

$\delta(Sx_n, Tz) > 0$ (Otherwise if $\delta(Sx_n, Tz) = 0$, then as $n \to \infty$, $\delta(A, Tz) = 0$ which implies $d(z, Tz) \leq \delta(A, Tz) = 0$, a contradiction) and

$$\tau + F(\delta(Sx_n, Tz)) \leq F\{\alpha d(fx_n, gz) + \beta D(fx_n, Sx_n) + \gamma D(gz, Tz) + \lambda D(fx_n, Tz) + LD(gz, Sx_n)\}.$$

Taking $n \to \infty$,

$$\tau + F(\delta(A, Tz)) \leq F\{\alpha d(z, gz) + 0 + 0 + \lambda D(z, Tz) + LD(gz, A)\},$$

i.e., $F(\delta(A, Tz)) < \tau + F(\delta(A, Tz)) \leq F\{(\alpha + \lambda + L)d(z, gz)\} \leq F(\delta(A, Tz))$, a contradiction, since $z \in A$, $F$ is strictly increasing function and $\tau > 0$.
Hence, $\delta(z, Tz) = 0$, i.e., $Tz = \{gz\} = \{z\}$.
Therefore, $z$ is a strict common fixed point of $f$, $g$, $T$ and $S$.
Now we claim that uniqueness of $z$, if not let $z$ and $w$ be two strict common fixed point.
Now putting $x = z$ and $y = w$ in condition (4),
$\delta(Sz, Tw) > 0$ (Otherwise if $\delta(Sz, Tw) = 0$ which implies $Sz = Tw = \{z\} = \{w\}$, i.e., $z = w$, a contradiction) and

$$\tau + F(\delta(Sz, Tw)) \leq F\{(\alpha d(fz, gw) + \beta D(fz, Sz) + \gamma D(gw, Tw) + \lambda D(fz, Tw) + LD(gw, Sz)\}.$$

or $F(d(z, w)) < \tau + F(\delta(Sz, Tw)) \leq (\alpha + \lambda + L)F(d(z, w)) \leq F(d(z, w))$, a contradiction, since $Sz = \{z\}$, $Tw = \{w\}$, $F$ is strictly increasing function and $\tau > 0$.
Therefore $z$ is a unique strict common fixed point.

If $f = g$ and $S = T$, we obtain the following corollary:
**Corollary 3** Let $f : X \to X$ be a single-valued self mapping and $T : X \to B(X)$ be a multivalued self mapping of a metric space $(X, d)$. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that $\forall x, y \in X$ we have:

$$\delta(Tx, Ty) > 0 \implies \tau + F(\delta(Tx, Ty)) \leq F\{\alpha d(fx, fy) + \beta D(fx, Tx) + \gamma D(fy, Ty) + \lambda D(fx, Ty) + LD(fy, Tx)\}, \quad (5)$$

where, $\alpha + \beta + \gamma + \lambda + L < 1$, $\alpha$, $\beta$, $\gamma$, $\lambda$, $L \geq 0$ and $f$ is strongly tangential with respect to $T$. Then $f$ and $T$ have a strict coincidence point. Moreover, $f$ and $T$ have a unique strict common fixed point if $f$ and $T$ are weakly compatible.

**Example 6** Let $X = [0, 16]$, $d(x, y) = |x - y|$ and $f$ and $T$ defined by

$$fx = \begin{cases} \frac{x+2}{2}, & 0 \leq x \leq 2 \\ 16, & 2 < x \leq 16 \end{cases} \qquad Tx = \begin{cases} \{2\}, & 0 \leq x \leq 2 \\ [2, \frac{5}{2}], & 2 < x \leq 16 \end{cases}$$

Consider the two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n = 2 - \frac{1}{n}$ and $y_n = 2$ for all $n \geq 1$. Clearly $\lim_{n\to\infty} fx_n = 2 \in \{2\} = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ty_n$, i.e., the pair $f$ is strongly tangential with respect to $T$. The point $z = 2$ is a strict coincidence point and so $fT2 = Tf2 = \{2\}$, i.e., $(f, T)$ is weakly compatible.
Further $f$ and $T$ satisfy Hardy-Rogers type $F$-contraction (5) for $\tau = \frac{3}{10}$, $F(x) = \frac{-1}{\sqrt{x}}$, $\alpha = \beta = \frac{1}{4}$, $\gamma = \frac{1}{6}$, $\lambda = \frac{1}{35}$ and $L = \frac{1}{49}$.
Hence all the conditions of Corollary 3 are satisfied and $z = 2$ is a unique strict common fixed point of $f$ and $T$. One may notice that here $f$ and $T$ are discontinuous mappings and $fX \not\subseteq TX$.

If $S$ and $T$ are single-valued mappings, we get the following corollary:

**Corollary 4** Let $f$, $g$, $S$ and $T : X \to X$ be single-valued self mappings of a metric space $(X, d)$. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that $\forall x, y \in X$ we have:

$$\begin{aligned}(Sx, Ty) > 0 \implies \tau + F(d(Sx, Ty)) \\ \leq F\{\alpha d(fx, gy) + \beta d(fx, Sx) + \gamma d(gy, Ty) + \lambda d(fx, Ty) + Ld(gy, Sx)\}, \quad (6)\end{aligned}$$

where, $\alpha+\beta+\gamma+\lambda+L < 1$, $\alpha$, $\beta$, $\gamma$, $\lambda$, $L \geq 0$ and pair $(f, g)$ is strongly tangential with respect to $(S, T)$. Then pairs $(f, S)$ and $(g, T)$ have a strict coincidence point. Moreover, $f$, $g$, $S$ and $T$ have a unique common fixed point if $(f, S)$ and $(g, T)$ are weakly compatible.

It is interesting to see here that by suitably choosing the values of $\alpha$, $\beta$, $\gamma$, $\lambda$, $L$ in (6) we get $F$ -contraction [22], Kannan contraction [11], Chatterjee contraction [4] and Reich contraction[14] (see also Tomar et al. [17] and Tomar and Ritu [19]).

**Remarks**

(i) Since $F$-contraction is proper generalization of ordinary contraction, our results for two hybrid pairs of self mappings generalize, extend and improve the results of Wardowski [22] and others existing in literature (for instance Chatterjee [4], Ćirić [6], Cosentino and Vetro [7], Hardy-Rogers [8], Kannan [11], Minak et al. [12], Reich [14], Wardowski and Dung [23], Tomar et al.( [17], [18], [20], [21]), Tomar and Ritu [19] and references therein without using completeness or closedness of space/subspace, containment requirement of range space and continuity of involved hybrid pair of mappings.

(ii) It is interesting to see that any supplementary condition is not assumed with the notion of strongly tangential to establish strict coincidence and strict common fixed point. For instance Chauhan et al. [5] used the notion of quasi-coincidentally commutativity with the notion of coincidentlly idempotent and Sintunavarat and Kumam [16] used the condition $Afa = Bgb$ (where a is a point of coincidence of $A$ and $f$ and $b$ is a point of coincidence of $B$ and $g$) with weak compatibility and hence reveal the significance of the notion of strongly tangential.

## 4. Application To Dynamic Programming

The purpose of this section is to prove the existence of solution for a system of functional equations arising in dynamic programming of multistage decision processes as an application of Theorem 1. Usually, a dynamical process consists of a state space and a decision space. The state space is the set of the initial state actions and transition model of the process; the decision space is the set of possible actions that are allowed for the process. It is well known that the dynamic programming provides useful tools for mathematical optimization and computer programming as well.

The existence of solution of functional equation arising in dynamic programming was first studied by Bellman [1] using famous Banach fixed- point theorem. There after many results of solutions and common solutions for some functional equations in dynamic programming were obtained using suitable fixed point theorems(for instance [1],[2],[3]).

Let $B(W)$ be the set of all bounded real-valued functions on $W$. We consider the operators $T_i$ and $A_i : B(W) \to B(W)$ given by

$$\begin{cases} T_i h(x) = \sup\limits_{y \in D} \{g(x,y) + G_i(x,y,h(\xi(x,y))\}, & \text{i=1,2,} \\ A_i k(x) = \sup\limits_{y \in D} \{\acute{g}_i(x,y) + \acute{G}_i(x,y,k(\xi(x,y))\}, & \text{i=1,2,} \end{cases} \tag{7}$$

where $x$ and $y$ signify the state and decision vectors. $\xi_i(x,y) : W \times D \to W, g$, $\acute{g} : W \times D \to R$, $G_i$, $\acute{G}_i : W \times D \times R \to R$ represent the transformations of the process, $T_i$ and $A_i$ denote the optimal return functions with the initial state $x, W \in U$ is a state space, $D \in V$ is a decision space and $U$, $V$ are Banach spaces, for $h_i, k_i \in B(W)$, $x \in W$ for i=1,2; these mappings are well-defined if the functions $g$ , $\acute{g}$, $G_i$ and $\acute{G}_i$ are bounded. Also, denote

$$\Theta(h,k) = \max\{d(A_1h, A_2k), d(A_1h, T_1h), d(A_2k, T_2k), \frac{1}{2}(d(A_1h, T_2k)+d(A_2k, T_1h))\},$$

for $h, k \in B(W)$. For an arbitrary $h \in B(W)$ define $\|h\| = \sup\limits_{x \in W} \|h\|$.

Clearly, $(B(W), \|.\|)$ is a Banach space endowed with metric $d$ defined by $d(h,k) = \sup_{x \in W} \mid h(x) - k(x) \mid$ where in convergence is uniform.

**Theorem 3** Let $T_i, A_i : B(W) \to B(W)$ be given by (7), for $i = 1, 2$. Suppose that the following hypotheses hold:

(1) There exists $\tau \in \mathbb{R}^+$ such that

$$|G_1(x,y,h(\xi(x,y))) - G_2(x,y,k(\xi(x,y)))| \leq e^{-\tau}\theta(h,k),$$

for all $x \in W, y \in D$;

(2) $g$, $\acute{g} : W \times D \to \mathbb{R}$ and $G_i$, $\acute{G}_i : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions, for $i = 1, 2$;

(3) There exists a sequence $\{h_n\}$ and $\{k_n\} \in B(W)$ and a function $h^* \in B(W)$ such that

$$\lim_{n \to \infty} T_1 h_n = \lim_{n \to \infty} T_2 k_n = A \in B(W),$$

whenever

$$\lim_{n \to \infty} A_1 h_n = \lim_{n \to \infty} A_2 k_n = h^* \in A$$
$$h^* \in A_1 \cap A_2.$$

(4) $A_1T_1h = T_1A_1h$, whenever $T_1h = \{A_1h\}$ and $T_2A_2h = A_2T_2h$, whenever $T_2h = \{A_2h\}$

Then the system of functional equations has a unique bounded solution.

**Proof** By hypothesis (3), $(A_1, A_2)$ is strongly tangential with respect to $(T_1, T_2)$. Now, let $\lambda$ be an arbitrary positive number, $x \in W$ and $h, k \in B(W)$. Then there exists $y_1, y_2 \in D$ such that

$$T_1h(x) < g(x,y_1) + G_1(x,y_1,h(\xi(x,y_1))) + \lambda, \tag{8}$$

$$T_2k(x) < g(x,y_2) + G_2(x,y_2,k(\xi(x,y_2))) + \lambda, \tag{9}$$

$$T_1h(x) \geq g(x,y_2) + G_1(x,y_2,h(\xi(x,y_2))), \tag{10}$$

$$T_2k(x) \geq g(x,y_1) + G_2(x,y_1,k(\xi(x,y_1))), \tag{11}$$

Next, by using (8) and (11), we obtain

$$T_1h(x) - T_2k(x) < G_1(x,y_1,h(\xi(x,y_1))) - G_2(x,y_1,k(\xi(x,y_1))) + \lambda$$

$$\leq| G_1(x, y_1, h(\xi(x, y_1))) - G_2(x, y_1, k(\xi(x, y_1))) | +\lambda$$
$$\leq e^{-\tau}\theta(h, k) + \lambda$$

Analogously, by using (9) and (10), we get

$$T_2k(x) - T_1h(x) < e^{-\tau}(\Theta(h, k)) + \lambda, \tag{12}$$

Hence

$$| T_1h(x) - T_2k(x) |< e^{-\tau}(\Theta(h, k)) + \lambda, \tag{13}$$

Notice that, the last inequality does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrarily. Implying thereby

$$d(T_1h, T_2k) \leq e^{-\tau}(\Theta(h, k)).$$

On taking logarithms, we can write

$$\tau + \ln d(T_1h, T_2k) \leq \ln(\theta(h, k)).$$

Moreover, in view of the hypotheses (4) the pairs $(A_1, T_1)$ and $(A_2, T_2)$ are weakly compatible. If we consider $F \in \mathfrak{F}$ defined by $F(t) = \ln t$, for each $t \in (0, \infty)$, then all the hypotheses of Theorem 1 are satisfied for $f = A_1$, $S = T_1$, $g = A_2$, $T = T_2$. So by using Theorem 1, the mappings have a unique common fixed point, i.e., the system of functional equations has a unique bounded solution.

It is interesting to point out that on the same lines Theorem 2 can also be utilized to find the solution of the system of functional equations arising in dynamic programming.

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