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EXISTENCE OF COMMON FIXED POINT IN SYMMETRIC SPACE WITH APPLICATION

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ABSTRACT. Motivated by the fact that most of the times techniques used to establish coincidence and common fixed point do not require triangle inequality of the distance function, an attempt has been made in this paper to obtain coincidence and common fixed point theorems for S and T-compatible of type (E) and S and T- sub-sequentially continuous pairs of self-mappings in a symmetric space. Examples are given to illustrate our results and an application is also furnished to demonstrate the applicability of results obtained.

1. INTRODUCTION

M. Fréchet [4] axiomatically framed the idea of distance at the beginning of nineteen century although its knowledge is as old as the history of civilization. Actually, appreciating the Euclidean distance between two points given by the absolute difference, Fréchet expressed and generalized the notion of distance in an abstract form. It is an imperative fact that the origination of the notion of metric unlocks a novel era to mathematical analysis and consequently the interconnected disciplines. Recently, the notion of F-contraction has fascinated the attentiveness of numerous researchers and by now there exists a substantial literature related to this notion. For instance: Minak et al. [5], Tomar et al. [11]-[12], Tomar and Ritu [14], Wardowski [15], Wardowski et al. [16] and many others. Aim of this paper is to establish some common fixed-point theorems for mappings satisfying Ćirić type F-contraction in symmetric (semi-metric) space using S and T-compatibility of type (E) and S and T-sub-sequential continuity. Motivation behind is the fact that most of the times techniques used to establish coincidence and common fixed point do not require triangle inequality of the distance function. Further this appears to be of fundamental significance in view of a traditionally noteworthy open question raised by Rhoades [7] whether or not there is a contractive condition which is convincing to establish a fixed point, but does not force the mapping to be continuous at the fixed point. In this paper, we postulate one more affirmative solution to an open question of Rhoades [[8], page 242] in a symmetric space. It may be witnessed

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that in most of the fixed-point theorems in the literature, the contractive condition forces the mapping to be continuous at the fixed point however continuity is neither assumed nor implied by the contractive definition. Apart from these, we give several illustrative examples that signify the inspiration of our explorations. In fact, the common fixed-point theorems presented here enrich and improve earlier known results on the topic in the literature. Also we utilize our result to establish the existence and uniqueness of the solution of an integral equation.

2. Preliminaries

A symmetric on a set X is a nonnegative real valued function d on $X \times X$ such that
(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$.
A set X together with a symmetric d is called a symmetric space.
Following Pant [6] reciprocal continuity in a symmetric space is defined as:
Definition 1 A pair of self mappings (S, T) in a symmetric space (X, d) is reciprocally continuous iff

$$\lim_{n\to\infty} STx_n = St$$

and

$$\lim_{n\to\infty} TSx_n = Tt,$$

whenever a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Following H. Bouhadjera and Godet-Thobie [3] sub-sequential continuity in a symmetric space is defined as:
Definition 2 A pair of self mappings (S, T) in a symmetric space (X, d) is sub-sequentially continuous iff there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$ and satisfy

$$\lim_{n\to\infty} STx_n = St$$

and

$$\lim_{n\to\infty} TSx_n = Tt.$$

It is worth mentioning here that continuous or reciprocally continuous mappings are sub-sequentially continuous but the converse may be not be true.

Following S. Beloul [2] weak sub-sequential continuity, S-sub-sequential continuity and T-sub-sequential continuity in a symmetric space are defined as:
Definition 3 A pair of self mappings (S, T) in a symmetric space (X, d) is weakly sub-sequentially continuous iff there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$ and satisfy

$$\lim_{n\to\infty} STx_n = St$$

or

$$\lim_{n\to\infty} TSx_n = Tt.$$

Definition 4 A pair of self mappings (S, T) in a symmetric space (X, d) is S-sub-sequentially continuous iff there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$ and satisfy $\lim_{n\to\infty} STx_n = St$.

Definition 5 A pair of self mappings (S, T) in a symmetric space (X, d) is T-sub-sequentially continuous iff there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$ and satisfy $\lim_{n\to\infty} TSx_n = Tt$.

Example 1 Let $X = [0, 1]$ be endowed with a symmetric d. Let S and T be defined as follows:

$$Sx = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ \frac{1-x}{4}, & \frac{1}{2} < x \le 1, \end{cases} \qquad Tx = \begin{cases} 1-x, & 0 \le x \le \frac{1}{2} \\ \frac{1}{8}, & \frac{1}{2} < x \le 1, \end{cases}$$

Clearly S and T are discontinuous at $\frac{1}{2}$. If we consider a sequence $\{x_n\}$, which is defined for each $n \ge 1$ by $x_n = \frac{1}{2}$, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \dfrac{1}{2}$. Also we have:

$$\lim_{n\to\infty} STx_n = \lim_{n\to\infty} S\frac{1}{2} = S\frac{1}{2}$$

and

$$\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} T\frac{1}{2} = T\frac{1}{2}.$$

So a pair of self mappings (S, T) is sub-sequentially continuous and hence weakly sub-sequentially continuous. Now let $\{y_n\}$ be a sequence defined for each $n \ge 1$ by $y_n = \frac{1}{2} - \frac{1}{n}$, we have

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = \frac{1}{2},$$

and

$$\lim_{n\to\infty} STy_n = \lim_{n\to\infty} S(\frac{1}{2} + \frac{1}{n}) = \frac{1}{8} \ne S\frac{1}{2}$$

but

$$\lim_{n\to\infty} TSy_n = \lim_{n\to\infty} T(\frac{1}{2} - \frac{1}{n}) = \frac{1}{2} = T\frac{1}{2}.$$

Hence, it is not reciprocally continuous.

Notice that reciprocally continuous or sub-sequentially continuous pair of mappings is weakly sub-sequentially continuous however the reverse implication is not essentially true. Also one may notice that a pair of self mappings (S, T) is T-sub-sequentially continuous but not S-sub-sequentially continuous.

Example 2 Let $X = [0, 4]$ be endowed with a symmetric d. Let S and T be defined as follows:

$$Sx = \begin{cases} \frac{4-x}{3}, & 0 \le x \le 1 \\ \frac{x}{2}, & 1 < x \le 4, \end{cases} \qquad Tx = \begin{cases} 2-x, & 0 \le x \le 1 \\ 3, & 1 < x \le 4, \end{cases}$$

Clearly S and T are discontinuous at 1. If we consider a sequence $\{x_n\}$, which is defined for each $n \ge 1$ by $x_n = 1$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 1.$$

Also we have:

$$\lim_{n\to\infty} STx_n = \lim_{n\to\infty} S1 = S1,$$

and

$$\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} T1 = T1.$$

But, if $\{y_n\}$ be a sequence defined for each $n \geq 1$ by $y_n = 1 - \frac{1}{n}$, we have

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = 1$$

and

$$\lim_{n\to\infty} STy_n = \lim_{n\to\infty} S(1 + \frac{1}{n}) = \frac{1}{2} \neq S1.$$

and

$$\lim_{n\to\infty} TSy_n = \lim_{n\to\infty} T(1 + \frac{1}{3n}) = 3 \neq T1.$$

So a pair of self mappings (S, T) is sub-sequentially continuous, S-sub-sequentially continuous and T-subsequentially continuous but not reciprocally continuous.

Example 3 Let $X = [0, 10]$ be endowed with a symmetric d. Let S and T be defined as follows:

$$Sx = \begin{cases} 8 - x, & 0 \leq x \leq 4 \\ \frac{x}{2}, & 4 < x \leq 10, \end{cases} \qquad Tx = \begin{cases} \frac{8+x}{3}, & 0 \leq x \leq 4 \\ 7, & 4 < x \leq 10, \end{cases}$$

Clearly S and T are discontinuous at 4. If we consider a sequence $\{x_n\}$, which is defined for each $n \geq 1$ by $x_n = 4 - \frac{1}{n}$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 4.$$

Also we have:

$$\lim_{n\to\infty} STx_n = \lim_{n\to\infty} S(4 - \frac{1}{3n}) = 4 = S4$$

and

$$\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} T(4 + \frac{1}{n}) = 7 \neq T4.$$

So a pair of self mappings (S, T) is S-sub-sequentially continuous but is neither T-sub-sequentially continuous nor reciprocally continuous.

Notice that appropriate change of position of S and T in the Definition 4 yields Definition 5. S-sub-sequentially continuous (or T-sub-sequentially continuous) pair of self mappings is weakly sub-sequentially continuous but not sub-sequentially continuous. For details on the variants of continuity one may refer to Tomar and Karapinar [13].
Following Singh and Singh [10] compatiblity of type (E), S-compatiblity of type (E) and T-compatiblity of type (E) in a symmetric space are defined as:

Definition 6 A pair of self mappings (S, T) in a symmetric space (X, d) is compatible of type (E), if

$$\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSx_n = St$$

and

$$\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} STx_n = Tt,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

Definition 7 A pair of self mappings (S,T) of a symmetric space (X,d) is S-compatible of type (E), if

$$\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} STx_n = Tt$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

Definition 8 A pair of self mappings (S,T) of a symmetric space (X,d) is T-compatible of type (E) if

$$\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSx_n = St$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

Example 4 Let $X = [0,1]$ be endowed with a symmetric d. Let S and T be defined as follows:

$$Sx = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ \frac{1+x}{4}, & \frac{1}{2} < x \le 1, \end{cases} \qquad Tx = \begin{cases} \frac{x}{2}, & 0 \le x \le \frac{1}{2} \\ \frac{1}{5}, & \frac{1}{2} < x \le 1, \end{cases}$$

We consider a sequence $\{x_n\}$, which is defined for each $n \ge 1$ by $x_n = \frac{1}{n}$. Clearly

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0$$

and also we have:

$$\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} STx_n = T0$$

and

$$\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSx_n = S0.$$

Then (S,T) is Compatible of type (E).

Example 5 Let $X = [0,10]$ be endowed with a symmetric d. Let S and T be defined as follows:

$$Sx = \begin{cases} 3-x, & 0 \le x \le 2 \\ \frac{1}{2}, & 2 < x \le 10, \end{cases} \qquad Tx = \begin{cases} \frac{5-x}{2}, & 0 \le x < 2 \\ \frac{1}{2}, & 2 \le x \le 10, \end{cases}$$

If we consider a sequence $\{x_n\}$, which is defined for each $n \ge 1$ by $x_n = 1 - \frac{1}{n}$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 2.$$

Also we have:

$$\lim_{n\to\infty} STx_n = \lim_{n\to\infty} S(2+\frac{1}{2n}) = \frac{1}{2} = T2$$

and

$$\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} S(2+\frac{1}{n}) = \frac{1}{2} = T2.$$

Similarly,

$$\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} T(2+\frac{1}{n}) = \frac{1}{2} \neq S2.$$

$$\lim_{n\to\infty} TTx_n = \lim_{n\to\infty} T(2+\frac{1}{2n}) = \frac{1}{2} \neq S2.$$

So a pair of self mappings (S, T) is T-compatible of type (E) but not compatible of type (E) and also does not commute at 2 as $ST2 = \frac{5}{2} \neq 2 = TS2$.
One may notice that T-compatiblity of type (E) does not reduce to weak compatibility at coincidence point. It is worth mentioning here that compatiblity of type (E) and S-compatibility of type (E) also does not reduce to weak compatibility at coincidence point like most of the weaker forms of commutativity. For details on this concept one may refer to Singh and Tomar[9]. Further notice that appropriate change of position of S and T in the Definition 7 yields Definition 8.
Let $\mathcal{F}$ be the family of all continuous functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying:

- F is strictly increasing.
- For each sequence $\{\alpha_n\}$ in X, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$, $n \in \mathbb{N}$.
- There exists $k \in (0, 1)$ satisfying $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Example (1) $F(t) = \ln t$,
(2) $F(t) = t + \ln t$,
(3) $F(t) = -\frac{1}{\sqrt{t}}$.

Definition 9 [15] A self mapping T on a metric space (X, d) is an F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

for all $x, y \in X$.

Definition 10 [5] A self mapping T on a metric space (X, d) is a Ćirić type F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(M(x, y)),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$, for all $x, y \in X$.

Wardowski and Dung [16] introduced it independently as F-weak contraction. Every F-contraction is an F-weak contraction however reverse implication does not hold.

3. Main results

First we define Ćirić type F-contraction for two pairs of mappings in a symmetric space as follows:

Definition 11 Two pairs of self mappings (A, S) and (B, T) on a symmetric space (X, d) are said to satisfy Ćirić type F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$d(Sx, Ty) > 0 \implies \tau + F(d(Sx, Ty)) \leq F(M(x, y)) \quad (1)$$

where

$$M(x, y) = \max\{d(Ax, By), d(Ax, Sx), d(By, Ty), \frac{1}{2}[d(Ax, Ty) + d(By, Sx)]\},$$

for all $x, y \in X$.

Theorem 1 Let A, B, S and T be self mappings of a symmetric space (X, d) such that:

(1) a pair (A, S) is S-sub-sequentially continuous and S-compatible of type (E) then A and S have a coincidence point.
(2) a pair (B, T) is T-sub-sequentially continuous and T-compatible of type (E) then B and T have a coincidence point.

If pairs of self mappings (A, S) and (B, T) satisfy Ćirić type F-contraction then A, B, S and T have a unique common fixed point in X.

Proof: Since a pair (A, S) is S-sub-sequentially continuous, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u,$$

for some $u \in X$ and satisfy

$$\lim_{n\to\infty} SAx_n = Su.$$

Also (A, S) is S-compatible of type (E), so

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S^2x_n = Au.$$

Consequently we obtain $Au = Su$, i.e., A and S has a coincidence point.
Similarly, since a pair (B, T) is T-sub-sequentially continuous and compatible of type (E), there exists a sequence $\{y_n\}$ such that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = v,$$

for some $v \in X$ and

$$\lim_{n\to\infty} TBy_n = Tv.$$

Also (B, T) is T-compatible of type (E) which implies

$$\lim_{n\to\infty} TBy_n = \lim_{n\to\infty} T^2y_n = Bv.$$

Consequently we obtain $Bv = Tv$, i.e., B and T has a coincidence point.
Now we claim that $Au = Bv$, if not using $x = u$ and $y = v$ in condition (1),

$$d(Su, Tv) > 0 \implies \tau + F(d(Su, Tv)) \leq F(\max\{d(Au, Bv), d(Au, Su), d(Bv, Tv), \frac{1}{2}[d(Au, Tv) + d(Bv, Su)]\}). \quad (2)$$

Therefore

$$F(d(Au,Bv)) < \tau + F(d(Su,Tv)) \leq F(d(Au,Bv)),$$

a contradiction, since F is strictly increasing function and $\tau > 0$.
So $d(Au,Bv) = 0$, i.e., $Au = Su = Bv = Tv$.
Now we claim that $u = Au$, if not using $x = x_n$ and $y = v$ in condition (1),

$$d(Sx_n,Tv) > 0 \implies \tau + F(d(Sx_n,Tv)) \leq F(\max\{d(Ax_n,Bv), d(Ax_n,Sx_n), d(Bv,Tv), \frac{1}{2}[d(Ax_n,Tv) + d(Bv,Sx_n)]\}). \tag{3}$$

Taking lim $n \to \infty$, we get

$$\tau + F(d(u,Tv)) \leq F(\max\{d(u,Au), d(u,u), \frac{1}{2}(d(u,Au) + d(Au,u))\}).$$

Therefore,

$$F(d(u,Au)) < \tau + F(d(u,Au)) \leq F(d(u,Au)),$$

a contradiction, since F is strictly increasing function and $\tau > 0$.
Then $d(u,Au) = 0$, i.e., $u = Au$. Therefore $u = Au = Su$.
Now we claim that $v = Bv$, if not using $x = u$ and $y = y_n$ in condition (1),

$$d(Su,Ty_n) > 0 \implies \tau + F(d(Su,Ty_n)) \leq F(\max\{d(Au,By_n), d(Au,Su), d(By_n,Ty_n), \frac{1}{2}[d(Au,Ty_n) + d(By_n,Su)]\})). \tag{4}$$

Taking lim $n \to \infty$, we get

$$\tau + F(d(Su,v)) \leq F(\max\{d(Au,v), d(v,v), \frac{1}{2}[d(Au,v) + d(v,Su)]\}).$$

Therefore,

$$F(d(Bv,v)) < \tau + F(d(Bv,v)) \leq F(d(Bv,v)),$$

a contradiction, since F is strictly increasing function and $\tau > 0$.
Then $d(v,Bv) = 0$, i.e., $v = Bv$. Therefore $v = Bv = Tv$.
Hence, u is a common fixed point for A, B, S and T.
The uniqueness of common fixed point of A, B, S and T is an easy consequence of (1). This completes the proof.

Now, we conclude our main result by furnishing two interesting examples to demonstrate Theorem 1 besides exhibiting its superiority over earlier relevant results.

Example 6 Let $X = [0,4]$ and symmetric $d(x,y) = (x-y)^2$. Let self-mappings A, B, S and T on X be defined as follows:

$$Ax = \begin{cases} 1, & 0 \leq x \leq 1 \\ \frac{x+4}{2}, & 1 < x \leq 4, \end{cases} \quad Bx = \begin{cases} 1, & 0 \leq x \leq 1 \\ \frac{9-x}{2}, & 1 < x \leq 4, \end{cases}$$

$$Sx = \begin{cases} 1, & 0 \leq x \leq 1, \\ \frac{x}{10}, & 1 < x \leq 4, \end{cases} \quad Tx = \begin{cases} 1, & 0 \leq x \leq 1 \\ \frac{x}{8}, & 1 < x \leq 4. \end{cases}$$

Consider a sequence $\{x_n\}$ for all $n \geq 1$ such that $x_n = 1$. It is clear that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1.$$

So

$$\lim_{n\to\infty} SAx_n = S1,$$

i.e., (A,S) is S- sub-sequentially continuous. Also,

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S^2x_n = 1 = A1.$$

Hence (A,S) is S-compatible of type (E).
Similarly, consider a sequence $\{y_n\}$ for all $n \geq 1$ such that $y_n = 1 - \frac{1}{n}$, it is clear that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1,$$

such that

$$\lim_{n\to\infty} TBy_n = T1,$$

i.e. (B,T) is T- sub-sequentially continuous. Also,

$$\lim_{n\to\infty} TBy_n = \lim_{n\to\infty} T^2y_n = 1 = B1.$$

Hence (B,T) is T-compatible of type (E).
For $x, y \in [0,1]$, we have:

$$d(Sx,Ty) = (1-1)^2 = 0.$$

For $x \in [0,1]$ and $y \in (1,4]$, we have:

$$\tau + \ln(d(Sx,Ty)) = \frac{1}{5} + 2\ln(1-\frac{y}{8}) \leq 2\ln\frac{40-5y}{8} = \ln(d(By,Ty)).$$

For $x \in (1,4]$ and $y \in [0,1]$, we have:

$$\tau + \ln(d(Sx,Ty)) = \frac{1}{5} + 2\ln(1-\frac{y}{10}) \leq 2\ln\frac{10+2x}{5} = \ln(d(Sx,Ax)).$$

For $x, y \in (1,4]$, we have:

$$\tau + \ln(d(Sx,Ty)) = \frac{1}{5} + 2\ln\frac{5x-y}{40} \leq 2\ln\frac{10+2x}{5} = \ln(d(Sx,Ax)).$$

Hence pairs (A,S) and (B,T) satisfy Ćirić type F-contraction (1) for $\tau = \frac{1}{5}$ and $F = \ln x$ and 1 is the unique common fixed point of A, B, S and T. One may notice that none of the mappings is continuous and neither $AX \not\subseteq SX$ nor $BX \not\subseteq TX$.

Example 7 Let $X = [0,10]$ and symmetric $d(x,y) = (x-y)^2$. Let self-mappings A, B, S and T on X be defined as follows:

$$Ax = \begin{cases} 2-x, & 0 \leq x \leq 1 \\ 10, & 1 < x \leq 10, \end{cases} \quad Bx = \begin{cases} \frac{4-x}{3}, & 0 \leq x \leq 1 \\ 7, & 1 < x \leq 10, \end{cases}$$

$$Sx = \begin{cases} \frac{11-x}{10}, & 0 \leq x \leq 1, \\ \frac{1}{10}, & 1 < x \leq 10, \end{cases} \quad Tx = \begin{cases} \frac{12-x}{11}, & 0 \leq x \leq 1 \\ \frac{1}{10}, & 1 < x \leq 10. \end{cases}$$

Consider a sequence $\{x_n\}$ for all $n \geq 1$ such that $x_n = 1$. It is clear that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1.$$

So

$$\lim_{n\to\infty} SAx_n = S1,$$

i.e. (A, S) is S- sub-sequentially continuous. Also,

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S^2x_n = 1 = A1.$$

Hence, (A, S) is S-compatible of type (E), but it is neither reciprocally continuous nor compatible of type (E) as there exists a sequence $z_n = 1 - \frac{1}{n}$ such that $\lim_{n\to\infty} Az_n = \lim_{n\to\infty} Sz_n = 1$, but $\lim_{n\to\infty} SAz_n \neq S1$ and $\lim_{n\to\infty} ASz_n \neq A1$. Also, $\lim_{n\to\infty} SAz_n = \lim_{n\to\infty} SSz_n \neq A1$ and $\lim_{n\to\infty} ASz_n = \lim_{n\to\infty} AAz_n \neq S1$.
Similarly, consider a sequence $\{y_n\}$ for all $n \geq 1$ such that $y_n = 1$, it is clear that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1,$$

such that

$$\lim_{n\to\infty} TBy_n = T1,$$

i.e., (B, T) is T- sub-sequentially continuous. Also,

$$\lim_{n\to\infty} TBy_n = \lim_{n\to\infty} T^2y_n = 1 = B1.$$

Hence, (B, T) is T-compatible of type (E), but it is neither reciprocally continuous nor compatible of type (E) as there exists a sequence $u_n = 1 - \frac{1}{n}$ such that $\lim_{n\to\infty} Bu_n = \lim_{n\to\infty} Tu_n = 1$, but $\lim_{n\to\infty} TBu_n \neq T1$ and $\lim_{n\to\infty} BTu_n \neq B1$. Also, $\lim_{n\to\infty} TBu_n = \lim_{n\to\infty} TTu_n \neq B1$ and $\lim_{n\to\infty} BTu_n = \lim_{n\to\infty} BBu_n \neq T1$.
For $x, y \in [0, 1]$, we have:

$$\tau + \ln(d(Sx, Ty)) = \frac{10}{9} + \ln\left(\frac{1 - 11x + 10y}{110}\right)^2 \leq \ln\left(\frac{2 - 3x + y}{3}\right)^2 = \ln(d(Ax, By)).$$

For $x \in [0, 1]$ and $y \in (1, 10]$, we have:

$$\tau + \ln(d(Sx, Ty)) = \frac{10}{9} + 2\ln(1 - \frac{x}{10}) \leq 2\ln 6.9 = 3.87 = \ln(d(By, Ty)).$$

For $x \in (1, 10]$ and $y \in [0, 1]$, we have:

$$\tau + \ln(d(Sx, Ty)) = \frac{10}{9} + 2\ln(\frac{109 - 10y}{110}) \leq 2\ln 9.9 = 4.585 = \ln(d(Ax, Sx)).$$

For $x, y \in (1, 10]$, we have:

$$d(Sx, Ty) = 0.$$

Hence pairs (A, S) and (B, T) satisfy Ćirić type F-contraction (1) for $\tau = \frac{10}{9}$ and $F = \ln x$ and 1 is the unique common fixed point of A, B, S and T. One may notice that none of the mappings is continuous and neither $AX \nsubseteq SX$ nor $BX \nsubseteq TX$.

It is interesting to note that these examples cannot be covered by all those coincidence and common fixed point theorems which require containment of range space of both the pairs, continuity/reciprocal continuity/sub-sequential continuity requirement of self mappings along with completeness (or closedness) of underlying space. Further the notions of S and T-compatiblity of type (E) are more general than other weaker forms of commutativity as it does not reduce to the weak compatibility even at the coincidence point [9].

Definition 12 A pair of self mappings (A, S) on a symmetric space (X, d) is said to satisfy Ćirić type F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$d(Sx, Sy) > 0 \implies \tau + F(d(Sx, Sy)) \leq F(M(x, y)) \tag{5}$$

where $M(x, y) = \max\{d(Ax, Ay), d(Ax, Sx), d(Ay, Sy), \frac{1}{2}[d(Ax, Sy) + d(Ay, Sx)]\}$, for all $x, y \in X$.

If $A = B$ and $S = T$, we obtain the following corollary:
Corollary 1 Let A and S be self mappings in a symmetric space (X, d) such that:

(1) (A, S) is S-sub-sequentially continuous and S-compatible of type (E) then A and S have a coincidence point.

If pair of self mappings (A, S) satisfy Ćirić type F-contraction (5) then A and S have a unique common fixed point in X.

If $F(t) = \ln t$, we obtain the following corollary:
Corollary 2 Let A, B, S and T be self mappings in a symmetric space (X, d) such that:

(1) (A, S) is S-sub-sequentially continuous and S-compatible of type (E), then A and S have a coincidence point.
(2) (B, T) is T-sub-sequentially continuous and T-compatible of type (E), then B and T have a coincidence point.

If there exists $\tau > 0$ such that

$$d(Sx, Ty) > 0 \implies \tau + \ln(d(Sx, Ty)) \leq \ln(M(x, y)) \tag{6}$$

where

$$M(x, y) = \max\{d(Ax, By), d(Ax, Sx), d(By, Ty), \frac{1}{2}[d(Ax, Ty) + d(By, Sx)]\}$$

for all $x, y \in X$, then A, B, S and T have a unique common fixed point in X.

It is interesting to point out here that all these results remain true even if we assume (A, S) to be A-sub-sequentially continuous and A-compatible of type (E) and (B, T) to be B-sub-sequentially continuous and B-compatible of type (E).

Theorem 2 Let A, B, S and T be self mappings in a symmetric space (X, d) such that

(1) (A, S) is weakly sub-sequentially continuous and compatible of type (E), then A and S have a coincidence point.
(2) (B, T) is weakly sub-sequentially continuous and compatible of type (E), then B and T have a coincidence point.

If pairs of self mappings (A, S) and (B, T) satisfy Ćirić type F-contraction (5) then A, B, S and T have a unique common fixed point in X.

Proof: Proof of Theorem 2 follows on the similar lines as of Theorem 1.

If $A = B$ and $S = T$, we obtain the following corollary:
Corollary 3 Let A and S be self mappings in a symmetric space (X, d) such that,

(1) (A, S) is weakly sub-sequentially continuous and compatible of type (E), then A and S have a coincidence point.

If a pair of self mappings (A, S) satisfies Ćirić type F-contraction (5) then A and S have a unique common fixed point in X.

Remark 1 Batra et al. [1] proved unique coincidence point result for a pair of self mappings satisfying F-g contraction by taking containment of range space of involved mappings, completeness of space along with continuity and commutativity of both the mappings. We have established existence and uniqueness of coincidence and common fixed point for two pairs of discontinuous self mappings without exploiting containment of range space of involved mappings and completeness of underlying space which is symmetric. Moreover both the pairs are S and T-compatible of type (E), which is weaker than commutativity of a pair of mappings. Further Ćirić type F-contraction used is more general than $F-g$ contraction used by Batra et al. [1].

Remark 2 For different values of F we get different contractions and hence different common fixed point theorems. Further F-contraction is proper generalization of ordinary contraction and using the fact that every metric space is a symmetric space. Our results generalize, extend and improve the results of Wardowski [15] and others existing in literature (for instance: Batra et al. [1], Beloul [2], Bouhadjera and Thobie [3], Minak et al. [5], Wardowski [15], Wardowski et al. [16] and references therein).

Remark 3 We now show that the mappings A and S used are discontinuous at a common fixed point $u(Au = Su = u)$. Let a pair (A, S) satisfies Ćirić type F-contraction and there exists a sequence $\{x_n\}$ in X such that $\lim_n Ax_n = \lim_n Sx_n = u$, for some $u \in X$. If possible, suppose A is continuous. So, $\lim_n ASx_n = Au = u$ and $\lim_n AAx_n = Au = u$. Using $x = Ax_n$ and $y = u$ in Ćirić type F-contraction, we get: $d(SAx_n, Su) > 0 \implies \tau + F(d(SAx_n, Su)) \leq F(\max\{d(AAx_n, Au), d(AAx_n, SAx_n), d(Au, Su), \frac{1}{2}d(AAx_n, Su) + d(Au, SAx_n))\})$, i.e., $\tau + F(d(SAx_n, Su)) \leq d(Su, SAx_n)$, which is a contradiction. Therefore A is discontinuous at a common fixed point. If we assume S to be continuous, following similar pattern we may show that S is also discontinuous at common fixed point. So we give one more answer to the open problem posed by Rhoades [7] regarding continuity of mappings at fixed point. ?

4. Application in Integral equation

Consider the following integral equation:

$$u(t) = h(t) + \int_0^\lambda K_i(t, s, u(s))ds, i = 1, 2, \tag{7}$$

where $t, s \in [0, \lambda]$, $\lambda \in \mathbb{R}^+$, $K_i : [0, \lambda] \times [0, \lambda] \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$. Assume that $X = \mathbb{R}([0, \lambda])$ be the set of all functions defined on $[0, \lambda]$.
Now, we define
(i)

$$\phi : X \times X \to \mathbb{R}^+$$

(ii) $d(x, y) = sup_{l \in [0,\lambda]}(x(t) - y(t))^2$. Then (X, d) is a symmetric space.
(iii) $\theta(x, y) = \max\{d(Ax, Ay), d(Ax, Sx), d(Ay, Sy), \frac{1}{2}(d(Ax, Sy) + d(Ay, Sx))\}$.

Theorem 3 Let A, $S : [0, \infty) \to [0, \infty)$. Suppose the following hypotheses hold:
(i) there exists a function $\phi : [0, \lambda] \times [0, \lambda] \to [0, +\infty]$, such that

$$| K_i(t, s, u(t)) - K_i(t, s, w(t) | \leq \phi(t, s)F(\theta(u, w)), i = 1, 2,$$

for each $u, v \in \mathbb{R}$ and each $t, s \in [0, \lambda]$ and F be strictly increasing function.
(ii) $\sup_{t \in [0,\lambda]} \int_0^\lambda \phi(t, s)ds \leq \eta$ for some $\eta < [0, 1)$,
(iii)There exists a sequence $\{x_n\}$ such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = t$ for some t in X and satisfy $\lim\limits_{n\to\infty} ASx_n = At$ or $\lim\limits_{n\to\infty} SAx_n = St$ and $\lim\limits_{n\to\infty} SSx_n = \lim\limits_{n\to\infty} SAx_n = At$ or $\lim\limits_{n\to\infty} ASx_n = \lim\limits_{n\to\infty} AAx_n = St$.
Then the system of equations (7) have a unique common solution in $[0, \infty)$.
Proof: Consider the two mappings A and S such that

$$Su(t) = h(t) + \int_0^\lambda K_1(t, s, u(s))ds$$

and

$$Aw(t) = h(t) + \int_0^\lambda K_2(t, s, w(s))ds.$$

Now,
$| Su(t) - Sw(t) | = | \int_0^\lambda [K_1(t, s, u(s) - K_1(t, s, w(s)]ds |$

$$\leq \int_0^\lambda \phi(t, s)F(\theta(u, w))ds \leq F(\theta(u, w)) \int_0^\lambda \phi(t, s)ds \leq \eta F(\theta(u, w)).$$

Consequently all the hypotheses of Corollary 3 hold. Hence A and S have a unique common fixed point and so the system of equations (6) have a unique common solution.

Conclusion Discontinuities appear everywhere. For instance in different biological, industrial and economic phenomena involving threshold operations which are discontinuous. In particular, neurons in a neural net either fires (function value = 1) or does not fire (function value =0) conditional to the fact that whether the input crosses a certain threshold or not. Numerous industrial censors, band passes filters and the diode also work in this manner. Motivated by these facts we provide yet new solutions to the once open problem on the existence of a contractive mapping which possesses a common fixed point in symmetric space but is not continuous at the common fixed point. It is worth mentioning here that even a contractive mapping does not have a fixed point in a complete metric space. For instance, if $T : X \to X$ defined by $Tx = x + \frac{1}{1+e^x}$, for all $x \in X$, where X is a set of nonnegative real numbers.

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