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GLOBAL 2-RAINBOW DOMINATION IN GRAPHS

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ABSTRACT. A 2-rainbow dominating function (2RDF) $g : V \to \mathcal{P}(A)$ $\big($ where $\mathcal{P}(A)$ is the power set of the set of two colors $A = \{1, 2\}\big)$ of a graph $G = (V, E)$ is defined to be satisfying the condition that for every vertex $v \in V$ with $g(v) = \phi$ we have $\bigcup_{u \in N(v)} g(u) = A$. The minimum value of $w(g) = \sum_{v \in V} |f(v)|$ among all such functions g of G is called the 2-rainbow domination number of G and is denoted by $\gamma_{r2}(G)$. A set $S \subseteq V$ is a global dominating set of a graph G if S dominates both G and its complement $\overline{G}$. The minimum cardinality $\gamma_g(G)$ of a global dominating set of G is called the global domination number of the graph G. In this paper, we introduce the global 2-rainbow domination number $\gamma_{gr2}(G)$ of a graph G, study some of its properties, determine its exact values for some specific graphs and we characterize the graphs G with $\gamma_{gr2}(G) = p$, where p is the number of vertices of G.

1. Introduction

All graphs considered here are finite, undirected without loops and multiple edges. For a graph $G = (V, E)$, let V and E denote the set of all vertices and edges of G with $|V| = p$ and $|E| = q$, respectively. The open neighborhood and the closed neighborhood of a vertex $v \in V$ are defined by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. The cardinality of $N(v)$ is called the degree of the vertex v and denoted by $deg(v)$ in G. The maximum and the minimum degrees in G are denoted respectively by $\Delta(G)$ and $\delta(G)$. That is $\Delta(G) = \max_{v \in V} |N(u)|$, $\delta(G) = \min_{v \in V} |N(u)|$. For more terminology and notations about graph, we refer the reader to [1, 10].

A subset D of V is called dominating set if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that v is adjacent to u. The minimum cardinality of a dominating set in G is called the domination number of G and is denoted by $\gamma(G)$. For more details about domination of graphs, we refer the reader to [11].

In [14], the concept of domination of a graph G has extended to be a domination of the graph G and its complement $\overline{G}$ and is called the global domination of G. A set $S \subseteq V$ is a global dominating set of a graph G if S dominates both G and its

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complement $\overline{G}$. The minimum cardinality $\gamma_g(G)$ of a global dominating set of G is called the global domination number of the graph G.

The Roman domination of graphs has introduced in [6]. A function $f : V \to \{0, 1, 2\}$ is called a Roman dominating function (RDF) of a graph G, if each vertex $v \in V$ with $f(v) = 0$ is adjacent to at least one vertex $u \in V$ for which $f(u) = 2$. The minimum weight $w(f) = \sum_{x \in V} f(x)$ among all the RDFs of G is called the Roman domination number of G and is dented by $\gamma_R(G)$. In [13], the authors have extended the concept of global domination number to the Roman domination number $\gamma_{gR}(G)$ of a graph G and study some of its properties.
Proposition 1[13] For any graph G, $\gamma_g(G) \leq \gamma_{gR}(G) \leq 2\gamma_g(G)$.

The rainbow domination of graphs has introduced in [2] and has been studied extensively by several authors in [3, 4, 5, 7, 8, 9, 12, 15, 16]. A 2-rainbow dominating function (2RDF) on a graph G is a function $g : V \to \mathcal{P}(A)$ (where $\mathcal{P}(A)$ is the power set of the set of two colors $A = \{1, 2\}$) satisfying the condition that for every vertex $v \in V$ with $g(v) = \phi$ we have $\bigcup_{u \in N(v)} g(u) = A$. The weight $w(g)$ or simply $g(V(G))$ of a function g is defined by $g(V(G)) = \sum_{v \in V} |g(v)|$. The 2-rainbow domination number $\gamma_{r2}(G)$ of a graph G is defined to be the minimum value of $g(V(G))$ among all the 2RDFs g of G. In this paper, we introduce the global 2-rainbow domination number of a graph G that we denote it by $\gamma_{gr2}(G)$ as follows: we define a global 2-rainbow dominating function (G2RDF) $f : V \to \mathcal{P}(A)$ on G to be a 2RDF for both G and $\overline{G}$. The global 2-rainbow domination number $\gamma_{gr2}(G)$ of G is the minimum value of $w(f) = f(V(G)) = \sum_{v \in V} |f(v)|$ among all the G2RDFs f of G.

2. Notation and Definitions

The distance between two vertices u and v in a connected graph G is the number of edges in a shortest path connecting them. The eccentricity of a vertex v is the greatest distance between v and any other vertex and denoted by $e(v)$. The diameter $diam(G)$ of G is the greatest eccentricity of a vertex in G.

The union of two graphs G and H is the graph obtained by combine their vertex sets and edge sets, namely $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$. The Cartesian product of two graphs G and H, where $|V(G)| = p_1$, $|V(H)| = p_2$ and $|E(G)| = q_1$, $|E(H)| = q_2$ is denoted by $G \,\square\, H$ has the vertex set $V(G) \times V(H)$ and, two vertices (u, u') and (v, v') are connected by an edge if and only if either ([$u = v$ and $u'v' \in E(H)$]) or ([$u' = v'$ and $uv \in E(G)$]). The corona product $G \circ H$ of two graphs G and H, where $|V(G)| = p_1$, $|V(H)| = p_2$ and $|E(G)| = q_1$, $|E(H)| = q_2$ is the graph obtained by taking $|V(G)|$ copies of H and joining each vertex of the i-th copy with vertex $u \in V(G)$.

A maximal complete subgraph of a graph G is called a clique. The clique number $\omega(G)$ of a graph G is the maximum order among the complete subgraphs of G. A set S of vertices is called independent if no two vertices in S are adjacent. Throughout this paper, we denote to the path, cycle, complete and wheel graphs by P_p, C_p, K_p and W_p, respectively. $K_{r,m}$ is the complete bipartite graph on $r + m$ vertices.

3. Some Properties

Definition 1 A global 2-rainbow dominating function of a graph $G = (V, E)$ is a function f that assigns to each vertex a set of colors from the set $A = \{1, 2\}$

such that for each vertex $v \in V$ with $f(v) = \phi$ we have $\bigcup_{u \in N_G(v)} f(u) = A$ and $\bigcup_{u \in N_{\overline{G}}(v)} f(u) = A$, where $N_{\overline{G}}(v)$ is the open neighborhood set of v in $\overline{G}$. The global 2-rainbow domination number $\gamma_{gr2}(G)$ of G is the minimum of $w(f) = f\big(V(G)\big) = \sum_{v \in V} |f(v)|$ over all such functions of G.

For a graph $G = (V, E)$, let $f : V \to \mathcal{P}(A)$ be a G2RDF of G and let $(V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ be the partition of V induced by f, where $V_\phi = \{v \in V : f(v) = \phi\}$, $V_{\{1\}} = \big\{v \in V : f(v) = \{1\}\big\}$, $V_{\{2\}} = \big\{v \in V : f(v) = \{2\}\big\}$ and $V_A = \{v \in V : f(v) = A\}$. Clearly that there exists a one to one correspondence between the functions $f : V \to \mathcal{P}(A)$ and the ordered partition $(V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ of V. Thus we will write $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$.

Proposition 2 For any graph G, $\gamma_g(G) \leq \gamma_{gr2}(G) \leq \gamma_{gR}(G) \leq 2\gamma_g(G)$.
Proof. By using Proposition 1, we need only to prove that $\gamma_g(G) \leq \gamma_{gr2}(G)$ and $\gamma_{gr2}(G) \leq \gamma_{gR}(G)$. Let $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ be a γ_{gr2}-function of G. Then clearly that $V_{\{1\}} \cup V_{\{2\}} \cup V_A$ dominates V_ϕ, so $V_{\{1\}} \cup V_{\{2\}} \cup V_A$ is a global dominating set of G. Thus

$$\gamma_g(G) \leq |V_{\{1\}} \cup V_{\{2\}} \cup V_A| \leq |V_{\{1\}}| + |V_{\{2\}}| + |V_A| \leq |V_{\{1\}}| + |V_{\{2\}}| + 2|V_A| = \gamma_{gr2}(G).$$

For the other inequality, suppose $f = (V_0, V_1, V_2)$ be a γ_{gR}-function of G. Define the function $h : V \to \mathcal{P}(A)$ by

$$h(v) = \begin{cases} \phi, & v \in V_0; \\ \{1\}\, or\, \{2\}, & v \in V_1; \\ A, & v \in V_2. \end{cases}$$

It is easy to see that the function h is a G2RDF of G. Hence, $\gamma_{gr2}(G) \leq w(h) = |V_1| + 2|V_2| = \gamma_{gR}(G)$.

Proposition 3 Let G be a graph. Then $\gamma_g(G) = \gamma_{gr2}(G)$ if and only if $G = K_p$ or $G = \overline{K_p}$.
Proof. Suppose $\gamma_g(G) = \gamma_{gr2}(G)$. Let $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ be a γ_{gr2}-function of G. Then $|V_{\{1\}}| + |V_{\{2\}}| + |V_A| = |V_{\{1\}}| + |V_{\{2\}}| + 2|V_A|$ which means $|V_A| = 0$. Therefore $\gamma_g(G) = p$. Since $\gamma_g(G) = p$ if and only if $G = K_p$ or $G = \overline{K_p}$ (see [14]). Hence the result holds.
The converse is clear.

Proposition 4 Let G be a graph. Then $\gamma_{gr2}(G) = \gamma_{r2}(G)$ if and only if there exists a γ_{r2}-function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ of G such that for every vertex $v \in V_\phi$ there exists either a vertex $u \in V_A$ such that $u \notin N(v)$ or two vertices $x \in V_{\{1\}}$, $y \in V_{\{2\}}$ such that $x, y \notin N(v)$.
Proof. Let $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ be a γ_{r2}-function of G. Suppose f satisfies the given condition, clearly that f is a G2RDF of G. Then $\gamma_{gr2}(G) \leq \gamma_{r2}(G)$. Hence, $\gamma_{gr2}(G) = \gamma_{r2}(G)$.
Conversely, we have $\gamma_{gr2}(G) = \gamma_{r2}(G)$. For any γ_{r2}-function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ of G, suppose there exists a vertex $v \in V_\phi$ such that either $V_A \subseteq N(v)$ or at least $V_{\{1\}} \subseteq N(v)$. Then $\bigcup_{u \in N_{\overline{G}}(v)} f(v) \neq A$. Therefore, $\gamma_{r2}(G) < \gamma_{gr2}(G)$, a contradiction. Hence the result holds.

Proposition 5 For any graph G of order $p \geq 4$ with $\Delta(G) = p - 1$ and $\delta(G) = 1$, we have $\gamma_{gr2}(G) = 4$.
Proof. Let u and v be two vertices of G such that $deg(u) = p - 1$ and $deg(v) = 1$. Define $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(u) = f(v) = A$ and $f(x) = \phi, \forall x \in V(G) \backslash \{u, v\}$. Clearly that $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ is a G2RDF of G with a minimum weight because we cannot assign more than $p - 2$ vertices by ϕ in G under a function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$. Hence, $\gamma_{gr2}(G) = 4$.

Observation 1 For any graph G on $p \geq 4$ vertices, $4 \leq \gamma_{gr2}(G) \leq p$.

4. Exact values of some specific graphs

In this section, we determine the exact values of γ_{gr2} for some standard graphs like paths, cycles, complete, complete bipartite and wheel graphs and also for $G \,\Box\, K_2$, $G \circ \overline{K_n}$, where G is a connected graph and $\overline{K_n}$ is the null graph with n vertices.

Proposition 6 [3]

(1) $\gamma_{r2}(P_p) = \lfloor \frac{p}{2} \rfloor + 1$.
(2) For $p \geq 3$, $\gamma_{r2}(C_p) = \lfloor \frac{p}{2} \rfloor + \lceil \frac{p}{4} \rceil - \lfloor \frac{p}{4} \rfloor$.

Theorem 1

(1) $\gamma_{gr2}(P_p) = \begin{cases} p, & p = 2,3,4; \\ 4, & \text{p=5}; \\ \lfloor \frac{p}{2} \rfloor + 1, & p \geq 6. \end{cases}$
(2) $\gamma_{gr2}(C_p) = \begin{cases} p, & p = 3,4; \\ 4, & \text{p=5}; \\ \lfloor \frac{p}{2} \rfloor + \lceil \frac{p}{4} \rceil - \lfloor \frac{p}{4} \rfloor, & p \geq 6. \end{cases}$
(3) $\gamma_{gr2}(K_p) = p$.
(4) $\gamma_{gr2}(K_{r,m}) = 4$, where $r + m \geq 4$.
(5) For $p \geq 5$, $\gamma_{gr2}(W_p) = \begin{cases} 5, & p = 5,7,8,9; \\ 4, & p = 6; \\ 6, & p \geq 10. \end{cases}$

Proof. We only prove (5) and (1)$-$(4) are obvious. Let $V(W_p) = \{v, v_1, \ldots, v_{p-1}\}$, where v is the center vertex and let $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ be a γ_{gr2}-function of W_p. For $p = 5$, we define $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(u) = \{1\}$ for all $u \in V(W_p)$ and for $p = 7$, we define $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(v) = \{1\}$, $f(v_1) = f(v_4) = A$ and $f(u) = \phi$ for all $u \in V(W_p) \setminus \{v, v_1, v_4\}$ and for $p = 8$, we define $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(v) = f(v_1) = \{1\}$, $f(v_2) = f(v_4) = f(v_7) = \{2\}$ and $f(v_3) = f(v_5) = f(v_6) = \phi$, also when $p = 9$, we define $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(v) = f(v_1) = \{1\}$, $f(v_2) = f(v_5) = f(v_8) = \{2\}$ and $f(u) = \phi$ for all $u \in V(W_p) \setminus \{v, v_1, v_2, v_5, v_8\}$. It is clear that, f is a γ_{gr2}-function of W_p with $w(f) = 5$. Hence, $\gamma_{gr2}(W_p) = 5$, for $p = 5, 7, 8, 9$.
Now, for $p = 6$, we define $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(v) = f(v_1) = \{1\}$, $f(v_2) = f(v_5) = \{2\}$ and $f(v_3) = f(v_4) = \phi$, which is a γ_{gr2}-function of W_6 with $w(f) = 4$. Hence, $\gamma_{gr2}(W_6) = 4$. Finally, for $p \geq 10$, we define $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(v) = f(v_1) = A$, $f(v_2) = f(v_{p-1}) = \{2\}$ and $f(u) = \phi$

for all $u \in V(W_p) \setminus \{v, v_1, v_2, v_{p-1}\}$. Hence, $\gamma_{gr2}(W_p) = 6$, for $p \geq 10$.

Rainbow domination in a graph G has a natural connection with the study of $\gamma(G \square K_k)$ with $k \geq 1$. If the vertex set of K_k is $\{x_1, x_2, \ldots, x_k\}$, then there is a one-to-one correspondence between the set of k-rainbow dominating functions of G and the dominating sets of $G \square K_k$. For a given k-rainbow dominating function of G the set

$$D_f = \bigcup_{v \in V} \left(\bigcup_{i \in f(v)} \{(v, x_i)\} \right),$$

is a dominating set of $G \square K_k$. The reverse correspondence is clear [2].
Observation 2[2] For any graph G and $k \geq 1$, $\gamma_{rk}(G) = \gamma(G \square K_k)$.

Actually, the result in Observation 2, is not always true for $\gamma_{gr2}(G)$ and $\gamma_g(G \square K_2)$. In the following theorem we show that when the equality between $\gamma_{gr2}(G)$ and $\gamma_g(G \square K_2)$ holds.
Theorem 2 Let G be a connected graph. Then $\gamma_{gr2}(G) = \gamma_g(G \square K_2)$ if and only if $\gamma_{gr2}(G) = \gamma_{r2}(G)$.
Proof. By using Observation 2, it is enough if we prove that $\gamma(G \square K_2) = \gamma_g(G \square K_2)$ for any connected graph G. Since G is connected, then it is clear that, any γ-set D of $G \square K_2$ must contain vertices from the two copies of G (let us consider $|G| = n \geq 3$ because $n = 2$ is trivial). Thus any vertex $x \in V(G \square K_2) \setminus D$ has at least a vertex $y \in D$ such that $y \notin N(x)$ (see Figure 1). Therefore, D is a global dominating set of $G \square K_2$. Hence, $\gamma_{r2}(G) = \gamma(G \square K_2) = \gamma_g(G \square K_2)$.

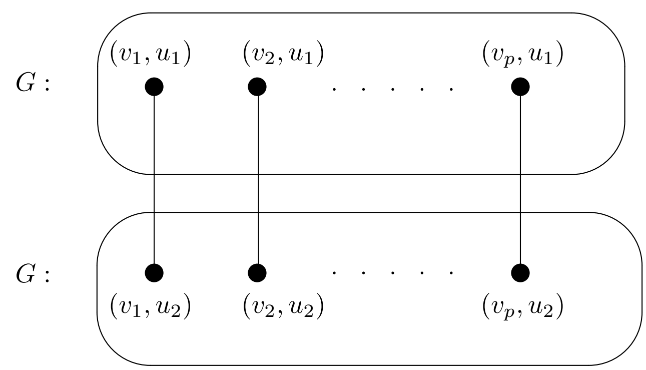

FIGURE 1. $G \square K_2$, where $V(K_2) = \{u_1, u_2\}$

Theorem 3 For any connected graph G on $p \geq 2$ vertices,

$$\gamma_{gr2}(G \circ \overline{K_n}) = \begin{cases} p + \gamma(G), & \text{if } p \geq 3 \text{ and } n = 1; \\ 2p, & \text{otherwise.} \end{cases}$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $V(\overline{K_n}) = \{u_1, u_2, \ldots, u_n\}$ as in Figure 2. Without loss of generality, to define a G2RDF of $G \circ \overline{K_n}$ we have three cases:
Case 1. We can define a function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(u_{ij}) = \phi$ for all $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, n$ and $f(v) = A$, $\forall\, v \in V(G)$. Therefore f is a G2RDF

of $G \circ \overline{K_n}$ with $w(f) = 2p$.

Case 2. We can define a function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(v) = \{2\}$, $\forall\, v \in D$, $f(v) = \phi$, $\forall\, v \in V(G) - D$, for some $D \subseteq V(G)$ and for the leaves u_{ij}, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, n$ by $f(u_{ij}) = \{2\}$ if $v_i \in D$ and $f(u_{ij}) = \{1\}$ if $v_i \in V(G) - D$. Clearly that f is a G2RDF of $G \circ \overline{K_n}$ with $w(f) = p + |D|$ if and only if D is a dominating set of G. Hence, the smallest weight of a function f in this case is when D is a γ-set of G.

Case 3. We can define a function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(v) = A$, $\forall\, v \in D$, $f(v) = \phi$, $\forall\, v \in V(G) - D$, where $D \subseteq V(G)$ and $f(u_{ij}) = \phi$ if $v_i \in D$ and $f(u_{ij}) = \{1\}$ if $v_i \in V(G) - D$. Therefore, f is a G2RDF of $G \circ \overline{K_n}$ with $w(f) = p + |D|$ if and only if D is a dominating set of G (note that, if $\gamma(G) = 1$ with $|G| = p \geq 3$, then we have to label at least one vertex of the leaves u_{ij} by $\{2\}$ when $v_i \in V(G) - D$). Thus, the smallest weight of a function f in this case is when $\gamma(G) = |D|$.

Hence,

$$\gamma_{gr2}(G \circ \overline{K_n}) = \begin{cases} p + \gamma(G), & \text{if } p \geq 3 \text{ and } n = 1; \\ 2p, & \text{otherwise.} \end{cases}$$

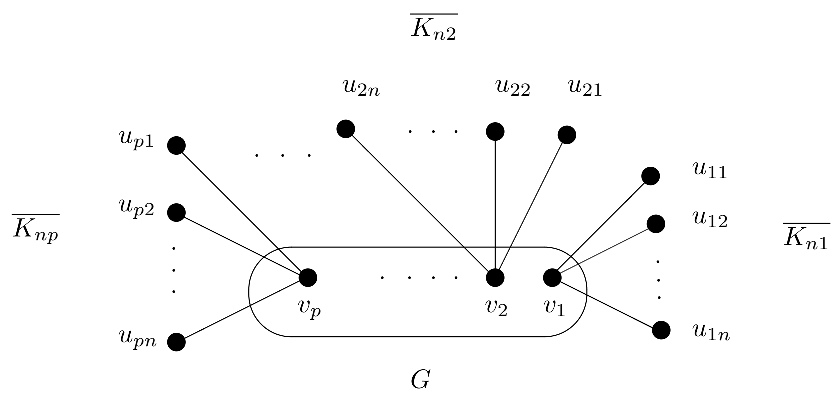

FIGURE 2. $G \circ \overline{K_n}$

5. GRAPHS WITH $\gamma_{gr2} = p$

In this section, we characterize graphs G with $\gamma_{gr2}(G) = p$.

Proposition 7 For any graph G on $p \leq 4$ vertices, $\gamma_{gr2}(G) = p$.

Proof. For all graphs of order $p \leq 3$ the proof is clear because in this case we cannot define any γ_{gr2}-function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ with $f(v) = \phi$ for any $v \in V(G)$. Suppose now $p = 4$. If $G \cong K_4$, then $\gamma_{gr2}(G) = 4$. Also, we have the following cases:

Case 1. If $G \cong K_4 - e$ (e is an edge of G), then $G \cong H_1$ (see Figure 3). Thus for H_1, we have two options to define a function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ of G with minimum weight. Either $f(v_2)$ and $f(v_4)$ equal to singleton sets and $f(v_1) = \phi$, $f(v_3) = A$ and vice versa, or $f(v_i)$ equal to singleton sets for all $i = 1, 2, 3, 4$. Hence, $\gamma_{gr2}(G) = p$.

Case 2. If $G \cong K_4 - 2e$, then either $G \cong C_4$ or $G \cong H_2$ (Figure 3). For $G \cong C_4$, clearly that $\gamma_{gr2}(G) = p$ (Theorem 1), and for $G \cong H_2$, we have three options to define a function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ with minimum weight. Either $f(v_3) = A$, $f(v_4) = \phi$ and $f(v_1)$, $f(v_2)$ equal to different singleton sets, or $f(v_1) = \phi$, $f(v_4) = A$ and $f(v_2)$, $f(v_3)$ equal to different singleton sets $\big($with the same thing for $f(v_2)$ instead of $f(v_1)\big)$, or $f(v_i)$ equal to singleton sets for all $i = 1, 2, 3, 4$. Hence, $\gamma_{gr2}(G) = p$.
Case 3. If $G \cong K_4 - 3e$, then either $G \cong P_4$ or $G \cong S_4$ or $G \cong C_3 \cup K_1$. Hence, $\gamma_{gr2}(G) = p$.
Case 4. If $G \cong K_4 - 4e$, then either $G \cong K_2 \cup K_2$ or $G \cong P_3 \cup K_1$. Then clearly that, $\gamma_{gr2}(G) = p$.
Note that all the other graphs of four vertices are clear.

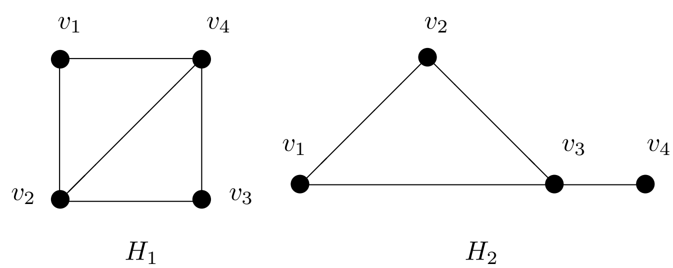

FIGURE 3. $H_1 \cong K_4 - e$ and $H_2 \cong K_4 - 2e$

We consider all the graphs from now to the end of the paper to be connected on p vertices.
Theorem 4[13] For any graph G with $\gamma_{gR}(G) = p$, $diam(G) \leq 3$.

Theorem 5 Any graph G on p vertices with $\gamma_{gr2}(G) = p$ has diameter less than or equal three.
Proof. The proof is straightforward by Proposition 2 and Theorem 4.

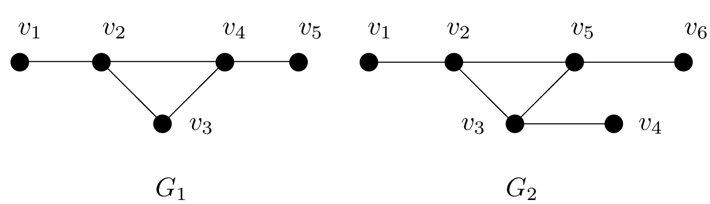

FIGURE 4. Graphs G_1 and G_2

Theorem 6 [13] Let G be a graph with $diam(G) = 3$. Then $\gamma_{gR}(G) = p$ if and only if G is one of the graphs P_4, G_1, G_2, where G_1, G_2 are given in Figure 4.

According to Proposition 2 and Theorem 6, we have the following theorem.
Theorem 7 Let G be a graph with $diam(G) = 3$. Then $\gamma_{gr2}(G) = p$ if and only if $G \cong P_4$.
Proof. By Propositions 2, 7 and Theorem 6, we get the same result about P_4. But $\gamma_{gr2}(G_1)$ and $\gamma_{gr2}(G_2)$ do not equal p, which we are going to clarify in the following. For G_1 define the function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(v_1) = f(v_2) = \{1\}$, $f(v_4) = f(v_5) = \{2\}$ and $f(v_3) = \phi$ which is a G2RDF of G_1, then $\gamma_{gr2}(G_1) \neq p$, and for G_2 define the function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(v_1) = f(v_4) = \{1\}$, $f(v_5) = f(v_6) = \{2\}$ and $f(v_2) = f(v_3) = \phi$ which is a G2RDF of G_2, then $\gamma_{gr2}(G_2) \neq p$ (see Figure 4).

In the following, we study the graphs G of $diam(G) = 2$.
Definition 2 Let $G = (V, E)$ be a graph with $diam(G) = 2$. We Consider $F_1 \subseteq V(G)$ induces a maximum clique in G, $F_2 = V(G) \setminus F_1$, where $|F_i| = p_i$, $i = 1, 2$ and $V(F_1) = \{y_1, y_2, \ldots, y_{p_1}\}$, $V(F_2) = \{x_1, x_2, \ldots, x_{p_2}\}$.

Theorem 8 Let $\omega(G) = 2$. Then $\gamma_{gr2}(G) = p$ if and only if G is one of the graphs P_3, C_4 and $K_{1,3}$.
Proof. Since $diam(G) = 2$ and $\omega(G) = 2$, then $|F_1| = p_1 = 2$. Therefore, G is a free-triangle graph of diameter two. Suppose $p \leq 4$. Then by Proposition 7 and Theorem 1[part (4)], the result is satisfied for C_4 and $K_{1,m}$ with $m = 2$ or $m = 3$. For the other free-triangle graphs of diameter two (here $p \geq 5$), we have the following cases:
Case 1. Suppose G has a vertex v of degree $p-1$. Then $G \cong K_{1,m}$ with $m \geq 4$. In this case we can define a G2RDF of G with $\gamma_{gr2}(G) < p$ by labeling the center vertex v and one of its neighborhood by $A = \{1, 2\}$ and for all the other vertices in G by ϕ.
Case 2. Suppose now G has no vertex of degree $p-1$. Let v_1 and v_2 be two non-adjacent vertices in G which they have a maximum number of common neighbors among all the other vertices in G. Suppose u is a common neighbor for v_1 and v_2. Then there exist at least two non-adjacent vertices to u say x and y (recall that $p \geq 5$ and $diam(G) = 2$). Define the function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(u) = \phi$, $f(v_1) = f(x) = \{1\}$, $f(v_2) = f(y) = \{2\}$ and for all $w \in V(G) \setminus \{u, v_2, y\}$ by $f(w) = \{1\}$, clearly f is a G2RDF of G with $w(f) < p$. Hence the result.
The converse is clear.

Theorem 9 Let $\omega(G) \geq 3$ and F_2 induces a clique. Then $\gamma_{gr2}(G) = p$ if and only if $G \cong K_p - e$ or $\overline{G} \cong p_2K_2 \cup (p_1 - p_2)K_1$. Furthermore, if $\omega(G) = 3$ and $|F_2| = p_2 = 1$, then $G \cong K_4 - e$ or $G \cong K_4 - 2e$.
Proof. We have $p_1 \geq 3$. Thus we will discuss the proof according to $|F_2| = p_2$ into the following cases:
Case 1. Suppose $p_2 = 1$. Then $V(F_2) = \{x\}$.

(1) If $p_1 = 3$, then $|G| = 4$. Thus by Proposition 7, the results $G \cong K_4 - e$ or $G \cong K_4 - 2e$ hold (see Figure 3).
(2) Assume that $p_1 \geq 4$. We claim that the vertex x is non adjacent to exactly one vertex of F_1. For contrary, suppose x is non adjacent to two vertices of F_1 say y_1, y_2. We define $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(x) = \phi$, $f(y_1) = \{1\}$, $f(y_2) = \{2\}$, $f(y_3) = \{2\}$ and $f(y) = \{1\}$ for all $y \in V(F_1) \setminus \{y_1, y_2, y_3\}$.

Then f is a G2RDF of G with $w(f) < p$, a contradiction. Hence, $G \cong K_p - e$.

Case 2. Suppose $p_2 \geq 2$. We claim that each vertex of F_2 is non adjacent to exactly one vertex of F_1 and no two vertices of F_2 are non adjacent to the same vertex of F_1. For contrary, suppose x_1 in F_2 is non adjacent to y_1, y_2 in F_1. Define $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(x_1) = \phi$, $f(y_1) = \{1\}$, $f(y_2) = \{2\}$, $f(y) = \{1\}$ for all $y \in V(F_1) \setminus \{y_1, y_2\}$ and $f(x) = \{2\}$ for all $x \in V(F_2) \setminus \{x_1\}$. Clearly that f is a G2RDF of G with $w(f) < p$, a contradiction. Now, suppose there exists two vertices x_1 and x_2 of F_2 that are non adjacent to the vertex y_1 of F_1. Then the induced subgraph $\langle (F_1 \setminus \{y_1\}) \cup \{x_1, x_2\} \rangle$ is a clique of G of order $p_1 + 1$, which contradicts the maximality of F_1.
Assume that x_i is non adjacent to y_i, where $i = 1, 2, \ldots, p_2$ [recall that $p_2 \leq p_1$]. Therefore, $deg_G(x_i) = deg_G(y_i) = p_1 + p_2 - 2 = p - 2$ for all $i = 1, 2, \ldots, p_2$ and $deg_G(y_i) = p - 1$ for all $i = p_2 + 1, p_2 + 2, \ldots, p_1$. Hence, $\overline{G} \cong p_2 K_2 \cup (p_1 - p_2) K_1$. The converse is straight forward.

Theorem 10 Let G be a connected graph on $p \geq 5$ vertices. Suppose $\omega(G) \geq 4$ and F_2 induces an independent subgraph of G. Then $\gamma_{gr2}(G) = p$ if and only if $G \cong K_p - e$.
Proof. Since $\omega(G) \geq 4$, then each vertex $x_i \in V(F_2)$, $i = 1, 2, \ldots, p_2$ has at least one vertex in F_1 which they are non adjacent one to the other.
Claim 1. We claim that $p_2 = 1$. For contrary, suppose that $p_2 \geq 2$. Assume that x_1 is non adjacent to y_1, we define the function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(x_1) = \phi$, $f(y_1) = f(y_2) = \{1\}$ and $f(x) = f(y) = \{2\}$ for all $x \in V(F_2) \setminus \{x_1\}$ and $y \in V(F_1) \setminus \{y_1, y_2\}$. Clearly that f is a G2RDF of G with $w(f) < p$, which a contradiction. Then our claim is true. Hence, $V(F_2) = \{x\}$.
Claim 2. Now, we claim that x is non adjacent to exactly one vertex in F_1. This claim has proved in the proof of Theorem 9 (Case 1). Hence, $G \cong K_p - e$.
The other side is clear.

Theorem 11 Let $\omega(G) = 3$ and F_2 induces an independent subgraph of G. Then $\gamma_{gr2}(G) = p$ if and only if $G \cong K_4 - e$ or $G \cong K_4 - 2e$.
Proof. The proof is same as Theorem 10, with some different in Claim 2. Since Claim 1 holds, then $|G| = 4$. Thus from the proof of Proposition 7, we have only two graphs satisfy our conditions which are $G \cong H_1 = K_4 - e$ and $G \cong H_2 = K_4 - 2e$. The converse is clear.

Theorem 12 Let $\omega(G) \geq 3$ and F_2 be neither induce a clique nor independent. Then $\gamma_{gr2}(G) \neq p$.
Proof. Since F_2 be neither induce a clique nor independent and $\omega(G) \geq 3$, then $p_2 \geq 3$ and hence each vertex $x_i \in V(F_2)$, $i = 1, 2, \ldots, p_2$ has at least one vertex in F_1 and an other vertex in F_2 which it is non adjacent to both. Suppose x_1 is non adjacent to $x_2 \in V(F_2)$ and $y_1 \in V(F_1)$. We define the function $f = (V_\phi, V_{\{1\}}, V_{\{2\}}, V_A)$ by $f(x_1) = \phi$, $f(y_1) = f(y_2) = \{1\}$ and $f(x) = f(y) = \{2\}$ for all $x \in V(F_2) \setminus \{x_1\}$ and $y \in V(F_1) \setminus \{y_1, y_2\}$. Clearly that f is a G2RDF of G with $w(f) < p$. Hence, $\gamma_{gr2}(G) \neq p$.

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