Electronic Journal of Mathematical Analysis and Applications
Vol. 7(1) Jan. 2019, pp. 102-115.
ISSN: 2090-729X(online)
http://fcag-egypt.com/Journals/EJMAA/

ON SOLUTIONS OF THE RECURSIVE EQUATIONS $x_{n+1} = x_{n-1}^p/x_n^p$ $(p > 0)$ VIA FIBONACCI-TYPE SEQUENCES

ÖZKAN ÖCALAN AND OKTAY DUMAN

Abstract. In this paper, by using the classical Fibonacci sequence and the golden ratio, we first give the exact solution of the nonlinear recursive equation $x_{n+1} = x_{n-1}/x_n$ with respect to certain powers of the initial values x_{-1} and x_0. Then we obtain a necessary and sufficient condition on the initial values for which the equation has a non-oscillatory solution. Later we extend our all results to the recursive equations $x_{n+1} = x_{n-1}^p/x_n^p$ $(p > 0)$ in a similar manner. We also get a characterization for unbounded positive solutions. At the end of the paper we analyze all possible positive solutions and display some graphical illustrations verifying our results.

1. Introduction

We consider the following second order nonlinear recursive equation

$$x_{n+1} = \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \cdots, \tag{1}$$

with any nonzero initial values x_{-1} and x_0. We should note that this equation and its generalization have been studied extensively (see, for instance, [1, 4, 5, 6, 8, 9, 11]). However, to the best of our knowledge, there is no information in the literature about the exact solution of Eq. (1). In this paper, we find an interesting connection between the exact solution of Eq. (1) and the classical Fibonacci sequence. This connection enables us to obtain necessary and sufficient condition for a non-oscillatory positive solution of Eq. (1), which are convergent monotonically to the equilibrium point 1. Later, we extend our all results to solutions of the following nonlinear recursive equations

$$x_{n+1} = \left(\frac{x_{n-1}}{x_n}\right)^p, \quad p > 0 \text{ and } n = 0, 1, \cdots, \tag{2}$$

with any nonzero initial values x_{-1} and x_0. For recent improvements about (2), see the papers [2, 3, 7, 10, 12, 13].

We first recall some basic concepts used in the paper. For all other details, we refer the book by Grove and Ladas [9] (see also [10]).

2010 *Mathematics Subject Classification.* Primary: 39A10, 39A21; Secondary: 11B39.

Key words and phrases. Recursive equations, Difference equations, Semi-cycle solutions, Non-oscillatory solutions, Fibonacci sequence, Golden ratio.

Submitted April 12, 2018.

A difference (recursive) equation of order $(k+1)$ is an equation of the form of

$$x_{n+1}=F\left(x_{n},x_{n-1},\cdots,x_{n-k}\right),\quad n=0,1,\cdots, \tag{3}$$

where F is a continuous function mapping some set J^{k+1} into J. The set J is usually an interval of real numbers, or a union of intervals. A solution of Eq. (3) is a sequence $\{x_n\}_{n=-k}^{\infty}$ satisfying (3) for all $n=0,1,\cdots$. If we prescribe a set of $(k+1)$ initial conditions $x_{-k},x_{-k+1},\cdots,x_0\in J$, then we can write

$$\begin{aligned}
x_1 &= F\left(x_0,x_{-1},\cdots,x_{-k}\right),\\
x_2 &= F\left(x_1,x_0,\cdots,x_{-k+1}\right),\\
&\cdots
\end{aligned}$$

which enables the existence of the solution $\{x_n\}_{n=-k}^{\infty}$ uniquely determined by the initial conditions. A solution of Eq. (3) which is constant for all $n\geq -k$ is said to be an equilibrium solution of (3). If $x_n=\bar{x}$ for all $n\geq -k$ is an equilibrium solution of (3), then $\bar{x}$ is called an equilibrium point.

A positive semi-cycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of Eq. (3) consists of a string of terms $\{x_l,x_{l+1},\cdots,x_m\}$, all greater than or equal to $\overline{x}$, with $l\geq -k$ and $m\leq\infty$ and such that

$$\text{either } l=-k \text{ or } l>-k \text{ and } x_{l-1}<\overline{x}$$

and

$$\text{either } m=\infty \text{ or } m<\infty \text{ and } x_{m+1}<\overline{x}.$$

A negative semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ of Eq. (3) consists of a string of terms $\{x_l,x_{l+1},\cdots,x_m\}$, all less than $\overline{x}$, with $l\geq -k$ and $m\leq\infty$ and such that

$$\text{either } l=-k \text{ or } l>-1 \text{ and } x_{l-1}\geq\overline{x}$$

and

$$\text{either } m=\infty \text{ or } m<\infty \text{ and } x_{m+1}\geq\overline{x}.$$

Finally, a solution $\{x_n\}_{n=-k}^{\infty}$ of Eq. (3) is said to be non-oscillatory about $\bar{x}$ if there exists $N\geq -k$ such that

$$\text{either } x_n>\overline{x} \text{ for all } n\geq N$$

or

$$x_n<\overline{x} \text{ for all } n\geq N.$$

Otherwise, $\{x_n\}_{n=-k}^{\infty}$ is called oscillatory about $\bar{x}$.

Our strategy for this paper is as follows:

- In Section 2, we get the exact solution of (1) by means of the Fibonacci sequence and the golden ratio, and obtain a necessary and sufficient condition on the initial values x_{-1} and x_0 for which Eq. (1) has a non-oscillatory solution.
- In Section 3, we extend our all results to the solutions of Eq. (2). We also get a characterization for unbounded positive solutions.
- In the last section, we analyze all possible positive solutions and display some graphical illustrations verifying our results.

2. Non-Oscillatory Solutions of Eq. (1)

We first obtain the exact solution of Eq. (1) with the help of the classical Fibonacci sequence defined by

$$\begin{cases} f_{n+2} = f_n + f_{n+1}, & n = 0, 1, 2, \cdots \\ f_0 = 1 \text{ and } f_1 = 1. \end{cases} \tag{4}$$

It is easy to check that

$$f_n = \frac{\varphi_1^{n+1} - \varphi_2^{n+1}}{\sqrt{5}}, \ n = 0, 1, 2, \cdots, \tag{5}$$

where

$$\begin{aligned} \varphi_1 &= \frac{1+\sqrt{5}}{2} \text{ (the golden ratio)}, \\ \varphi_2 &= \frac{1-\sqrt{5}}{2} \end{aligned}$$

Then we get the following result.

Theorem 1. *For any nonzero initial values x_{-1} and x_0, the exact solution of Eq. (1) is*

$$x_n = \begin{cases} \dfrac{x_{-1}^{f_{n-1}}}{x_0^{f_n}}, & \text{if } n = 1, 3, 5, \cdots \\ \\ \dfrac{x_0^{f_n}}{x_{-1}^{f_{n-1}}}, & \text{if } n = 2, 4, 6, \cdots, \end{cases} \tag{6}$$

where f_n is the n-th Fibonacci number given by the formula (5).

Proof. Consider (1) by taking $n = 0, 1, 2, \cdots$ as follows:

$$\begin{aligned} n &= 0 \Rightarrow x_1 = \frac{x_{-1}}{x_0} \\ n &= 1 \Rightarrow x_2 = \frac{x_0}{x_1} = \frac{x_0^2}{x_{-1}}, \\ n &= 2 \Rightarrow x_3 = \frac{x_1}{x_2} = \frac{x_{-1}/x_0}{x_0^2/x_{-1}} = \frac{x_{-1}^2}{x_0^3}, \\ n &= 3 \Rightarrow x_4 = \frac{x_2}{x_3} = \frac{x_0^2/x_{-1}}{x_{-1}^2/x_0^3} = \frac{x_0^5}{x_{-1}^3}, \\ & \cdots \end{aligned}$$

If we continue this process and also consider (4), then the solution in (6) immediately follows from a simple induction. □

Now we need the following well-known properties of the Fibonacci numbers:

$$\lim_{n\to\infty} \frac{f_n}{f_{n-1}} = \varphi_1 \tag{7}$$

and

$$\frac{f_{2n-1}}{f_{2n-2}} < \varphi_1 < \frac{f_{2n}}{f_{2n-1}}, \quad n = 1, 2, 3, \cdots, \tag{8}$$

where φ_1 is the golden ratio as stated before.

The next result is a special case of Lemma 4.4 in [9, p. 78] (see also [1]).

Lemma 1. *If $\{x_n\}_{n=-1}^{\infty}$ is a positive solution of Eq.* (1) *which consists of a single semi-cycle, then $\{x_n\}_{n=-1}^{\infty}$ converges monotonically to the equilibrium point $\overline{x} = 1$.*

Now, we are ready to study the semi-cycle analysis of Eq. (1).

Theorem 2. *For Eq.* (1) , *the following statements hold true:*

(i) *A positive solution of Eq.* (1) *consists of a single positive semi-cycle if and only if*

$$x_0 \geq 1 \quad and \quad x_{-1} = x_0^{\varphi_1}. \tag{9}$$

(ii) *A positive solution of Eq.* (1) *consists of a single negative semi-cycle if and only if*

$$0 < x_0 < 1 \quad and \quad x_{-1} = x_0^{\varphi_1}. \tag{10}$$

Furthermore, in both cases, the solution $\{x_n\}$ converges monotonically to 1.

Proof. From the similarity, we just prove (i).

Necessary. Assume that a positive solution of Eq. (1) consists of a single positive semi-cycle, which means that $\{x_n\}$ is a non-oscillatory solution. From (6), we may write that

$$x_{-1}^{f_{n-1}} \geq x_0^{f_n} \text{ for } n = 1, 3, 5, \cdots$$

and

$$x_0^{f_n} \geq x_{-1}^{f_{n-1}} \text{ for } n = 2, 4, 6, \cdots$$

Then, we get

$$x_0^{f_n/f_{n-1}} \leq x_{-1} \text{ for } n = 1, 3, 5, \cdots$$

and

$$x_0^{f_n/f_{n-1}} \geq x_{-1} \text{ for } n = 2, 4, 6, \cdots$$

Taking limit as $n \to \infty$ on the both sides of the last inequalities and using the property (7), we observe that

$$x_{-1} = x_0^{\varphi_1}.$$

And also, from the assumption, it must be $x_0 \geq 1$. Hence, the proof of (9) is completed.

Sufficiency. Assume that (9) holds. Then, it follows from (6) that, for $n = 1, 2, 3, \cdots$,

$$x_{2n-1} = x_0^{\varphi_1 f_{2n-2} - f_{2n-1}}$$

and

$$x_{2n} = x_0^{f_{2n} - \varphi_1 f_{2n-1}}.$$

Using the property (8) we get

$$x_n \geq 1 \text{ for } n = 1, 2, 3, \cdots.$$

And, from the assumption, $x_0 \geq 1$ and $x_{-1} \geq 1$, we see that $x_n \geq 1$ for all $n \geq -1$. Furthermore, Lemma 1 implies that the solution in (i) converges decreasingly to 1. Therefore, the proof is completed. □

Remark 1. *From Lemma* 1 *and Theorem* 2*, we can say that Eq.* (1) *has a positive non-oscillatory solution which is convergent to* 1 *if and only if $x_0 > 0$ and $x_{-1} = x_0^{\varphi_1}$.*

3. Non-Oscillatory Solutions of Eq. (2)

For a given $p > 0$, define the Fibonacci-type sequences as follows:

$$\begin{cases} f_{n+2}(p) = p\left(f_n(p) + f_{n+1}(p)\right), & n = 0, 1, 2, \cdots, \\ f_0(p) = 1 \text{ and } f_1(p) = p \end{cases} \tag{11}$$

and

$$\begin{cases} g_{n+2}(p) = p\left(g_n(p) + g_{n+1}(p)\right), & n = 0, 1, 2, \cdots, \\ g_0(p) = p \text{ and } g_1(p) = p^2. \end{cases} \tag{12}$$

Then, by using the characteristic equation of (11) and (12), one can observe that

$$f_n(p) = \frac{\left(\varphi_1(p)\right)^{n+1} - \left(\varphi_2(p)\right)^{n+1}}{\sqrt{p^2 + 4p}}, \quad n = 0, 1, 2, \cdots \tag{13}$$

and

$$g_n(p) = p\frac{\left(\varphi_1(p)\right)^{n+1} - \left(\varphi_2(p)\right)^{n+1}}{\sqrt{p^2 + 4p}}, \quad n = 0, 1, 2, \cdots, \tag{14}$$

where

$$\begin{aligned} \varphi_1(p) &= \frac{p + \sqrt{p^2 + 4p}}{2} \quad \text{(say, } p\text{-golden ratio)} \\ \varphi_2(p) &= \frac{p - \sqrt{p^2 + 4p}}{2}. \end{aligned}$$

Then, using the idea as in Theorem 1, we get the next result for the exact solution of Eq. (2) for each $p > 0$.

Theorem 3. *For any nonzero initial values x_{-1} and x_0 and for every $p > 0$, the exact solution of Eq.* (2) *is*

$$x_n = \begin{cases} \dfrac{x_{-1}^{g_{n-1}(p)}}{x_0^{f_n(p)}}, & \text{if } n = 1, 3, 5, \cdots \\[2ex] \dfrac{x_0^{f_n(p)}}{x_{-1}^{g_{n-1}(p)}}, & \text{if } n = 2, 4, 6, \cdots, \end{cases} \tag{15}$$

where $f_n(p)$ and $g_n(p)$ are the n-th Fibonacci-type numbers given by (13) *and* (14)*, respectively.*

Proof. If we take $n = 0, 1, 2, \cdots$ in Eq. (2), then we may write that

$$\begin{aligned} n &= 0 \Rightarrow x_1 = \frac{x_{-1}^p}{x_0^p} \\ n &= 1 \Rightarrow x_2 = \frac{x_0^p}{x_1^p} = \frac{x_0^p}{x_{-1}^{p^2}/x_0^{p^2}} = \frac{x_0^{p^2+p}}{x_{-1}^{p^2}}, \\ n &= 2 \Rightarrow x_3 = \frac{x_1^p}{x_2^p} = \frac{x_{-1}^{p^2}/x_0^{p^2}}{x_0^{p^3+p^2}/x_{-1}^{p^3}} = \frac{x_{-1}^{p^3+p^2}}{x_0^{p^3+2p^2}}, \\ n &= 3 \Rightarrow x_4 = \frac{x_2^p}{x_3^p} = \frac{x_0^{p^3+p^2}/x_{-1}^{p^3}}{x_{-1}^{p^4+p^3}/x_0^{p^4+2p^3}} = \frac{x_0^{p^4+3p^3+p^2}}{x_{-1}^{p^4+2p^3}}, \\ &\cdots \end{aligned}$$

If we continue this process and also consider (11) and (12), then the solution in (15) is obtained easily by an induction. □

Now we need the following lemmas.

Lemma 2. *For every $p > 0$, we have*

$$\lim_{n\to\infty} \frac{f_n(p)}{g_{n-1}(p)} = \frac{\varphi_1(p)}{p},$$

where $\varphi_1(p)$ is the p-golden ratio, as stated before.

Proof. Use (13) and (14). □

Lemma 3. *For every $p > 0$, we have*

$$\frac{f_{2n-1}(p)}{g_{2n-2}(p)} < \frac{\varphi_1(p)}{p} < \frac{f_{2n}(p)}{g_{2n-1}(p)}, \quad n = 1, 2, 3, \cdots .$$

Proof. From (13) and (14), observe that, for all $n = 1, 2, 3, \cdots$,

$$\frac{\varphi_1(p)}{p} g_{2n-2}(p) - f_{2n-1}(p) = -\left(\varphi_2(p)\right)^{2n-1} > 0$$

and

$$f_{2n}(p) - \frac{\varphi_1(p)}{p} g_{2n-1}(p) = \left(\varphi_2(p)\right)^{2n} > 0,$$

whence the result. □

Combining the above facts and following the same lines as in Theorem 2 for each $p > 0$ (just replace f_{n-1} and f_n by $g_{n-1}(p)$ and $f_n(p)$, respectively), we arrive the next result; so we omit its proof.

Theorem 4. *For every $p > 0$, the following statements hold true:*

(*i*) *A positive solution of Eq.* (2) *consists of a single positive semi-cycle if and only if*

$$x_0 \geq 1 \quad \text{and} \quad x_{-1} = x_0^{\frac{\varphi_1(p)}{p}}.$$

(*ii*) *A positive solution of Eq.* (2) *consists of a single negative semi-cycle if and only if*

$$0 < x_0 < 1 \quad \text{and} \quad x_{-1} = x_0^{\frac{\varphi_1(p)}{p}}.$$

Furthermore, in both cases, the solution $\{x_n\}$ converges monotonically to 1.

Remark 2. *Observe that, for $p = 1$, all Fibonacci-type sequences used above and the corresponding golden ratios are equivalent, i.e.*

$$f_n = f_n(1) = g_n(1), \quad n = 0, 1, 2, \cdots$$

and

$$\varphi_1 = \varphi_1(1) = \frac{1 + \sqrt{5}}{2}.$$

One can check that, for every $p > 0$, Eq. (2) *has the following positive non-oscillatory solution, which is convergent to* 1, *for given initial values x_{-1} and x_0*

such that $x_{-1} = x_0^{\varphi_1(p)/p}$:

$$x_n := x_n(p, x_0) = \begin{cases} x_0^{\frac{\varphi_1(p)}{p} g_{n-1}(p) - f_n(p)}, & \text{if } n = 1, 3, 5, \cdots \\ \\ x_0^{f_n(p) - \frac{\varphi_1(p)}{p} g_{n-1}(p)}, & \text{if } n = 2, 4, 6, \cdots . \end{cases} \tag{16}$$

Then, for each $p > 0$*, the corresponding solution in* (16) *converges decreasingly to* 1 *for* $x_0 > 1$ *while it converges increasingly to* 1 *for* $0 < x_0 < 1$*. Hence, we can say that, for given* $p, x_{-1}, x_0 > 0$*, if* $\{x_n\}_{n=-1}^{\infty}$ *is a solution of Eq.* (2)*, then*

$$\{x_n\}_{n=-1}^{\infty} \text{ is non-oscillatory } \Leftrightarrow \ x_{-1} = x_0^{\frac{\varphi_1(p)}{p}}. \tag{17}$$

This answers the open problem for $\alpha = 0$ *and* $p > 0$ *posed by Stević (see* [12, p. 2]*). Obviously, if* $x_{-1} = x_0 = 1$*, then for every* $p > 0$*, we get the equilibrium solution* $x_n = 1$ *for* $n \geq -1$*. See the next section for more details.*

Before closing this section, we focus on the unbounded solutions of Eq. (2). First observe that

$$\varphi_1(p) > 1 \Leftrightarrow p > \frac{1}{2}. \tag{18}$$

Then (18) implies that

$$\text{if } p > \frac{1}{2}, \text{ then } f_n(p) \to \infty \text{ and } g_n(p) \to \infty \text{ as } n \to \infty. \tag{19}$$

Theorem 5. *Let* $x_{-1}, x_0 > 0$ *and* $p > 1/2$ *be given. Assume that* $\{x_n\}_{n=-1}^{\infty} := \{x_n(p, x_{-1}, x_0)\}_{n=-1}^{\infty}$ *is a solution of Eq.* (2)*. Then,*

$$\{x_n\}_{n=-1}^{\infty} \text{ is unbounded } \Leftrightarrow \ x_{-1} \neq x_0^{\frac{\varphi_1(p)}{p}}.$$

Furthermore, in this case,

$$\text{either } \lim_{n\to\infty} x_{2n-1} = 0 \text{ and } \lim_{n\to\infty} x_{2n} = +\infty, \tag{20}$$

or

$$\lim_{n\to\infty} x_{2n-1} = +\infty \text{ and } \lim_{n\to\infty} x_{2n} = 0. \tag{21}$$

Proof. Necessity. Assume that the solution $\{x_n\}_{n=-1}^{\infty}$ is unbounded. Then, it follows from (15) and (19) that $\{x_n\}_{n=-1}^{\infty}$ cannot be non-oscillatory. Hence, by (17), we get $x_{-1} \neq x_0^{\varphi_1(p)/p}$.

Sufficiency. Assume now that $x_{-1} \neq x_0^{\varphi_1(p)/p}$ holds. Then, we consider the following possible cases:

(a) $x_0 > 1$ and $x_{-1} > x_0^{\varphi_1(p)/p}$,
(b) $x_0 > 1$ and $1 \leq x_{-1} < x_0^{\varphi_1(p)/p}$,
(c) $x_0 \geq 1$ and $0 < x_{-1} < 1$ or $x_0 > 1$ and $0 < x_{-1} \leq 1$,
(d) $0 < x_0 < 1$ and $x_{-1} \geq 1$ or $0 < x_0 \leq 1$ and $x_{-1} > 1$,
(e) $0 < x_0 < 1$ and $x_0^{\varphi_1(p)/p} < x_{-1} \leq 1$,
(f) $0 < x_0 < 1$ and $0 < x_{-1} < x_0^{\varphi_1(p)/p}$.

In the case of (a), there exists a number $a > 1$ such that

$$1 < x_0^{\frac{\varphi_1(p)}{p}} < x_0^{a\frac{\varphi_1(p)}{p}} \leq x_{-1}.$$

Then, we may write from (15) that, for every $n = 1, 2, 3, \cdots$,

$$x_{2n-1} = \frac{x_{-1}^{g_{2n-2}(p)}}{x_0^{f_{2n-1}(p)}} \geq \frac{x_0^{a \frac{\varphi_1(p)}{p} g_{2n-2}(p)}}{x_0^{f_{2n-1}(p)}} = x_0^{a \frac{\varphi_1(p)}{p} g_{2n-2}(p) - f_{2n-1}(p)}. \tag{22}$$

We also get

$$a \frac{\varphi_1(p)}{p} g_{2n-2}(p) - f_{2n-1}(p) = a \left(\frac{\varphi_1(p)}{p} g_{2n-2}(p) - f_{2n-1}(p) \right) + (a-1) f_{2n-1}(p).$$

From (13), (14), (19) the right hand side of the last inequality goes to the infinity as $n \to \infty$. Since $x_0 > 1$ and $a > 1$, (22) implies that $x_{2n-1} \to \infty$ as $n \to \infty$. Similarly, from (15),

$$0 < x_{2n} = \frac{x_0^{f_{2n}(p)}}{x_{-1}^{g_{2n-1}(p)}} \leq \frac{x_0^{f_{2n}(p)}}{x_0^{a \frac{\varphi_1(p)}{p} g_{2n-1}(p)}} = x_0^{f_{2n}(p) - a \frac{\varphi_1(p)}{p} g_{2n-1}(p)},$$

which yields that $x_{2n} \to 0$ as $n \to \infty$ due to (19).

Since the case of (b) is a symmetric position of (a), it is omitted.

The cases of (c) and (d) are straightforward from the definition in (15).

By using a similar idea as in (a) and (b) one can also prove that if (e) or (f) holds, then we easily get either (20) or (21). Therefore the proof is completed. $\square$

We should note that if we take $p = 1$ in Theorem 5, then wee see that Eq. (1) has unbounded positive solutions if and only if $x_{-1} \neq x_0^{\varphi_1}$. In this case, either (20) or (21) is satisfied.

The case of $0 < p \leq 1/2$ is little bit complicated. A natural question arises: Is there any unbounded positive solution of Eq. (2) for $0 < p \leq 1/2$? We also answer this question after the following discussion.

- If $0 < p < 1/2$, then we see from (13) and (14) that $\lim_{n\to\infty} f_n(p) = \lim_{n\to\infty} g_n(p) = 0$. In this case, (15) implies that $\lim_{n\to\infty} x_n = 1$ for every initial values $x_{-1}, x_0 > 0$. Hence, if $0 < p < 1/2$, every positive solution of Eq. (2) is oscillatory and convergent to 1 for all choices of initial values $x_{-1}, x_0 > 0$ provided that $x_{-1} \neq x_0^{\varphi_1(p)/p}$.
- If $p = 1/2$, then we get $\lim_{n\to\infty} f_n(1/2) = 2/3$ and $\lim_{n\to\infty} g_n(1/2) = 1/3$. In this case, by (15) we observe that

$$\lim_{n\to\infty} x_{2n-1} = \frac{x_{-1}^{1/3}}{x_0^{2/3}} \text{ and } \lim_{n\to\infty} x_{2n} = \frac{x_0^{2/3}}{x_{-1}^{1/3}}$$

  for every initial values $x_{-1}, x_0 > 0$. Therefore, we get the following result.

Corollary 1. *The following statements hold true:*

(i) *Eq.* (2) *has positive prime period-2 solutions if and only if* $p = 1/2$.

(ii) *Let* $p = 1/2$ *and* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.* (2). *Then,*

$$\{x_n\}_{n=-1}^{\infty} \text{ is periodic with period 2 if and only if } x_{-1} = \frac{1}{x_0} \text{ with } x_0 \neq 1.$$

Proof. (i) *Necessity.* Suppose that $\{x_n\}_{n=-1}^{\infty}$ is a positive prime period-2 solution of Eq. (2). Then, we get

$$x_{n+1} = x_{n-1} \text{ and } x_{n+2} = x_n.$$

Using this fact and also considering (2), we see that

$$x_{n+2} = x_n = \left(\frac{x_n}{x_{n+1}}\right)^p = \left(\frac{x_n}{x_{n-1}}\right)^p = \frac{1}{x_{n+1}} = \frac{1}{x_{n-1}},$$

which gives $x_n = 1/x_{n-1}$. Hence

$$x_{n-1}^{2p} = x_{n-1},$$

which implies $p = 1/2$.

Sufficiency. If $p = 1/2$, then by taking $x_{-1} = \frac{1}{x_0}$ with $x_0 > 0$ and $x_0 \neq 1$, it follows from Eq. (2) that

$$\begin{aligned} n &= 0 \Rightarrow x_1 = \left(\frac{x_{-1}}{x_0}\right)^{1/2} = \frac{1}{x_0} \\ n &= 1 \Rightarrow x_2 = \left(\frac{x_0}{x_1}\right)^{1/2} = x_0, \\ n &= 2 \Rightarrow x_3 = \left(\frac{x_1}{x_2}\right)^{1/2} = \frac{1}{x_0}, \\ n &= 3 \Rightarrow x_4 = \left(\frac{x_2}{x_3}\right)^{1/2} = x_0, \\ &\cdots \end{aligned}$$

Hence the corresponding positive solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (2) satisfies the relation

$$x_{n-1} = \frac{1}{x_n} \text{ for } n = 0, 1, 2, \cdots.$$

If we use the above fact in Eq. (2), then we get

$$x_{n+2} = \left(\frac{x_n}{x_{n+1}}\right)^{1/2} = \left(\frac{x_n}{(x_{n-1}/x_n)^{1/2}}\right)^{1/2} = x_n,$$

whence the result.

(ii) *Necessity.* Assume that $\{x_n\}_{n=-1}^{\infty}$ is a period-2 solution of Eq. (2). Since $x_{n+1} = x_{n-1}$ for every $n = 0, 1, 2, \cdots$, we get

$$x_{n+1} = \left(\frac{x_{n-1}}{x_n}\right)^{1/2} = x_{n-1},$$

which implies that

$$x_{n-1} = \frac{1}{x_n} \text{ for } n = 0, 1, 2, \cdots \tag{23}$$

Taking $n = 0$ in (23), the necessity part of (ii) follows immediately.

Sufficiency. It is clear from (i). □

Finally, the above discussion shows that if $0 < p \leq 1/2$, then every positive solution of Eq. (2) must be bounded, which clarifies the problem stated above.

4. Graphical Illustrations

So far we have seen that, for each $p, x_{-1}, x_0 > 0$, Eq. (2) has positive non-oscillatory (and so convergent) solutions, 2-periodic solutions and unbounded solutions. Now we analyze them with respect to the position of p and the initial values x_{-1}, x_0.

We first consider the non-oscillatory solutions in (16), i.e., the case of $x_{-1} = x_0^{\varphi_1(p)/p}$.

Now let $x_0 > 0$ ($x_0 \neq 1$) be fixed in (16). Then, after some calculations we see that

$$\frac{\partial}{\partial p} x_n(p, x_0) = \frac{n\left(\varphi_2(p)\right)^{n-1} \ln x_0}{x_0^{(\varphi_2(p))^n}} h(p) \text{ for } n = 1, 3, 5, \cdots,$$

and

$$\frac{\partial}{\partial p} x_n(p, x_0) = -n\left(\varphi_2(p)\right)^{n-1} x_0^{(\varphi_2(p))^n} \left(\ln x_0\right) h(p) \text{ for } n = 2, 4, 6, \cdots,$$

where

$$h(p) = \frac{(p+2)\sqrt{p^2+4p} - \left(p^2+4p\right)}{2p(p+4)}.$$

Since $h(p) > 0$ for every $p > 0$, we observe that

$$\text{if } x_0 > 1, \text{ then } \frac{\partial}{\partial p} x_n(p, x_0) > 0 \text{ for every } n = 1, 2, 3, \cdots \text{ and } p > 0 \tag{24}$$

and

$$\text{if } 0 < x_0 < 1, \text{ then } \frac{\partial}{\partial p} x_n(p, x_0) < 0 \text{ for every } n = 1, 2, 3, \cdots \text{ and } p > 0. \tag{25}$$

From (24) and (25), we can say that:

- if $x_0 > 1$, then the corresponding solution in (16) (for $n = 1, 2, 3, \cdots$) is strictly increasing with respect to $p > 0$. This means that, for each fixed $x_0 > 1$, if one increases p, then the solution converges to the equilibrium point 1 more slowly (see Figure 1);
- if $0 < x_0 < 1$, then the corresponding solution in (16) (for $n = 1, 2, 3, \cdots$) is strictly decreasing with respect to $p > 0$. Therefore, in this case, for bigger values of p, we get solutions converging more slowly to 1 (see Figure 2).

Let $p > 0$ be fixed. Then, one can also check that the corresponding solution in (16) is strictly increasing with respect to $x_0 > 0$ (see Figures 3 and 4).

Now we consider the oscillatory solutions and periodic solutions of Eq. (2).

Take $p = 0.2.$, $x_{-1} = 0.3$ and $x_0 = 2$. Then, we know that the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ in (15) is convergent to 1 by oscillating around 1, which is indicated in Figure 5.

Taking $p = 0.5$, $x_{-1} = 0.4$ and $x_0 = 1.2$, one can see the positive prime period-2 solution indicated in Figure 6.

Finally, for $p = 1/2$, if we take $x_{-1} = 1/x_0 = 1.4$, then we get the 2-periodic solution of Eq. (2) in Figure 7.

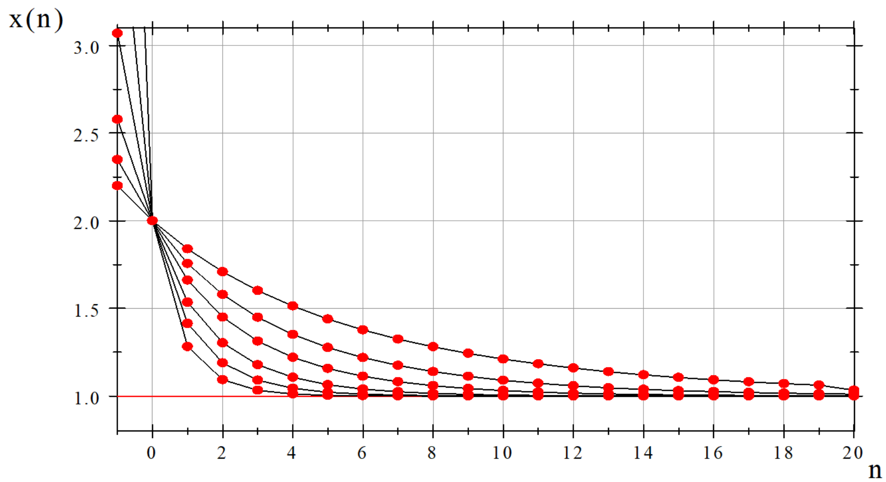

FIGURE 1. Graphs of non-oscillatory solutions corresponding to the values $x_0 = 2$, $x_{-1} = x_0^{\varphi_1(p)/p}$ and $p = 0.2,\ 0.5,\ 1,\ 2,\ 3.5,\ 6.4$.

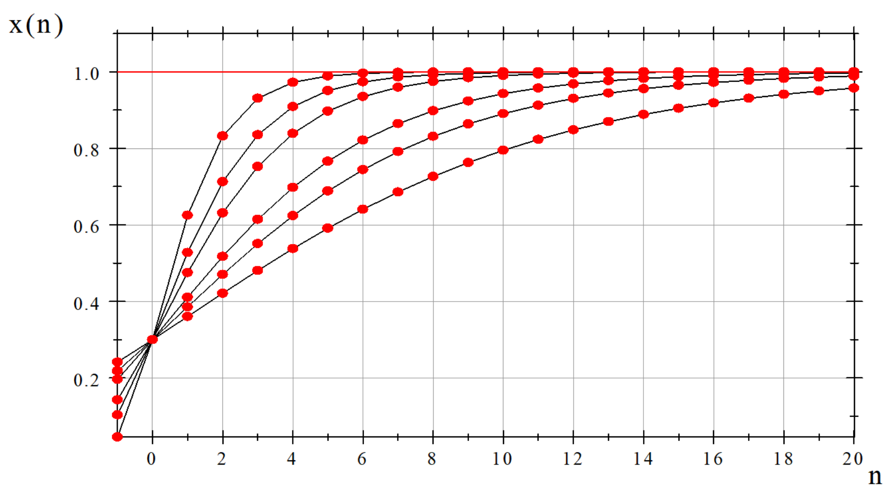

FIGURE 2. Graphs of non-oscillatory solutions corresponding to the values $x_0 = 0.3$, $x_{-1} = x_0^{\varphi_1(p)/p}$ and $p = 0.25,\ 0.6,\ 1,\ 2.1,\ 3,\ 4.7$.

REFERENCES

[1] A. M. Amleh, E. A. Grove, G. Ladas and D. A. Georgiou, On the recursive sequence $x_{n+1} = \alpha + (x_{n-1}/x_n)$. J. Math. Anal. Appl. 233, 2, 790-798, 1999.

[2] K. S. Berenhaut and S. Stević, A note on positive non-oscillatory solutions of the difference equation $x_{n+1} = A + (x_{n-k}/x_n)^p$. J. Difference Equ. Appl. 12, 5, 495-499, 2006.

[3] K. S. Berenhaut and S. Stević, The behaviour of the positive solutions of the difference equation $x_n = A + (x_{n-2}/x_{n-1})^p$. J. Difference Equ. Appl. 12, 9, 909-918, 2006.

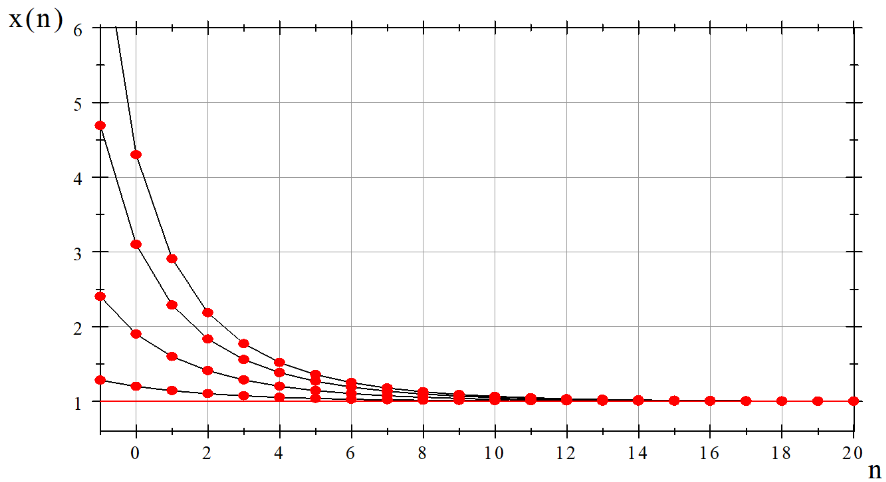

FIGURE 3. Graphs of non-oscillatory solutions corresponding to the values $p = 2$ and $x_0 = 1.2$, 1.9, 3.1, 4.3, and $x_{-1} = x_0^{\varphi_1(p)/p}$.

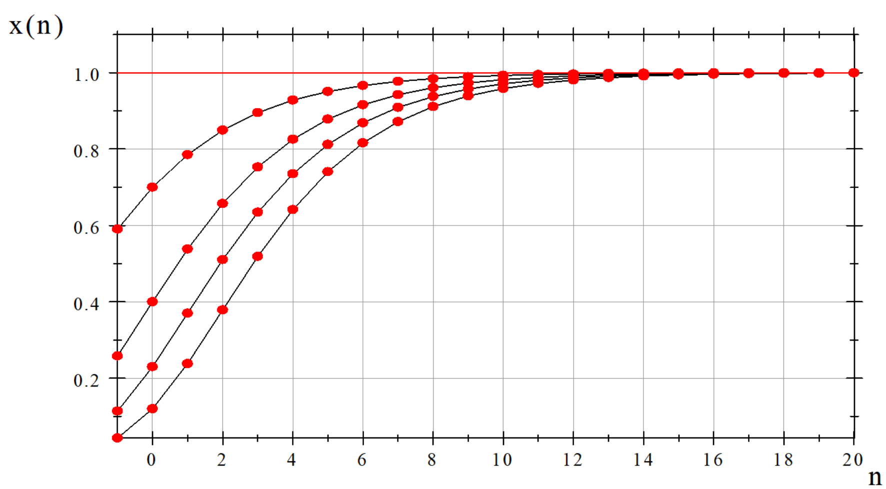

FIGURE 4. Graphs of non-oscillatory solutions corresponding to the values $p = \sqrt{2}$ and $x_0 = 0.12, 0.23, 0.4, 0.7$, and $x_{-1} = x_0^{\varphi_1(p)/p}$.

[4] E. Camouzis, and R. DeVault, The forbidden set of $x_{n+1} = p + x_{n-1}/x_n$. Special Session of the American Mathematical Society Meeting, Part II (San Diego, CA, 2002). J. Difference Equ. Appl. 9, 8, 739-750, 2003.

[5] R. DeVault, C. Kent and W. Kosmala, On the recursive sequence $x_{n+1} = p + x_{n-k}/x_n$. Special Session of the American Mathematical Society Meeting, Part II (San Diego, CA, 2002). J. Difference Equ. Appl. 9, 8, 721-730, 2003.

[6] R. DeVault, V. L. Kocic and D. Stutson, Global behavior of solutions of the nonlinear difference equation $x_{n+1} = p_n + x_{n-1}/x_n$. J. Difference Equ. Appl. 11, 8, 707-719, 2005.

[7] H. M. El-Owaidy, A. M. Ahmed and M. S. Mousa, On asymptotic behaviour of the difference equation $x_{n+1} = \alpha + (x_{n-1}/x_n)^p$. J. Appl. Math. Comput. 12, 1-2, 31-37, 2003.

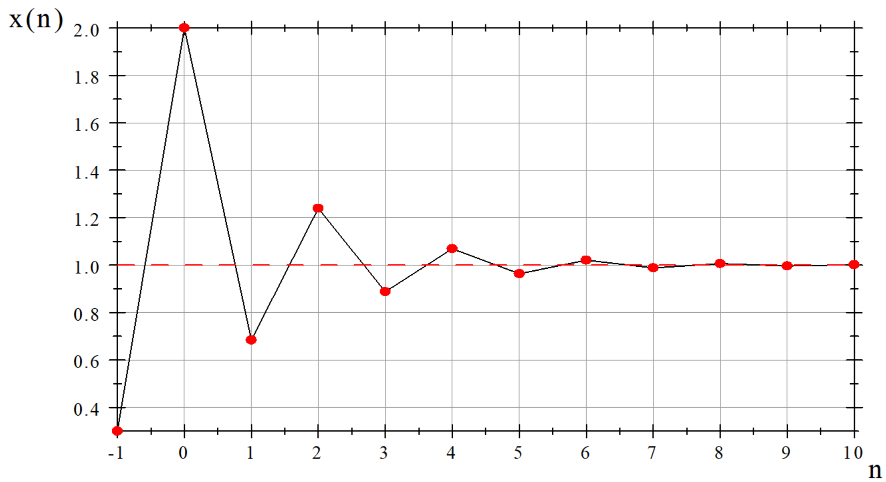

Figure 5. Graph of the oscillatory solution corresponding to the values $p = 0.2$, $x_{-1} = 0.3$ and $x_0 = 2$.

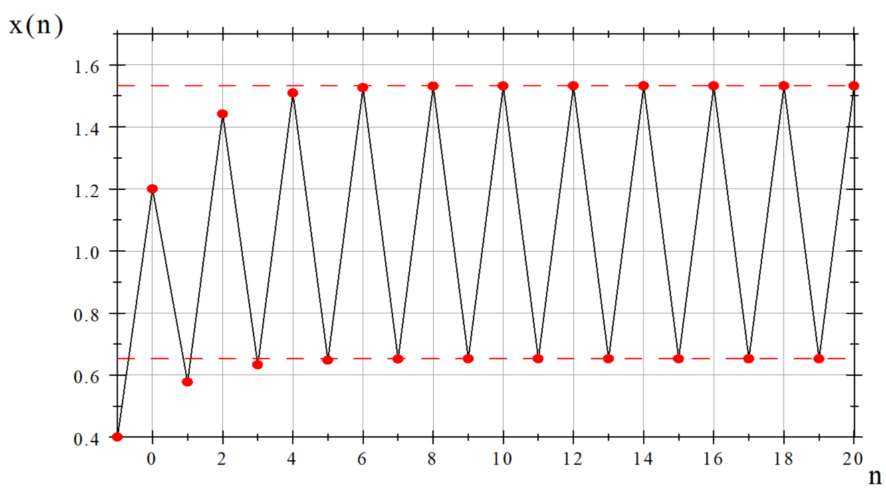

Figure 6. Graph of the solution corresponding to the values $x_{-1} = 0.4$, $x_0 = 1.2$ and $p = 0.5$.

[8] H. M. El-Owaidy, A. M. Ahmed and M. S. Mousa, On asymptotic behaviour of the difference equation $x_{n+1} = \alpha + x_{n-k}/x_n$. Appl. Math. Comput. 147, 1, 163-167, 2004.
[9] E. A. Grove and G. Ladas, Periodicities in nonlinear difference equations. Advances in Discrete Mathematics and Applications, 4. Chapman & Hall/CRC, Boca Raton, FL, 2005.
[10] M. R. S. Kulenović and G. Ladas, Dynamics of second order rational difference equations. With open problems and conjectures. Chapman & Hall/CRC, Boca Raton, FL, 2002.
[11] M. R. S. Kulenović, G. Ladas and C. B. Overdeep, On the dynamics of $x_{n+1} = p_n + x_{n-1}/x_n$. J. Difference Equ. Appl. 9, 11, 1053-1056, 2003.
[12] S. Stević, On the recursive sequence $x_{n+1} = \alpha + (x_{n-1}/x_n)^p$. J. Appl. Math. Comput. 18, 1-2, 229-234, 2005.

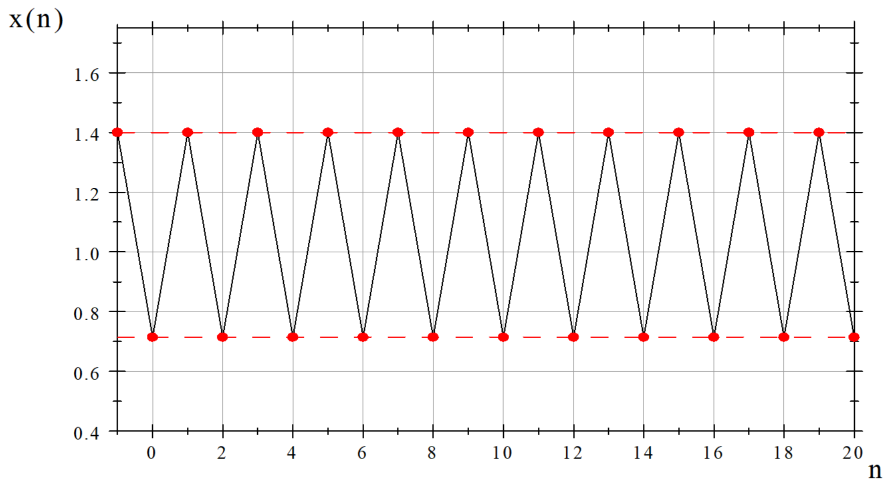

FIGURE 7. Graph of the 2-periodic solution corresponding to the values $p = 0.5$ and $x_{-1} = 1/x_0 = 1.4$.

[13] S. Stević, On the recursive sequence $x_{n+1} = A + x_n^p / x_{n-1}^r$. Discrete Dyn. Nat. Soc., Vol. 2007, Art. ID 40963, 9 pp.

Özkan Öcalan
Akdeniz University, Faculty of Science, Department of Mathematics, Antalya, Turkey
E-mail address: ozkanocalan@akdeniz.edu.tr

Oktay Duman
TOBB Economics and Technology University, Faculty of Arts and Sciences, Department of Mathematics, Ankara, Turkey
E-mail address: oduman@etu.edu.tr; okitayduman@gmail.com